Friction mediated by transient elastic linkages : extension to loads of bounded variation

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Abstract

In this work, we are interested in the convergence of a system of integro-differential equations with respect to an asymptotic parameter ε. It appears in the context of cell adhesion modelling [16, 15]. We extend the framework from [12, 13], strongly depending on the hypothesis that the external load f is in $\mathrm{Lip}([0,T])$ to the case where $f \in \mathrm{BV}(0,T)$ only. We show how results presented in [13] naturally extend to this new setting, while only partial results can be obtained following the comparison principle introduced in [12].

1 Introduction

Cell motility plays a central role in several important phenomenons in biology : cancer cell migration, neutrophils' extravasation, chemotaxis, etc. The present paper fits in the modelling framework presented in [17, 19, 10]. The adhesive dynamics of actin filaments are at the heart of the project : they contribute to lamellipodium's stabilization and allow the cell to attach to the substrate or the surrounding tissue. This paper contributes to the better mathematical understanding of a minimal model introduced first in [12], its aim is to extend results already obtained in [12, 13] to the case of *stiffer* external loads.

More precisely, we are interested in the motion of a single binding site, linked to a one-dimensional substrate and subjected to an external force f. As in [12, 13], the position of this binding site, denoted z_ε, solves a Volterra integral equation

$$\begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_0^\infty \left(z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a)\right) \rho_\varepsilon(a,t)\; da = f(t), & t \geq 0, \\ z_\varepsilon(t) = z_p(t), & t < 0. \end{cases} \tag{1}$$

The kernel ρ_ε above solves a non-local age-structured problem :

$$\begin{cases} \varepsilon\partial_t \rho_\varepsilon + \partial_a \rho_\varepsilon + \zeta_\varepsilon \rho_\varepsilon = 0, & t > 0,\ a > 0, \\ \rho_\varepsilon(a=0,t) = \beta_\varepsilon(t)\left(1 - \displaystyle\int_0^\infty \rho_\varepsilon(t,\tilde{a})\; d\tilde{a}\right), & t > 0, \\ \rho_\varepsilon(a, t=0) = \rho_{I,\varepsilon}(a), & a \geq 0, \end{cases} \tag{2}$$

where $\beta_\varepsilon \in \mathbb{R}_+$ (resp. $\zeta_\varepsilon \in \mathbb{R}_+$) is the kinetic on-rate (resp. off-rate) function. These possibly depend on the dimensionless parameter $\varepsilon > 0$. The past positions are stored in the Lipschitz function $z_p(t) \in \mathbb{R}$, prescribed for every $t < 0$.

Various mathematical issues related to this system have already been investigated [12, 13, 11]. In [12], the authors have introduced a specific Lyapunov functionnal in order to study the convergence

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of (2) when ε goes to 0. Indeed, due to the saturation effect in the non-local boundary condition in (2), neither the Generalized Relative Entropy [18, 5] nor more generic comparison principles [6] do apply. Then, concerning (1), under the assumptions that the force f is Lipschitz on $\mathbb{R}$, and because the kernel ρ_ε in (1) is non-negative, an extension of Gronwall's Lemma to integral equations, shows convergence of z_ε towards z_0 the solution of (3), the limit equation associated to (1). These two steps show that

$$\|z_\varepsilon - z_0\|_{C^0([0,T])} + \|\rho_\varepsilon - \rho_0\|_{C^0(]0,T];L^1(\mathbb{R}_+))} \to 0.$$

where z_0 it given by

$$\begin{cases} \mu_{1,0}(t)\partial_t z_0(t) = f(t), & t > 0 \\ z_0(0) = z_p(0) & t = 0 \end{cases} \tag{3}$$

where $\mu_{1,0}(t) := \int_0^\infty a\rho_0(a,t)da$, and ρ_0 solves :

$$\begin{cases} \partial_a \rho_0 + \zeta_0(a,t)\rho_0 = 0, & t > 0, a > 0, \\ \rho_0(a=0,t) = \beta_0(t)\left(1 - \int_0^\infty \rho_0(t,\tilde{a})\ d\tilde{a}\right), & t > 0. \end{cases} \tag{4}$$

In [13], the authors weakened some assumptions concerning the off-rate ζ_ε, by assuming that ζ_ε is not necessarily non-decreasing passed a certain age a_0. Then, they introduce a new variable u_ε related to z_ε which transforms (1) into a transport problem with a non-local source term :

$$\begin{cases} \varepsilon\partial_t u_\varepsilon + \partial_a u_\varepsilon = \dfrac{1}{\mu_{0,\varepsilon}(t)}\left(\varepsilon\partial_t f + \displaystyle\int_0^\infty \zeta_\varepsilon(\tilde{a},t)u_\varepsilon(\tilde{a},t)\rho_\varepsilon(\tilde{a},t)\ d\tilde{a}\right), & t > 0, a > 0, \\ u_\varepsilon(a=0,t) = 0, & t > 0, \\ u_\varepsilon(a,t=0) = u_{I,\varepsilon}(a) := \dfrac{z_\varepsilon(0) - z_p(-\varepsilon a)}{\varepsilon}, & a \geq 0, \end{cases} \tag{5}$$

where $\mu_{0,\varepsilon}(t) := \int_0^\infty \rho_\varepsilon(\tilde{a},t)\ d\tilde{a}$ and according to (1), it holds that

$$z_\varepsilon(0) = \frac{1}{\mu_{0,\varepsilon}(0)}\left(\int_0^\infty z_p(-\varepsilon a)\rho_{I,\varepsilon}(a)\ da + \varepsilon f(0)\right). \tag{6}$$

If $f \in \mathrm{Lip}(\mathbb{R})$, systems (1) and (5) are equivalent. Nevertheless, (5) admits a stability result that allows to show a weak-* convergence of $u_\varepsilon/(1+a)$ towards $u_0/(1+a)$ in $L^\infty(\mathbb{R}_+ \times (0,T))$, where u_0 is the solution of the limit problem

$$\begin{cases} \partial_a u_0 = \dfrac{1}{\mu_{0,0}}\displaystyle\int_0^\infty \zeta_0 u_0 \rho_0\ d\tilde{a}, & t > 0, a > 0, \\ u_0(a=0,t) = 0, & t > 0, \end{cases} \tag{7}$$

which in turn provides the strong convergence of z_ε in $C([0,T])$ towards z_0 solving (3).

In our analysis, however, when $f \in \mathrm{BV}((0,T))$, the derivative of f is neither a function nor it is bounded, since it is a Radon measure. Therefore, we cannot apply directly results from [12]. On the other hand, considering the framework exposed in [13] allows to give a meaning to (5), by approximation and show convergence when the approximation parameter δ and ε go to 0. Indeed, first we regularise the load f and analyze extensively the properties of the derivative in the BV setting and in the context of the Riemann-Stieltjes integral. Then starting from the regularized solution, we show how *a priori* estimates can be obtained uniformly with respect to δ and ε. Then we show that similar results can be obtained as in [13] in terms of convergence and consistency with the limit system. We show that in this new framework, the equivalence between (5) and (1) holds. For the particular case when the kernel ρ_ε is independent on time and on ε and under suitable hypotheses, we show error estimates to be compared with [12], the comparison principle being applied to the integral of the error's modulus.

The outline of the paper is as follows : in Section 2, collecting various results from the literature on BV-functions in one space dimension, we introduce the framework used in the rest of the paper. We make the link with the Riemann-Stieltjes integral, through a careful analysis of different definitions of BV-functions with respect to the boundary of the time domain $(0,T)$. In Section 3, we recall some results concerning (2) already established in [12]. Then in Section 4, we establish uniform (with respect to ε) *a priori* estimates for the regularized system in u_ε. After that, in Section 5, we show the weak convergence of u_ε towards u_0, the solution of the limit problem. This implies strong convergence of z_ε in $L^\infty(0,T)$ as stated in Theorem 8. We establish, in Section 6, a specific comparison principle for Volterra equations when the density ϱ is constant in time and does not depend on ε.

2 Notations and main assumptions

We denote $L^p_t L^q_a := L^p((0,T);L^q(\mathbb{R}_+))$ for any real $(p,q) \in [1,\infty]^2$ and

$$X_T := \left\{ g \in L^\infty_{loc}((0,T)\times\mathbb{R}_+) \; ; \; \sup_{t\in(0,T)} \|g(t,a)w(a)\|_{L^\infty_a} < \infty \right\} \tag{8}$$

where $w(a) := (1+a)^{-1}$. The space $\mathrm{Lip}(I)$ is the set of Lipschitz functions on the interval I.

Assumptions 1. For any $T > 0$ possibly infinite, we assume that :

i) The past condition z_p is L_{z_p}-Lipschitz on $\mathbb{R}_-$ *i.e.* :

$$|z_p(a_2) - z_p(a_1)| \le L_{z_p}|a_2 - a_1|, \quad \forall (a_2, a_1) \in \mathbb{R}_- \times \mathbb{R}_-.$$

ii) The function $\beta_\varepsilon(t)$ is in $L^\infty(0,T)$ and $\zeta_\varepsilon(a,t)$ is in $L^\infty(\mathbb{R}_+ \times (0,T))$.

iii) For limit functions $\beta_0 \in L^\infty_t$ and $\zeta_0 \in L^\infty_t L^\infty_a$ it holds that

$$\|\zeta_\varepsilon - \zeta_0\|_{L^\infty_{a,t}} \to 0 \quad \text{and} \quad \|\beta_\varepsilon - \beta_0\|_{L^\infty_t} \longrightarrow 0$$

as $\varepsilon \to 0$.

iv) There are upper and lower bounds such that

$$0 < \zeta_{\min} \le \zeta_\varepsilon(a,t) \le \zeta_{\max} \;\; \text{and} \;\; \beta_{\min} \le \beta_\varepsilon(a,t) \le \beta_{max} \; ,$$

for all $\varepsilon > 0$, $a \ge 0$ and $t > 0$.

Assumptions 2. The initial condition $\rho_{I,\varepsilon} \in L^\infty_a(\mathbb{R}_+)$ satisfies

i) positivity

$$\rho_{I,\varepsilon}(a) \ge 0 \; , \quad \text{a.e. in } \mathbb{R}_+ \; ,$$

moreover, on has also that the total initial population satisfies

$$0 < \int_{\mathbb{R}_+} \rho_{I,\varepsilon}(a)da < 1 \; ;$$

ii) boundedness of higher moments,

$$0 < \int_{\mathbb{R}_+} a^p \rho_{I,\varepsilon}(a)da < c_p \; , \quad \text{for } p = 1,2,$$

where c_p are positive constants depending only on p;

Next, we introduce definitions of functions with bounded variation in one dimension, as well as some related properties.

Definition 1. Let $f : (0,T) \to \mathbb{R}$ be a Lebesgue measurable function. The pointwise variation (or Jordan variation) of f on $(0,T)$ is

$$\operatorname{pvar}(f,(0,T)) := \sup_P \operatorname{var}(f,P) \tag{9}$$

where $\operatorname{var}(f,P) := \sum_{k=1}^n |f(t_k) - f(t_{k-1})|$ and $P = \{0 < t_0 < \cdots < t_n < T\}$ is a partition of $(0,T)$.

Moreover, we denote $\operatorname{BPV}((0,T)) := \{f \in \mathcal{L}((0,T)),\ \text{s.t } \operatorname{pvar}(f,(0,T)) < +\infty\}$, the space of measurable functions with pointwise bounded variation, see for example, [1, section 3.2], [9, chapter 2] and [7, section 2.2, 2.3]. The pointwise variation of f is clearly dependent on the value of f at each point of the domain, and it differs from one a.e.-representative of f to another. For this reason, for every measurable function f, one defines the essential pointwise variation :

$$\operatorname{epvar}(f,(0,T)) := \inf\{\operatorname{pvar}(g,(0,T)) : f(t) = g(t) \text{ a.e. } t \in (0,T)\} \tag{10}$$

In [9, Chapter 6], another functional space is defined :

Definition 2. Given an open interval $(0,T) \subset \mathbb{R}$, the space of functions with bounded variation $\operatorname{BV}((0,T))$ is defined as the space of all functions $f \in L^1((0,T))$ for which there exists a signed Radon measure μ_f such that

$$\int_{(0,T)} f\ \phi'\ dt = -\int_{(0,T)} \phi\ d\mu_f, \text{for every } \phi \in C_c^1((0,T)) \tag{11}$$

for all $\phi \in C_c^1((0,T))$. The measure μ_f is called the weak or distributional derivative of f.

Remark 1.

i) We define the total variation of $f \in L^1((0,T))$ by

$$\|Df\|((0,T)) = \sup\left\{-\int_{(0,T)} f\ \phi'\ dt,\ \ \phi \in C_c^1((0,T)), |\phi|_\infty \le 1\right\}. \tag{12}$$

Moreover, $f \in \operatorname{BV}((0,T))$ if $\|Df\|((0,T)) < +\infty$.

ii) Definitions (9) and (12) are not equivalent. For instance, the Dirichlet indicatrix function $\chi_{\mathbb{Q}\cap[0,1]}$ is not of pointwise bounded variation in $(0,1)$ in the sense of Definition 1 but is well defined in the sense of Definition 2. The equivalence between the two definitions holds up to a.e. equality. Moreover every integrable function $f : (0,T) \to \mathbb{R}$ such that $\operatorname{pvar}(f,(0,T)) < +\infty$, is in $\operatorname{BV}((0,T))$ and $\|Df\|((0,T)) \le \operatorname{pvar}(f,(0,T))$. On the other hand, if f belongs to $\operatorname{BV}((0,T))$, then f admits a right continuous representative $\bar{f}$ with bounded pointwise variation such that

$$\operatorname{pvar}(\bar{f},(0,T)) = \|Df\|((0,T)).$$

Fore more details, see, e.g., [9, theorem 7.3] and [7].

iii) Under the norm

$$\|f\|_{BV} := \|f\|_{L^1} + \operatorname{epvar}(f,(0,T)) < \infty$$

$\operatorname{BV}((0,T))$ is a Banach space.

Next, we provide existence of the left and right limits of functions with bounded variation [7, Proposition 2.2].

Lemma 1. Let $f \in \mathrm{BV}((0,T))$, Then both the limits

$$f(0^+) = \lim_{s\to 0, s>0} f(s) \quad \text{and} \quad f(T^-) = \lim_{s\to T, s<T} f(s) \quad \text{exist.}$$

Additionally, if f is integrable, the left and right limits are as follows:

Lemma 2. Suppose that $f \in \mathrm{BV}((0,T))$, then

$$f(0^+) = \lim_{\rho\to 0^+} \frac{1}{\rho}\int_0^\rho f(t)\ dt, \quad f(T^-) = \lim_{\rho\to 0^+} \frac{1}{\rho}\int_{T-\rho}^T f(t)\ dt.$$

Next, we present a result used in the proof of Proposition 3, which relates the pointwise variation to the Lebesgue measure:

$$\lambda(f,h,\Omega) := \int_{\{t\in\Omega: t+h\in\Omega\}} |f(t+h)-f(t)|\,dt,$$

Lemma 3. If f is in $\mathrm{BPV}((0,T))$, then $\lambda(f,h,(0,T))/|h|$ is bounded. Moreover,

$$\lambda(f,h,(0,T)) \le |h|\,\mathrm{pvar}(f,(0,T)).$$

For the proof we can see [9, Theorem 2.20].

Finally, in [7], the authors add a new notion of variation containing the boundary value in order to expand the total variation of f to $[0,T]$. This variation is defined as

$$\mathrm{varw}(f) := \sup_{\substack{\phi\in C_c^1([0,T])\\ |\phi|_\infty\le 1}} \left\{ \phi(T)f(T^-) - \phi(0)f(0^+) - \int_{(0,T)} f\ \phi'\ dt \right\} \tag{13}$$

Moreover, by summarizing the results of [7, Proposition 2.3, 2.6 and 2.7] all notions of variations coincide:

$$\mathrm{epvar}(f,(0,T)) = \|Df\|((0,T)) = \mathrm{varw}(f)$$

The previous result allows to extend Lemma 3 to $\mathrm{BV}((0,T))$ functions :

Lemma 4. If f is in $\mathrm{BV}((0,T))$, then $\lambda(f,h,(0,T))/|h|$ is bounded. Moreover,

$$\lambda(f,h,(0,T)) \le |h|\|Df\|((0,T))$$

Proof. By taking the infimum over almost every equal measurable functions, one has

$$\inf_{f=\tilde f \text{ a.e.}} \lambda(\tilde f,h,(0,T)) \le |h| \inf_{f=\tilde f \text{ a.e.}} \mathrm{pvar}(\tilde f,(0,T)) = |h|\,\mathrm{epvar}(f,(0,T)) = |h|\|Df\|((0,T))$$

Since the left hand side is a Lebesgue integral one has :

$$\inf_{f=\tilde f \text{ a.e.}} \lambda(\tilde f,h,(0,T)) = \lambda(f,h,(0,T))$$

which ends the proof. □

2.1 Data regularization

Theorem 1. For every $f \in \mathrm{BV}((0,T))$, there exists a sequence of smooth functions $(f_\delta)_\delta$ *in* $C^\infty((0,T))$ such that

$$\lim_{\delta\to 0}\int_{(0,T)} |f_\delta - f| dt = 0 \quad \text{and} \quad \lim_{\delta\to 0}\int_{(0,T)} |f'_\delta|\ dt = \|Df\|((0,T)).$$

Although the proof is classical (see for instance [20, Theorem 5.3.3 p.225]), we need the explicit form of f_δ in the rest of the paper. For this reason, we present in Section A.1 the proof of Theorem 1.

Lemma 5. Let $f \in \mathrm{BV}((0,T)) \cap L^\infty((0,T))$. Then the regularization function f_δ defined as (60) is bounded in $(0,T)$.

Next, we compare the left and right limits of f and its' approximation f_δ on the boundary:

Lemma 6. Let $f \in \mathrm{BV}((0,T))$ and f_δ defined as (60), then

$$f_\delta(0^+) = f(0^+) \text{ and } f_\delta(T^-) = f(T^-).$$

First we need the following result :

Proposition 1. Let $f \in \mathrm{BV}((0,T))$. For every $\delta > 0$, and $t_0 \in \{0,T\}$,

$$\lim_{\tau\to 0^+} \frac{1}{\tau}\int_{I_\tau\cap(0,T)} |f_\delta - f|\ dt = 0, \tag{14}$$

where $I_\tau = \{t \in \mathbb{R} : |t - t_0| < \tau\}$.

Proof. For a fixed $t_0 \in \{0,T\}$ and $t \in I_\tau \in (0,T)$, we have

$$f_\delta(t) - f(t) = \sum_{i=0}^{\infty}[\chi_{\delta_i} * (\phi_i f) - \phi_i f]$$

by the definition of supp ϕ_i (see (56)), we have $1/(j_0+i+1) < \tau < 1/(j_0+i-1)$ then $i > 1/\tau - j_0 - 1$. Since, $\mathbb{R}$ is an archimedean space then

$$\forall \tau > 0,\ \exists!\ i_0 := \left\lfloor \frac{1}{\tau} \right\rfloor - j_0 \text{ s.t } i_0 \le \frac{1}{\tau} < i_0 + 1 \tag{15}$$

which implies by using (58) that

$$\int_{I_\tau\cap(0,T)} |f_\delta - f|\ dt = \sum_{i=i_0}^{\infty}\int_{I_\tau\cap(0,T)} [\chi_{\delta_i} * (\phi_i f) - \phi_i f]\ dt \le \sum_{i=i_0}^{\infty} \delta 2^{-i}$$
$$\le \delta 2^{-i_0}\sum_{i=0}^{\infty} 2^{-i} = 2^{j_0+1}\ \delta\ 2^{-\lfloor \frac{1}{\tau} \rfloor} \qquad \square$$

then

$$\frac{1}{\tau}\int_{I_\tau\cap(0,T)} |f_\delta - f|\ dt \le C\ \delta\ \frac{2^{-\lfloor \frac{1}{\tau} \rfloor}}{\tau}$$

using again (15), we have

$$\frac{2^{-\lfloor \frac{1}{\tau} \rfloor}}{\tau} = \frac{\exp(-\lfloor \frac{1}{\tau} \rfloor \ln 2)}{\tau} \leq \frac{2\exp(-\frac{1}{\tau}\ln 2)}{\tau}$$

Finally, we conclude that

$$\lim_{\tau\to 0^+} \frac{1}{\tau} \int_{I_\tau \cap (0,T)} |f_\delta - f| \; dt = \lim_{\tau\to 0^+} \frac{2\exp(-\frac{1}{\tau}\ln 2)}{\tau} = 0.$$

Proof of Lemma 6. According to the Lemma 2,

$$\lim_{\tau\to 0^+} \frac{1}{\tau} \int_0^\tau |f_\delta(t) - f_\delta(0^+)| \; dt = 0 \quad \text{and} \quad \lim_{\tau\to 0^+} \frac{1}{\tau} \int_0^\tau |f(t) - f(0^+)| \; dt = 0.$$

Moreover, we have, thanks to Proposition 1

$$\lim_{\tau\to 0^+} \frac{1}{\tau} \int_0^\tau |f_\delta - f| \; dt = 0.$$

Thus, for all $\varepsilon' > 0$, there exist $\delta' > 0$ such that $0 < \tau < \delta'$ implies

$$\begin{aligned} \left|f_\delta(0^+) - f(0^+)\right| &= \frac{1}{\tau} \int_0^\tau \left|f_\delta(0^+) - f(0^+)\right| dt \\ &\leq \frac{1}{\tau} \int_0^\tau \left|f_\delta(0^+) - f_\delta(t)\right| dt + \frac{1}{\tau} \int_0^\tau |f_\delta(t) - f(t)| \, dt + \frac{1}{\tau} \int_0^\tau \left|f(t) - f(0^+)\right| dt \leq 3\varepsilon' \end{aligned}$$

which proves the required result. Similarly, we can prove that $f_\delta(T^-) = f(T^-)$. □

In the previous setting, the weak derivative of $f \in \mathrm{BV}((0,T))$ defines a linear continuous form on $C((0,T))$. In the next section, we show how to *extend* this measure on functions in $C([0,T])$.

2.2 Definition and basic properties of Stieltjes integral

The Riemann–Stieltjes integral (RS–integral) is a generalization of the Riemann integral. Let $\mathcal{P}$ a tagged partition of $[0,T]$, defined as

$$\mathcal{P} := \{(\xi_i, [t_{i-1}, t_i]) : 1 \leq i \leq n\} \tag{16}$$

where $0 = t_1 \leq \cdots \leq t_n = T$, and on each interval $[t_{i-1}, t_i]$ we choose a single value ξ_i, for $i \in \{1, \ldots, n\}$.

Definition 3. For any function f, $g : [0,T] \to \mathbb{R}$ and a partition $\mathcal{P}$, we define the Riemann-Stieltjes sum by

$$S(f, dg, \mathcal{P}, [0,T]) := \sum_i f(\xi_i)[g(t_i) - g(t_{i-1})].$$

Moreover, the RS-integral of f with respect to g

$$(RS) \int_{[0,T]} f(t) dg(t)$$

exists and has a value $I \in \mathbb{R}$, if, for every $\varepsilon > 0$, there exists $\delta > 0$, such that the mesh size $\max_i(t_i - t_{i-1}) < \delta$ and for every ξ_i in $[t_i, t_{i+1}]$,

$$|S(f, dg, \mathcal{P}, [0,T]) - I| < \varepsilon.$$

Lemma 7. Suppose that f is continuous on $[0,T]$ and g is of bounded pointwise variation on $[0,T]$, then

$$\left|\int_{[0,T]} f dg\right| \leq \|f\|_\infty \operatorname{pvar}(g,[0,T])$$

In the following Theorem we see that a Riemann-Stieltjes integral can be used to describe any bounded linear functional on $C([0,T])$ (see [3, Theorem 7.1.1] and [8, Theorem 4.4-1] for more details)

Theorem 2. Let $\Gamma_f \in (C([0,T]))'$, then there exist $g \in BPV([0,T])$ such that

$$\Gamma_f(\varphi) = \int_{[0,T]} \varphi dg, \qquad \forall \varphi \in C([0,T]).$$

Theorem 3. (Integration by parts). If one of the integrals $\int_{[0,T]} f dg$ and $\int_{[0,T]} g df$ exists, then the other exists as well, and we have

$$\int_{[0,T]} f dg + \int_{[0,T]} g df = [fg(t)]_{t=0}^{t=T}.$$

Moreover, If $f \in C^1([0,T])$ and $g \in BPV([0,T])$, then $df = f'dt$ in the second term of the left hand side.

For the proof cf [14, Theorem 5.52] and [2, Lemma 2].

Lemma 8. Let $f \in \mathrm{BV}((0,T))$. Then there exists $g \in \mathrm{BPV}([0,T])$ s.t

$$[f\varphi]_{t=0^+}^{t=T^-} - \int_{(0,T)} f\varphi' dx = \int_{[0,T]} \varphi dg, \quad \forall \varphi \in C^1([0,T])$$

s.t. $f(t) = g(t)$, a.e. $t \in (0,T)$

Proof. We regularize $f \in \mathrm{BV}((0,T))$ by $f_\delta \in C^\infty((0,T))$ as in Theorem 1, then we have :

$$\int_{t_k}^{s_k} f_\delta' \varphi dt + \int_{t_k}^{s_k} f_\delta \varphi' dt = [f_\delta \varphi]_{t=t_k}^{t=s_k} =: L_k, \quad \forall \varphi \in C^1([0,T])$$

where $s_k \to T^-$ and $t_k \to 0^+$. We define :

$$I_k := \int_{t_k}^{s_k} f_\delta' \varphi dt, \quad J_k := \int_{t_k}^{s_k} f_\delta \varphi' dt.$$

Thanks to Lebesgue's Theorem, one has that

$$\lim_{k\to\infty} I_k = I := \int_0^T f_\delta' \varphi dt, \quad \lim_{k\to\infty} J_k = J := \int_0^T f_\delta \varphi' dt$$

and thanks to Lemma 6 and the continuity of φ,

$$L_k = L := f(T^-)\varphi(T) - f(0^+)\varphi(0), \quad \forall k \in \mathbb{N}$$

So that we have :

$$\int_0^T f'_\delta \varphi dt + \int_0^T f_\delta \varphi' dt = [f\varphi]_{t=0^+}^{t=T^-}$$

If we set

$$\mathcal{I}_{f_\delta}(\varphi) := \int_0^T f'_\delta \varphi dt,$$

it is a linear continuous form on $C([0,T])$, since one has :

$$|\mathcal{I}_{f_\delta}(\varphi)| \le \|f'_\delta\|_{L^1((0,T))}\|\varphi\|_{L^\infty} \le \{\|Df\|((0,T)) + \delta\} \|\varphi\|_{L^\infty((0,T))} \tag{17}$$

where we used estimates from the proof of [20, Theorem 5.3.3]. Since it is a continuous linear form on $C([0,T])$, by Theorem 2 there exists $h_\delta \in \mathrm{BPV}([0,T])$ s.t.

$$\mathcal{I}_{f_\delta}(\varphi) = \int_{[0,T]} \varphi dh_\delta, \quad \forall \varphi \in C([0,T])$$

in the Stieljes' sense. But by using the integration by parts from Theorem 3, we have that

$$[f\varphi]_{t=0^+}^{t=T^-} - \int_{(0,T)} f_\delta \varphi' dt = \int_{[0,T]} \varphi dh_\delta = [h_\delta \varphi]_{t=0}^{t=T} - \int_{[0,T]} h_\delta \varphi' dt, \quad \forall \varphi \in C^1([0,T]) \tag{18}$$

which implies that

$$\int_{(0,T)} f_\delta \varphi' dt = \int_{(0,T)} h_\delta \varphi' dt, \quad \forall \varphi \in \mathcal{D}((0,T))$$

and then we can apply [9, Lemma 7.4] and conclude that there exists $c \in \mathbb{R}$, s.t.

$$f_\delta = h_\delta + c.$$

Then setting $g_\delta := h_\delta + c$ provides a function s.t.

$$[f_\delta \varphi]_0^T - \int_0^T f_\delta \varphi' dt = \int_0^T \varphi dg_\delta, \quad \forall \varphi \in C^1([0,T])$$

and s.t.

$$f_\delta(t) = g_\delta(t), \quad \text{a.e. } t \in (0,T).$$

Thanks to (17), $\mathcal{I}_{f_\delta}$ is a linear continuous form on $C([0,T])$ uniformly bounded with respect to δ. It can be identified via the Riesz representation theorem as a Radon measure μ_δ on $[0,T]$. Therefore, there exist $\mu \in M^1([0,T])$ and a sub-sequence μ_{δ_k} such that

$$\mu_{\delta_k} \overset{*}{\rightharpoonup} \mu$$

in $\sigma(M^1([0,T]), C([0,T]))$ with respect to the weak-* topology. By Theorem 2, there exists $h \in \mathrm{BPV}([0,T])$ s.t.

$$\mu(\varphi) = \int_0^T \varphi dh$$

where the left side is a Radon measure and the right hand side is the Riemann-Stieltjes integral. Because f_δ tends to f in the $L^1(0,T)$ topology, one has then that

$$[f\varphi]_{t=0^+}^{t=T^-} - \int_{(0,T)} f\varphi' dt = \int_{[0,T]} \varphi dh, \quad \forall \varphi \in C^1([0,T])$$

then using again integration by parts from Theorem 3, one concludes that

$$f(t) = h(t) + \tilde{c}, \quad \text{a.e. } t \in (0,T)$$

and setting $g = h + \tilde{c}$ ends the proof. □

Corollary 1. There exists a sub-sequence $(f_{\delta_k})_{k\in\mathbb{N}}$, s.t.

$$I_{f_{\delta_k}}(\varphi) := \int_0^T \varphi f'_{\delta_k} dt \to \int_0^T \varphi dg, \quad \forall \varphi \in C([0,T])$$

when $k \to \infty$

3 Mathematical background for the linkages' density

We list here some of the results proved in [12] used in the next sections of the paper.

Theorem 4. Let Assumptions 1 and 2 hold, then for every fixed $\varepsilon > 0$ there is a unique solution $\rho_\varepsilon \in C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$ of the problem (2). It satisfies (2) in the sense of characteristics, namely

$$\rho_\varepsilon(a,t) = \begin{cases} \beta_\varepsilon(t-\varepsilon a)\left(1 - \int_{\mathbb{R}_+} \rho_\varepsilon(\tilde{a}, t-\varepsilon a)\, d\tilde{a}\right) \exp\left(-\int_0^a \zeta_\varepsilon(\tilde{a}, t-\varepsilon(a-\tilde{a}))\, d\tilde{a}\right), & \forall a < \frac{t}{\varepsilon} \\ \rho_{I,\varepsilon}(a - \frac{t}{\varepsilon}) \exp\left(-\frac{1}{\varepsilon}\int_0^t \zeta_\varepsilon\left(\frac{\tilde{t}-t}{\varepsilon} + a, \tilde{t}\right)\, d\tilde{t}\right), & \forall a \geq \frac{t}{\varepsilon} \end{cases} \tag{19}$$

Moreover, it is a weak solution as well since it satisfies (cf [12, Lemma 2.1])

$$\begin{aligned} &\int_0^T \int_0^{+\infty} \rho_\varepsilon(a,t)\left(\varepsilon\partial_t\varphi + \partial_a\varphi - \zeta_\varepsilon\varphi\right) da\ dt - \varepsilon\int_0^{+\infty} \rho_\varepsilon(a,T)\varphi(a,T)\ da \\ &+ \int_0^T \rho_\varepsilon(a=0,t)\varphi(a=0,t)\ dt + \varepsilon\int_0^{+\infty} \rho_{I,\varepsilon}(a)\varphi(a,t=0)\ da = 0 \end{aligned} \tag{20}$$

for every $T > 0$ and test function $\varphi \in \mathcal{C}^\infty(\mathbb{R}_+^2) \cap L^\infty(\mathbb{R}_+^2)$. Now we define the moments of ρ_ε which we denote by $\mu_{p,\varepsilon}(t) := \int_{\mathbb{R}_+} a^p \rho_\varepsilon(a,t)da$, with $p = 1, 2$.

Lemma 9. Let Assumptions 1 and 2 hold, then the unique solution $\rho_\varepsilon \in C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$ of (2) satisfies

i) $\rho_\varepsilon(a,t) \geq 0$ for a.e (a,t) in $\mathbb{R}_+^2$,

ii) $\mu_{0,\min} \leq \mu_{0,\varepsilon}(t) < 1$, $\forall t \in \mathbb{R}_+$ where $\mu_{0,\min} < \min\left(\mu_{0,\varepsilon}(0), \frac{\beta_{\min}}{\beta_{\min}+\zeta_{\max}}\right)$

iii) $\mu_{p,\min} \leq \mu_{p,\varepsilon}(t) \leq k$, where , $\mu_{p,\min} = \min\left(\mu_{p,\varepsilon}(0), \dfrac{\mu_{p-1,\min}}{\zeta_{\max}}\right)$

The authors provide a Liapunov functional that reads :

$$\mathcal{H}[u] := \left|\int_0^\infty u(a)da\right| + \int_0^\infty |u(a)|da\ ,$$

thanks to which they obtain the following convergence result for ρ_ε :

Lemma 10. Let $\zeta_{\min} > 0$ be the lower bound to $\zeta_\varepsilon(a,t)$ according to Assumptions 1, and setting $\hat{\rho}_\varepsilon :=$ one has

$$\mathcal{H}[(\rho_\varepsilon - \rho_0)(\cdot,t)] \leq \mathcal{H}[\rho_{I,\varepsilon} - \rho_0(\cdot,0)] \exp\left(\frac{-\zeta_{\min} t}{\varepsilon}\right) + \frac{2}{\zeta_{\min}} \left\| \|R_\varepsilon\|_{L^1_a(\mathbb{R}_+)} + |M_\varepsilon| \right\|_{L^\infty_t(\mathbb{R}_+)} \tag{21}$$

with $R_\varepsilon := -\varepsilon\partial_t\rho_0 - \rho_0(\zeta_\varepsilon - \zeta_0)$ and $M_\varepsilon := (\beta_\varepsilon - \beta_0)(1 - \int_0^\infty \rho_0 da)$.

This ensures the convergence of ρ_ε that reads :

Theorem 5. Let ρ_ε the solution of the system (2) and let ρ_0 given by (4), then

$$\rho_\varepsilon \to \rho_0 \text{ in } C^0((0,\infty); L^1(\mathbb{R}_+)) \text{ as } \varepsilon \to 0 ,$$

where the convergence with respect to time is meant in the sense of uniform convergence on compact subintervals.

4 Existence, uniqueness and stability

Using the regularized function f_δ introduced in Theorem 1, we consider an approximation of (1) : we denote by $z_\varepsilon^\delta := z_\varepsilon^\delta(t)$ the function solving

$$\begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_0^\infty \left(z_\varepsilon^\delta(t) - z_\varepsilon^\delta(t-\varepsilon a)\right)\rho_\varepsilon(a,t)da = f_\delta(t), & t \le 0 \\ z_\varepsilon^\delta(t) = z_p(t), & t < 0 \end{cases} \tag{22}$$

We also define $u_\varepsilon^\delta(a,t)$, an approximation of the elongation variable u_ε, defined as the mild solution of

$$\begin{cases} \varepsilon\partial_t u_\varepsilon^\delta + \partial_a u_\varepsilon^\delta = \dfrac{1}{\mu_{0,\varepsilon}}\left(\varepsilon f_\delta' + \displaystyle\int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon da\right), & t>0, a>0 \\ u_\varepsilon^\delta(a=0,t) = 0, & t>0 \\ u_\varepsilon^\delta(a,t=0) = u_{I,\varepsilon}^\delta(a), & a \ge 0 \end{cases} \tag{23}$$

where

$$u_{I,\varepsilon}^\delta(a) := \frac{z_\varepsilon^\delta(0^+) - z_p(-\varepsilon a)}{\varepsilon} \tag{24}$$

and

$$z_\varepsilon^\delta(0^+) = \frac{1}{\mu_{0,\varepsilon}(0)}\left(\varepsilon f_\delta(0^+) + \int_0^\infty z_p(-\varepsilon a)\rho_{I,\varepsilon}(a)da\right). \tag{25}$$

More precisely, u_ε^δ is a solution of system (23) in the sense of characteristics, namely

$$u_\varepsilon^\delta(a,t) = \begin{cases} \int_0^a h(t-\varepsilon\tilde{a})\, d\tilde{a}, & \text{if } t > \varepsilon a \\ \int_0^{t/\varepsilon} h(t-\varepsilon\tilde{a})\, d\tilde{a} + u_{I,\varepsilon}^\delta(a - t/\varepsilon), & \text{if } t \le \varepsilon a, \end{cases} \tag{26}$$

where

$$h(t) := \frac{1}{\mu_{0,\varepsilon}}\left(\varepsilon\partial_t f_\delta + \int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon da\right).$$

By arguments similar to [12, Lemma 3], it is as well a weak solution of (23) *i.e.*

$$\begin{aligned} &-\int_0^T\int_0^\infty u_\varepsilon^\delta(\varepsilon\partial_t\varphi + \partial_a\varphi)\, da\, dt + \left[\int_0^\infty u_\varepsilon^\delta(s,a)\varphi(s,a)\, da\right]_{s=0}^{s=T} \\ &= \int_0^T \frac{1}{\mu_{0,\varepsilon}}\left(\varepsilon f_\delta' + \int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon\, da\right)\left(\int_0^\infty \varphi(t,\tilde{a})\, d\tilde{a}\right) dt \end{aligned} \tag{27}$$

for any function $\varphi \in C_c^\infty([0,T]\times\mathbb{R}_+)$. Although, problem (22) can be defined for weaker data (typically $L^1((0,T))$ or the space of Radon measures $M((0,T))$), the elongation problem (23), requires to give a meaning to the time derivative of f, which is more restrictive. Nevertheless, as we are mainly interested in convergence results, $f \in \mathrm{BV}((0,T))$ seems the weakest possible regularity to our knowledge.

Theorem 6. Let Assumptions 1 hold, and let ρ_ε be the unique solution of (2) then the system (23) has a unique solution $u_\varepsilon^\delta \in X_T$.

We are in the framework of [13, Theorem 6.1], but for sake of self-containtness, we recall in an abriged version the proof hereafter.

Proof. A Banach fixed point Theorem is used to prove this result. We define the mapping $\phi(v) = u$ such that by Duhamel's principle

$$u(a,t) = \begin{cases} \int_0^a G(t-\varepsilon\tilde{a})\ d\tilde{a}, & \text{if } t > \varepsilon a \\ \int_0^{t/\varepsilon} G(t-\varepsilon\tilde{a})\ d\tilde{a} + u_{I,\varepsilon}^\delta(a - t/\varepsilon), & \text{if } t \leq \varepsilon a. \end{cases} \tag{28}$$

where

$$G(t) := \frac{1}{\mu_{0,\varepsilon}} \left(\varepsilon f_\delta' + \int_0^\infty \zeta_\varepsilon(a,t) v(a,t) \rho_\varepsilon(a,t) da \right)$$

As in [13], a simple computation shows that

$$\|u\|_{X_T} \leq \|G\|_{L^\infty((0,T))} \frac{T}{T+\varepsilon} + \left\| \frac{u_{I,\varepsilon}^\delta(\cdot)}{1+a} \right\|_{L^\infty(\mathbb{R}_+)}$$

Moreover, since $\partial_t f_\delta \in L^\infty((0,T))$

$$\|G\|_{L^\infty((0,T))} \leq \frac{1}{\mu_{0,\min}} \varepsilon\ \|f_\delta'\|_{L^\infty((0,T))} + \frac{\zeta_{\max}(1+k)}{\mu_{0,\min}} \|v\|_{X_T}$$

where k is the upper bound of $\mu_{1,\varepsilon}$ proved in Lemma 9. Furthermore, by the same argument we can prove that ϕ is a contraction. Indeed, if $u_i = \phi(v_i)$ for $i \in \{1,2\}$

$$\|u_2 - u_1\|_{X_T} \leq C \frac{T}{T+\varepsilon} \|v_2 - v_1\|_{X_T}$$

for a constant $C > 0$. Then we can choose $T < \varepsilon/C$ and we obtain the existence of a local solution in time of (23), by Banach-Picard's fixed point theorem. As the contraction time does not depend on the initial data, we shall extend the same result by continuation. This shows existence and uniqueness in X_T for any $T > 0$. □

Lemma 11. If Assumptions 1 holds, then the solution of system (23) satisfies the uniform a priori estimates

$$\begin{aligned} \int_0^\infty \rho_\varepsilon(a,t) |u_\varepsilon^\delta(a,t)|\ da &\leq \int_0^t |f_\delta'|\ d\tilde{t} + \int_{\mathbb{R}_+} \rho_{I,\varepsilon} |u_{I,\varepsilon}^\delta|\ da \\ &\leq C \left(\|f\|_{\mathrm{BV}((0,T))}, \|(1+a)\rho_I\|_{L^1(\mathbb{R}_+)}, \|z_p'\|_{L^\infty(\mathbb{R}_-)} \right) \end{aligned} \tag{29}$$

where C is independent on ε and on δ.

Proof. Again, we proceed as in [13, Lemma 5.1], multiplying (23) by $\operatorname{sgn}(u_\varepsilon^\delta)$, testing against ρ_ε, and integrating with respect to a gives :

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_+} |u_\varepsilon^\delta| \rho_\varepsilon da + \int_{\mathbb{R}_+} |u_\varepsilon^\delta| \zeta_\varepsilon \rho_\varepsilon da \leq \varepsilon |f_\delta'| + \int_{\mathbb{R}_+} |u_\varepsilon^\delta| \zeta_\varepsilon \rho_\varepsilon da$$

the rigorous proof relies on arguments exposed in [12, Lemma 3.1] and is left to the reader. Finally, after integration with respect to time, we conclude that

$$\int_0^\infty \rho_\varepsilon(a,t)|u_\varepsilon^\delta(a,t)|da \le \int_0^t |f_\delta'|dt + \int_{\mathbb{R}_+} \rho_{I,\varepsilon}|u_{I,\varepsilon}^\delta|da$$

since

$$\begin{aligned}
|u_{I,\varepsilon}^\delta(a)| &\le \left|\frac{z_\varepsilon^\delta(0^+) - z_p(0)}{\varepsilon}\right| + \left|\frac{z_p(0) - z_p(-\varepsilon a)}{\varepsilon}\right| \\
&\le \frac{1}{\mu_{0,\varepsilon}(0)}\left|f_\delta(0^+) + \frac{1}{\varepsilon}\int_0^\infty (z_p(-\varepsilon a) - z_p(0))\rho_{I,\varepsilon}(a)\; da\right| + \left|\frac{z_p(0) - z_p(-\varepsilon a)}{\varepsilon}\right| \\
&\le \frac{1}{\mu_{0,\min}}\left(\left\|z_p'\right\|_{L^\infty(\mathbb{R}_-)}\mu_{1,\varepsilon}(0) + f_\delta(0^+)\right) + \left\|z_p'\right\|_{L^\infty(\mathbb{R}_-)} a \\
&\le \max\left\{\frac{1}{\mu_{0,\min}}\left\|z_p'\right\|_{L^\infty(\mathbb{R}_-)}\mu_{1,\varepsilon}(0) + f_\delta(0^+), \left\|z_p'\right\|_{L^\infty(\mathbb{R}_-)}\right\}(1+a),
\end{aligned}$$

the result follows. □

In order to establish the convergence of u_ε^δ in X_T, for a fixed ε, we introduce an intermediate variable w defined as

$$w(a,t) := u_\varepsilon^\delta(a,t) - \frac{f_\delta(t)}{\mu_{0,\varepsilon}(t)}. \tag{30}$$

It satisfies

$$\begin{cases} \varepsilon\partial_t w + \partial_a w = \dfrac{1}{\mu_{0,\varepsilon}}\left(\varepsilon\dfrac{f_\delta \partial_t \mu_{0,\varepsilon}}{\mu_{0,\varepsilon}} + \displaystyle\int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon da\right), & t > 0, a > 0, \\ w(a=0,t) = \dfrac{-\, f_\delta(t)}{\mu_{0,\varepsilon}(t)}, & t > 0, \\ w(a, t=0^+) = u_{I,\varepsilon}^\delta(a) - \dfrac{f_\delta(0^+)}{\mu_{0,\varepsilon}(0)}, & a \ge 0, \end{cases} \tag{31}$$

The following crucial result holds:

Lemma 12. For a fixed δ and ε, and under the Assumptions 1, the unknowns w and u_ε^δ, are uniformly bounded in X_T with respect to δ and ε.

Proof. Using arguments from [12, Lemma 2.1], one can show that w defined as

$$w(a,t) := \begin{cases} w(0, t - \varepsilon a) + \int_0^a G_w(t - \varepsilon\tilde{a})\; d\tilde{a}, & \text{if } t > \varepsilon a \\ w(a - t/\varepsilon, 0^+) + \int_0^{t/\varepsilon} G_w(t - \varepsilon\tilde{a})\; d\tilde{a}, & \text{if } t \le \varepsilon a. \end{cases} \tag{32}$$

is a weak solution of (31). In the latter definition $G_w(t) := \left\{\varepsilon\dfrac{f_\delta \partial_t \mu_{0,\varepsilon}}{\mu_{0,\varepsilon}^2} + \dfrac{1}{\mu_{0,\varepsilon}}\displaystyle\int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon^\delta da\right\}$. A simple computation shows that

$$\|w\|_{X_T} \le \|G_w\|_{L^\infty((0,T))} + \|w(0,.)\|_{L^\infty((0,T))} + \left\|\frac{w(.,0)}{1+a}\right\|_{L^\infty(\mathbb{R}_+)}.$$

It remains to estimate $\|G_w\|_{L^\infty(0,T)}$. For every fixed ε, $\mu_{0,\varepsilon}$ is a Lipschitz continuous function. Indeed, $\mu_{0,\varepsilon}$ satisfies

$$\varepsilon\partial_t\mu_{0,\varepsilon} - \beta_\varepsilon(1 - \mu_{0,\varepsilon}) + \int_{\mathbb{R}_+} \zeta_\varepsilon \rho_\varepsilon\; da = 0$$

and then

$$\|\varepsilon\partial_t\mu_{0,\varepsilon}\|_{L^\infty_t} \leq \|\beta_\varepsilon\|_{L^\infty_t} + \|\zeta_\varepsilon\|_{L^\infty_{t,a}}\|\rho_\varepsilon\|_{L^\infty_t L^1_a},$$

which shows, by using the result of Lemma 11 that G_w, and also w, are uniformly bounded in $L^\infty((0,T))$ with respect to δ and ε. Indeed,

$$\|G_w\|_{L^\infty((0,T))} \leq \frac{\|\varepsilon\partial_t\mu_{0,\varepsilon}\|_{L^\infty((0,T))}\|f_\delta\|_{L^\infty((0,T))}}{\mu_{0,\min}^2} + \frac{\zeta_{\max}}{\mu_{0,\min}}\int_{\mathbb{R}_+}\rho_\varepsilon|u^\delta_\varepsilon|da < +\infty.$$

Finally, we have that

$$\|u^\delta_\varepsilon\|_{X_T} \leq \|w\|_{X_T} + \frac{\|f_\delta\|_{L^\infty((0,T))}}{\mu_{0,\min}} < +\infty$$

which ends the proof. □

Previous stability estimates allow to show :

Theorem 7. Under Assumption 1, one has for any fixed $\varepsilon > 0$,

$$u^\delta_\varepsilon \rightharpoonup u_\varepsilon \text{ weakly-* in } X_T$$

as $\delta \to 0$, where u_ε solves the weak problem

$$\begin{aligned} &-\int_0^T\int_0^\infty u_\varepsilon(\varepsilon\partial_t+\partial_a)\varphi\ da\ dt + \varepsilon\left[\int_0^\infty u_\varepsilon(a,s)\varphi(a,s)\ da\right]_{s=0}^{s=T} \\ &= \varepsilon\int_0^T \frac{\int_0^\infty \varphi(\tilde a,t)d\tilde a}{\mu_{0,\varepsilon}}\ dg + \int_0^T \frac{\int_0^\infty \zeta_\varepsilon\rho_\varepsilon u_\varepsilon\ da}{\mu_{0,\varepsilon}}\left(\int_0^\infty \varphi(\tilde a,t)\ d\tilde a\right) \end{aligned} \tag{33}$$

for any $\varphi \in C^1_c([0,T]\times\mathbb{R}_+)$.

Proof. The uniform bound on u^δ_ε in X_T, proved in Lemma 12, implies that

$$\frac{u^\delta_\varepsilon}{1+a} \overset{*}{\rightharpoonup} \frac{u_\varepsilon}{1+a}$$

in $L^\infty((0,T)\times\mathbb{R}_+)$ in the weak-* sense and the limit function u_ε belongs X_T. For every $\psi \in L^\infty((0,T)\times\mathbb{R}_+)$, we have $\zeta_\varepsilon(1+a)\rho_\varepsilon\psi \in L^1((0,T)\times\mathbb{R}_+)$ and then

$$\int_0^T\int_0^\infty \zeta_\varepsilon\rho_\varepsilon u^\delta_\varepsilon\psi\ da\ dt \to \int_0^T\int_0^\infty \zeta_\varepsilon\rho_\varepsilon u_\varepsilon\psi\ da\ dt.$$

By Corollary 1, the first term of right-hand in (27) tends to $\int_0^T\int_0^\infty \varphi(t,\tilde a)d\tilde a/\mu_{0,\varepsilon}\ dg$ as $\delta \to 0$, for any $\varphi \in \mathcal{C}^1_c([0,T]\times\mathbb{R}_+)$. Regarding the second term of the right hand side in (27), one has that $\zeta_\varepsilon\rho_\varepsilon/\mu_{0,\varepsilon} \in L^1((0,T)\times\mathbb{R}_+)$ and this leads, thanks again to the weak-* convergence above to write :

$$\int_0^t \frac{1}{\mu_{0,\varepsilon}}\int_0^\infty \zeta_\varepsilon\rho_\varepsilon u^\delta_\varepsilon\ da\ d\tilde t \longrightarrow \int_0^t \frac{1}{\mu_{0,\varepsilon}}\int_0^\infty \zeta_\varepsilon\rho_\varepsilon u_\varepsilon\ da\ d\tilde t \ \text{ as } \delta\to 0, \tag{34}$$

which ends the proof. □

The latter theorem allows to prove a convergence result when returning to the z_ε variable :

Proposition 2. Under the same assumptions as above, it holds that

$$z_\varepsilon^\delta \;\to\; z_\varepsilon \text{ strongly in } L^\infty((0,T)) \text{ as } \delta \to 0,$$

where z_ε satisfies

$$z_\varepsilon(t) = z_\varepsilon(0^+) + \int_0^t \frac{\varepsilon}{\mu_{0,\varepsilon}}\, dg + \int_0^t \frac{1}{\mu_{0,\varepsilon}} \int_0^\infty \zeta_\varepsilon u_\varepsilon \rho_\varepsilon \; da \; d\tilde{t} \tag{35}$$

which is also a solution of (1).

Before showing this result, we make some comments : if u_ε^δ is a solution of (23) then z_ε^δ defined as

$$z_\varepsilon^\delta(t) := z_\varepsilon^\delta(0^+) + \int_0^t \frac{1}{\mu_{0,\varepsilon}(\tilde{t})} \left(\varepsilon f_\delta'(\tilde{t}) + \int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon da \right) d\tilde{t} \tag{36}$$

solves (22). Conversely, if z_ε^δ solves (22) then u_ε^δ, given by

$$u_\varepsilon^\delta(a,t) = \begin{cases} \dfrac{z_\varepsilon^\delta(t) - z_\varepsilon^\delta(t-\varepsilon a)}{\varepsilon}, & \text{if } t > \varepsilon a \\ \dfrac{z_\varepsilon^\delta(t) - z_p(t-\varepsilon a)}{\varepsilon}, & \text{if } t \le \varepsilon a \end{cases} \tag{37}$$

is a solution of (23). For more details see [13, Lemma 4.1 and Lemma 4.2].

Proof. First, by using (34) in the proof of Theorem 7, and Lemma 6, we have that $f_\delta(0^+) = f(0^+)$ and z_ε^δ given by (36) converge strongly in $L^\infty((0,T))$ to z_ε which verifies (35). Using [13, Lemma 4.2], if z_ε^δ is defined as (36) it solves (22). Multiplying (22) by a test function $\varphi \in L^1(0,T)$ gives :

$$\frac{1}{\varepsilon}\int_0^T \int_0^{+\infty} (z_\varepsilon^{\delta_k}(t) - z_\varepsilon^{\delta_k}(t-\varepsilon a))\; \rho_\varepsilon(a,t)\; \varphi(t)\; da\; dt = \int_0^T f_{\delta_k}(t)\; \varphi(t)\; dt \tag{38}$$

As $z_\varepsilon^{\delta_k}$ converges strongly in $L^\infty(0,T)$ to z_ε, the difference $z_\varepsilon^{\delta_k}(t) - z_\varepsilon^{\delta_k}(t-\varepsilon a)$ converges almost every where for any fixed (a,t) in $\mathbb{R}_+ \times (0,T)$ towards $z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a)$. Thanks to the L^∞ bounds on $u_\varepsilon^{\delta_k}/(1+a)$, and the bounds in $L^1(\mathbb{R}_+ \times (0,T))$ on the first moment of ρ_ε, there exists an integrable majorizing function $g(a,t)$ on $\mathbb{R}_+ \times (0,T)$ s.t.

$$\left| z_\varepsilon^{\delta_k}(t) - z_\varepsilon^{\delta_k}(t-\varepsilon a) \right| |\varphi(t)| \rho_\varepsilon(a,t) \le g(a,t)$$

uniformly for every k. Thus one can apply the Lebesgue's Theorem in the right hand side of (38). Since f_δ converges in $L^1(0,T)$ the convergence occurs in (38) for every $\varphi \in \mathcal{D}(0,T)$ and thus almost everywhere in $(0,T)$ and thus z_ε solves (1). □

5 Weak convergence when ε goes to zero

Next, we prove the weak convergence of u_ε from which we deduce the strong convergence of z_ε.

Theorem 8. Under the same assumptions as above, one has

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly-* in } X_T$$

as $\varepsilon \to 0$, where u_0 satisfies (7) and

$$\int_0^\infty u_0(a,t)\ \rho_0(a,t)\ da = f(t)\ a.e\ t\ \in (0,T).$$

Furthermore, it also holds that

$$z_\varepsilon \to z_0 \text{ strongly in } L^\infty((0,T)) \text{ as } \varepsilon \to 0.$$

Proof. The proof follows the same steps as in [13, Theorem 6.2]. First, by Lemma 12, u_ε^δ is uniformly bounded in X_T with respect to δ and ε, and therefore u_ε is uniformly bounded in X_T with respect to ε, then u_ε is weakly convergent to u_0 in X_T. On the other hand, Theorem 3.2 and Lemma 3.4 imply that

$$(1+a)\rho_\varepsilon \to (1+a)\rho_0$$

strongly in $L^1((0,T)\times\mathbb{R}_+)$. These arguments justify that for every $\psi \in L^\infty((0,T)\times\mathbb{R}_+)$ one has

$$\int_0^T\int_0^\infty \zeta_\varepsilon\rho_\varepsilon u_\varepsilon\psi\ da\ dt \to \int_0^T\int_0^\infty \zeta_0\rho_0 u_0\psi\ da\ dt.$$

Indeed, one has

$$\int_0^T\int_0^\infty \{\zeta_\varepsilon\rho_\varepsilon u_\varepsilon - \zeta_0\rho_0 u_0\}\,\psi\ da\ dt = \int_0^T\int_0^\infty (\zeta_\varepsilon-\zeta_0)\rho_\varepsilon u_\varepsilon\psi\ da\ dt$$
$$+\int_0^T\int_0^\infty \zeta_0(\rho_\varepsilon-\rho_0)u_\varepsilon\psi\ da\ dt + \int_0^T\int_0^\infty \zeta_0\rho_0\,(u_\varepsilon-u_0)\,\psi\ da\ dt$$

As $\zeta_\varepsilon \to \zeta_0$ by Assumptions 1, and thanks to the weak convergence of u_ε, both terms on the right-hand side tend to zero as $\varepsilon \to 0$. Note that this implies the weak convergence of $\int_{\mathbb{R}_+}\zeta_\varepsilon\rho_\varepsilon u_\varepsilon\ da$ in $L^1((0,T))$, since we can choose $\psi \in L^\infty((0,T))$. Moreover, thanks to Lemma 7 we obtain that

$$\left|\int_{[0,T]} \frac{\int_0^{+\infty}\varphi\ da}{\mu_{0,\varepsilon}}\ dg\right| \le \frac{C\|\varphi\|_{L^\infty_{t,a}}}{\mu_{0,\min}}\ \mathrm{pvar}(g,[0,T]) \le C\|f\|_{\mathrm{BV}((0,T))}.$$

As in [13, Theorem 6.2], passing to the limit in the weak formulation (33) we obtain

$$-\int_0^T\int_0^\infty u_0\partial_a\psi\ da\ dt = \int_0^T\int_0^\infty \frac{\zeta_0\rho_0u_0\psi}{\mu_{0,0}}\ da\ dt$$

which implies that u_0 satisfies

$$\begin{cases} \partial_a u_0 = \dfrac{1}{\mu_{0,0}}\displaystyle\int_0^\infty \zeta_0(\tilde a,t)u_0(\tilde a,t)\rho_0(\tilde a,t)\ d\tilde a, & t>0, a>0,\\ u_0(a=0,t)=0, & t>0, \end{cases} \tag{39}$$

Similarly, we have the weak convergence of $\int_0^\infty u_\varepsilon(t,a)\ \rho_\varepsilon(t,a)\ da$ towards $\int_0^\infty u_0(t,a)\ \rho_0(t,a)\ da$ in $L^1((0,T))$. Hence, one concludes that u_0 satisfies also

$$\int_0^\infty u_0(t,a)\ \rho_0(t,a)\ da = f(t), \quad \text{a.e. } t\in(0,T).$$

As the the right hand side of (39) does not depend on age, one has that $u_0 = \gamma(t)\ a$, where in order to satisfy the last compatibility condition implies that

$$\gamma(t) \int_{\mathbb{R}_+} a\rho_0(a,t)da = f(t), \quad \text{a.e. } t \in (0,T).$$

Thus $u_0(a,t) = f(t)/\mu_{1,0}(t)a$ for almost every $t \in (0,T)$ and every $a \in \mathbb{R}_+$. Using again the weak convergence of $\int_{\mathbb{R}_+} \zeta_\varepsilon \rho_\varepsilon u_\varepsilon\ da$ in $L^1((0,T))$ combined with the strong convergence of $\mu_{0,\varepsilon}$ allows to pass to the limit in the third term of (35).

Moreover,

$$\begin{aligned} |z_\varepsilon(0^+) - z_p(0)| &= \frac{1}{\mu_{0,\varepsilon}(0)} \left| \int_0^\infty (z_p(-\varepsilon a) - z_p(0))\, \rho_{I,\varepsilon}(a)\ da + \varepsilon f(0^+) \right| \\ &\leq \frac{\varepsilon k \|z_p'\|_{L^\infty(\mathbb{R}-)}}{\mu_{0,\min}} + \frac{\varepsilon |f(0^+)|}{\mu_{0,\min}} \longrightarrow 0 \text{ as } \varepsilon \to 0 \end{aligned}$$

where k is the constant from Lemma 9.

All together this provides that z_0 solves :

$$z_0(t) = z_p(0) + \int_0^t \frac{f(\tau)}{\mu_{1,0}(\tau)} \frac{\int_{\mathbb{R}_+} \zeta_0(a,\tau)\rho_0(a,\tau)da}{\mu_{0,0}(\tau)} d\tau \tag{40}$$

but because ρ_0 solves (4), one has that $a\rho_0$ solves :

$$\partial_a(a\rho_0) - \rho_0 + a\zeta_0(a,t)\rho_0 = 0,$$

which after integration in time shows that

$$\frac{\int_{\mathbb{R}_+} a\zeta_0(a,t)\rho_0(a,t)da}{\mu_{0,0}(t)} = 1$$

and this shows in turn that (40) reduces to

$$z_0(t) = z_0(0) + \int_0^t \frac{f(\tau)}{\mu_{1,0}(\tau)} d\tau$$

which is the integrated version of (3). □

6 A comparaison principle

In this section, we give error estimates between z_ε and z_0, the solution of the problem

$$\begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_0^\infty (z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a))\, \varrho(a)\ da = f(t), & t \geq 0, \\ z_\varepsilon(t) = z_p(t), & t < 0, \end{cases} \tag{41}$$

where ϱ is constant in time and satisfies

$$\begin{cases} \partial_a \varrho + \zeta(a)\varrho = 0, & a > 0, \\ \varrho(0) = \beta \left(1 - \displaystyle\int_0^\infty \varrho(\tilde{a})\ d\tilde{a}\right), \end{cases} \tag{42}$$

where the data of (41) and (42) satisfy

Assumptions 3.

i) $f \in \mathrm{BV}((0,T))$,

ii) $z_p \in \mathrm{Lip}(\mathbb{R}_-)$,

iii) $\beta \in \mathbb{R}_+^*$,

iv) $\zeta \in L^\infty(\mathbb{R}_+)$ such that

$$0 < \zeta_{\min} \leq \zeta(a) \leq \zeta_{\max}, \quad \text{a.e. } a \in \mathbb{R}_+.$$

Setting $\hat{z}_\varepsilon(t) := z_\varepsilon(t) - z_0(t)$, it solves :

$$\hat{z}_\varepsilon(t) = \frac{1}{\mu_0} \int_0^{\frac{t}{\varepsilon}} \hat{z}_\varepsilon(t - \varepsilon a) \varrho(a) da + \tilde{h}_\varepsilon(t)$$

where

$$\begin{aligned} \tilde{h}_\varepsilon(t) = & \frac{\varepsilon}{\mu_0} \int_0^{t/\varepsilon} \left(\frac{z_0(t) - z_0(t - \varepsilon a)}{\varepsilon} - a \partial_t z_0(t) \right) \varrho(a) \, da \\ & + \frac{\varepsilon}{\mu_0} \int_{t/\varepsilon}^{+\infty} \left(\frac{z_0(t) - z_0(0)}{\varepsilon} - a \partial_t z_0(t) \right) \varrho(a) \, da + \frac{1}{\mu_0} \int_{t/\varepsilon}^{+\infty} (z_p(t - \varepsilon a) - z_p(0)) \varrho(a) \, da \end{aligned} \tag{43}$$

where $\mu_0 := \displaystyle\int_0^{+\infty} \varrho(a) \, da$ then

$$|\hat{z}_\varepsilon(t)| \leq \frac{1}{\mu_0} \int_0^{\frac{t}{\varepsilon}} |\hat{z}_\varepsilon(t - \varepsilon a)| \varrho(a) da + |\tilde{h}_\varepsilon(t)| \tag{44}$$

Then integrating in time and setting

$$\hat{Z}_\varepsilon(t) := \int_0^t |\hat{z}_\varepsilon(\tau)| \, d\tau$$

one has that :

$$\hat{Z}_\varepsilon(t) = \int_0^t |\hat{z}_\varepsilon(\tau)| \, d\tau \leq \frac{1}{\mu_0} \int_0^t \int_0^{\frac{\tau}{\varepsilon}} |\hat{z}_\varepsilon(\tau - \varepsilon a)| \, \varrho(a) \, da \, d\tau + \int_0^t |\tilde{h}_\varepsilon(\tau)| \, d\tau$$

then, we change the order of integration and the domain of integration becomes $D' := \{(a, \tau) \in (0, t/\varepsilon) \times (\varepsilon a, t)\}$. We use the change of variable $\tilde{t} = \tau - \varepsilon a$ in order to write :

$$\begin{aligned} \int_0^t \int_0^{\frac{\tau}{\varepsilon}} |\hat{z}_\varepsilon(\tau - \varepsilon a)| \, \varrho(a) \, da \, d\tau &= \int_0^{\frac{t}{\varepsilon}} \int_{\varepsilon a}^t |\hat{z}_\varepsilon(\tau - \varepsilon a)| \, d\tau \, \varrho(a) \, da \\ &= \int_0^{\frac{t}{\varepsilon}} \int_0^{t - \varepsilon a} |\hat{z}_\varepsilon(\tilde{t})| \, d\tilde{t} \, \varrho(a) \, da = \int_0^{\frac{t}{\varepsilon}} \hat{Z}_\varepsilon(t - \varepsilon a) \, \varrho(a) \, da \end{aligned}$$

So that finally $\hat{Z}_\varepsilon$ solves :

$$\hat{Z}_\varepsilon(t) \leq \frac{1}{\mu_0} \int_0^{\frac{t}{\varepsilon}} \hat{Z}_\varepsilon(t - \varepsilon a) \, \varrho(a) \, da + \int_0^t \tilde{h}_\varepsilon(\tau) \, d\tau = \int_0^t \hat{Z}_\varepsilon(\tilde{a}) K_\varepsilon(t - \tilde{a}) \, d\tilde{a} + \int_0^t \tilde{h}_\varepsilon(\tau) \, d\tau$$

where $K_\varepsilon(\tilde{a}) := \frac{1}{\varepsilon \mu_0} \varrho\left(\frac{\tilde{a}}{\varepsilon}\right)$ is the kernel of the integral operator. We use a comparison principle [6, the Generalised Gronwall Lemma 8.2 p. 257] and construct a majorizing function U_ε of the form $U_\varepsilon(t) = \varepsilon(K_0 + K_1 t)$ where K_0 and K_1 are suitably chosen, such that $U_\varepsilon \geq |\hat{Z}_\varepsilon|$ and $U_\varepsilon \sim \varepsilon$. The following two lemmas are required in order to apply this comparison principle :

Lemma 13. The Volterra kernel K_ε satisfies :

$$\|K_\varepsilon\|_{L^\infty(0,T)} := \operatorname*{ess\,sup}_{t\in(0,T)} \int_0^t |K_\varepsilon(\tilde{a})|\, d\tilde{a} < 1$$

Proof. To prove this result, we need to show that

$$0 \leq \int_0^t |K_\varepsilon(\tilde{a})| \;\, d\tilde{a} = \frac{\int_0^{\frac{t}{\varepsilon}} \varrho(a) \; da}{\int_0^{+\infty} \varrho(a) \; da} < 1. \tag{45}$$

The kernel ϱ solves (42), thus it can be explicitly computed as

$$\varrho(a) = \frac{\beta}{1+\beta I} \exp\left(- \int_0^a \zeta(s) ds \right)$$

one has the lower bound :

$$\varrho(a) \geq \frac{\beta}{1+\beta I} \exp\left(-\zeta_{\max} a\right) > 0, \quad \forall a \in \mathbb{R}_+.$$

This in turn shows that

$$\int_{\frac{t}{\varepsilon}}^\infty \varrho(a) da > 0$$

which is equivalent to the claim. □

Lemma 14. Consider the expectation value of a given density ϱ with respect to the tail $a > t/\epsilon$,

$$A_1[\varrho](t) := \frac{\int_0^{+\infty} a\varrho(a+\frac{t}{\epsilon}) \text{ da}}{\int_0^{+\infty} \varrho(a+\frac{t}{\epsilon}) \text{ da}} \tag{46}$$

then under Assumptions 3, one has

$$A_1[\varrho](t) \leq \frac{\zeta_{\max}}{\zeta_{\min}^2}.$$

Proof. Setting

$$q(a,t) := \frac{\varrho(a+\frac{t}{\epsilon})}{\varrho(\frac{t}{\epsilon})}, \tag{47}$$

it solves

$$\partial_a q(a,t) + \zeta(a + t/\epsilon) \; q(a,t) = 0, \quad q(0,t) = 1.$$

This problem admits an explicit solution of the form

$$q(a,t) = \exp\left(- \int_0^a \zeta(\bar{a} + t/\epsilon) \text{ d}\bar{a} \right) = \exp\left(- \int_{t/\epsilon}^{a+t/\epsilon} \zeta(\hat{a}) \text{ d}\hat{a} \right). \tag{48}$$

Then, we shall rewrite (46) as :

$$A_1[\varrho](t) := \frac{\int_0^{+\infty} aq(a,t) \text{ da}}{\int_0^{+\infty} q(a,t) \text{ da}}$$

by using hypothesis iv) from Assumptions 3, one has

$$\exp(-\zeta_{\max} a) \leq q(a,t) \leq \exp(-\zeta_{\min} a)$$

This gives :

$$\int_0^{+\infty} q(a,t)\ \mathrm{d}a \geq \int_0^{+\infty} \exp(-\zeta_{\max} a)\ \mathrm{d}a = \frac{1}{\zeta_{\max}}$$

and

$$\int_0^{+\infty} a q(a,t) \mathrm{d}a \leq \int_0^{+\infty} a \exp(-\zeta_{\min} a) \mathrm{d}a = \frac{1}{\zeta_{\min}^2}$$

which shows the final upper bound. □

Proposition 3. Under Assumptions 3, for $0 < t < T$ one has the estimates :

$$\tilde{H}_\varepsilon(t) := \int_0^t \left| \tilde{h}_\varepsilon(\tau) \right| d\tau \leq \varepsilon^2 C_1$$

where C_1 depends on μ_2,μ_1,$\|\partial_t z_0\|_{\mathrm{BV}((0,T))}$ and on $\|z_p\|_{\mathrm{Lip}(\mathbb{R}_-)}$ but not on ε.

Proof. Recalling the definition of $\tilde{h}_\varepsilon$ in (43), we split $\int_0^t \left| \tilde{h}_\varepsilon(\tau) \right| d\tau$ into three parts. First, we define

$$I_1 := \frac{\varepsilon}{\mu_0} \int_0^t \int_0^{\tau/\varepsilon} \left| \frac{z_0(\tau) - z_0(\tau - \varepsilon a)}{\varepsilon} - a \partial_t z_0(\tau) \right| \varrho(a)\ da\ d\tau$$

Since $\partial_t z_0 \in \mathrm{BV}((0,T))$, then I_1 can be written in the form

$$I_1 = \frac{1}{\mu_0} \int_0^t \int_0^{\tau/\varepsilon} \left| \int_{\tau - \varepsilon a}^{\tau} \left(\partial_t z_0(\tilde{t}) - \partial_t z_0(\tau) \right)\ d\tilde{t} \right|\ \varrho(a)\ da\ d\tau$$

by switching the integration order between τ and a, and using the change of variable $\tilde{t} = \tau + h$, we get that

$$\begin{aligned}
I_1 &\leq \frac{1}{\mu_0} \int_0^{t/\varepsilon} \int_{\varepsilon a}^{t} \int_{\tau - \varepsilon a}^{\tau} \left| \partial_t z_0(\tilde{t}) - \partial_t z_0(\tau) \right| d\tilde{t}\ d\tau\ \varrho(a)\ da \\
&\leq \frac{1}{\mu_0} \int_0^{t/\varepsilon} \int_{\varepsilon a}^{t} \int_{-\varepsilon a}^{0} \left| \partial_t z_0(\tau + h) - \partial_t z_0(\tau) \right| dh\ d\tau\ \varrho(a)\ da \qquad (49) \\
&\leq \frac{1}{\mu_0} \int_0^{t/\varepsilon} \int_{-\varepsilon a}^{0} \int_{\varepsilon a}^{t} \left| \partial_t z_0(\tau + h) - \partial_t z_0(\tau) \right|\ d\tau\ dh\ \varrho(a)\ da
\end{aligned}$$

and thus applying Lemma 3, one has the estimate of the inner integral of the latter right hand side :

$$\int_{\varepsilon a}^{t} \left| \partial_t z_0(\tau + h) - \partial_t z_0(\tau) \right|\ d\tau \leq |h| \|\partial_t z_0\|_{\mathrm{BV}}, \quad \forall h \in (-\varepsilon a, 0)$$

which implies that

$$I_1 \leq \frac{\|\partial_t z_0\|_{\mathrm{BV}}}{\mu_0} \int_0^{t/\varepsilon} \int_{-\varepsilon a}^{0} |h|\ dh\ \varrho(a)\ da \leq \frac{\varepsilon^2 \|\partial_t z_0\|_{BV}}{2\mu_0} \int_0^{t/\varepsilon} a^2 \varrho(a)\ da.$$

Next, we set:

$$\begin{aligned}
I_2 =& \frac{\varepsilon}{\mu_0} \int_0^t \int_{\tau/\varepsilon}^{+\infty} \left| \frac{z_0(\tau) - z_0(0)}{\varepsilon} - a \partial_t z_0(\tau) \right| \varrho(a)\ da\ d\tau \\
\leq& \frac{1}{\mu_0} \int_0^t \int_{\tau/\varepsilon}^{+\infty} \left| \int_0^{\tau} \partial_t z_0(\tilde{t}) d\tilde{t} - \tau \partial_t z_0(\tau) \right| \varrho(a)\ da\ d\tau \quad + \frac{1}{\mu_0} \int_0^t \int_{\tau/\varepsilon}^{+\infty} |\tau - \varepsilon a| \left| \partial_t z_0(\tau) \right| \varrho(a)\ da\ d\tau \\
=:& I_{2,1} + I_{2,2}
\end{aligned}$$

As in the estimates of I_1, first, one switches the order of integration and then one integrates on

$$D := \{(a,\tau) \in (0,t/\varepsilon) \times (0,\varepsilon a)\} \cup \{(a,\tau) \in (t/\varepsilon,+\infty) \times (0,t)\}, \tag{50}$$

and one makes the change of variable $\tilde{t} = \tau + h$ in order to obtain

$$\begin{aligned} I_{2,1} &= \frac{1}{\mu_0}\int_0^{t/\varepsilon}\int_0^{\varepsilon a}\int_{-\tau}^0 |\partial_t z_0(\tau+h) - \partial_t z_0(\tau)|\, dh\; d\tau\; \varrho(a)\; da \\ &+ \frac{1}{\mu_0}\int_{t/\varepsilon}^{+\infty}\int_0^{t}\int_{-\tau}^0 |\partial_t z_0(\tau+h) - \partial_t z_0(\tau)|\, dh\; d\tau\; \varrho(a)\; da \\ &= \frac{1}{\mu_0}\int_0^{t/\varepsilon}\int_{-\varepsilon a}^{0}\int_{-h}^{\varepsilon a} |\partial_t z_0(\tau+h) - \partial_t z_0(\tau)|\, d\tau\; dh\; \varrho(a)\; da \\ &+ \frac{1}{\mu_0}\int_{t/\varepsilon}^{+\infty}\int_{-t}^{0}\int_{-h}^{t} |\partial_t z_0(\tau+h) - \partial_t z_0(\tau)|\, d\tau\; dh\; \varrho(a)\; da \end{aligned}$$

also, by using Lemma 4, we get that

$$\begin{aligned} I_{2,1} &\le \frac{\|\partial_t z_0\|_{\mathrm{BV}}}{\mu_0}\int_0^{t/\varepsilon}\int_{-\varepsilon a}^0 |h|\; dh\; \varrho(a)\; da + \frac{\|\partial_t z_0\|_{\mathrm{BV}}}{\mu_0}\int_{t/\varepsilon}^{+\infty}\int_{-t}^0 |h|\; dh\; \varrho(a)\; da \\ &\le \frac{\varepsilon^2\|\partial_t z_0\|_{BV}}{2\mu_0}\int_0^{t/\varepsilon} a^2\varrho(a)\; da + \frac{\|\partial_t z_0\|_{BV}}{2\mu_0}\int_{t/\varepsilon}^{+\infty} t^2\varrho(a)\; da \\ &\le \frac{\varepsilon^2\|\partial_t z_0\|_{BV}}{2\mu_0}\int_0^{t/\varepsilon} a^2\varrho(a)\; da + \frac{\varepsilon^2\|\partial_t z_0\|_{BV}}{2\mu_0}\int_{t/\varepsilon}^{+\infty} a^2\varrho(a)da \\ &\le \frac{\varepsilon^2\mu_2\|\partial_t z_0\|_{BV}}{\mu_0}. \end{aligned}$$

The second term $I_{2,2}$ is estimated in the same way as $I_{2,1}$. We have

$$\begin{aligned} I_{2,2} &= \frac{1}{\mu_0}\int_0^t\int_{\tau/\varepsilon}^{+\infty} |\tau-\varepsilon a|\,|\partial_t z_0(\tau)|\,\varrho(a)\; da\; d\tau \\ &= \frac{1}{\mu_0}\left\{\int_0^{t/\varepsilon}\int_0^{\varepsilon a} + \int_{t/\varepsilon}^{+\infty}\int_0^t |\tau-\varepsilon a|\,|\partial_t z_0(\tau)|\; d\tau\; \varrho(a)\; da\right\} \\ &\le \frac{\varepsilon^2\|\partial_t z_0\|_\infty}{2\mu_0}\int_0^{t/\varepsilon} a^2\varrho(a)\; da + \frac{\varepsilon^2\|\partial_t z_0\|_\infty}{2\mu_0}\int_{t/\varepsilon}^{+\infty} a^2\varrho(a)\; da \le \frac{\varepsilon^2\mu_2\|\partial_t z_0\|_\infty}{\mu_0}. \end{aligned}$$ □

Finally, by similar computations, one has

$$\frac{1}{\mu_0}\int_0^t\int_{\tau/\varepsilon}^{+\infty} |z_p(\tau-\varepsilon a) - z_p(0)|\,\varrho(a)\; da\; d\tau \le \varepsilon^2\|z_p\|_{\mathrm{Lip}(\mathbb{R}_-)}\frac{\mu_2}{\mu_0}$$

Theorem 9. Under Assumptions 3, z_ε tends to z_0, the solution of (3), strongly in $L^1(0,T)$ as ε goes to zero. Moreover, there exists a generic constant C depending only on the data of the problem but not on ε, such that :

$$\|z_\varepsilon - z_0\|_{L^1(0,T)} \le \varepsilon C.$$

Proof. We have proved in Lemma 13 that the Volterra kernel K_ε is non-positive and bounded (with a bound strictly less than one) in the sense of [6, Definition 2.2 p. 227 and Proposition 2.7 p.231] so [6, the Generalised Gronwall Lemma 8.2 p. 257] applies. First observe, by using Proposition 3, that

$$\hat{Z}_\varepsilon(t) - \int_0^t \hat{Z}_\varepsilon(\tilde{a}) K_\varepsilon(\tilde{a}) \; d\tilde{a} \leq \tilde{h}_{1,\varepsilon}(t) + \tilde{h}_{2,\varepsilon}(t)$$

We construct a function U_ε which satisfies,

$$U_\varepsilon(t) - \int_0^t U_\varepsilon(\tilde{a}) K_\varepsilon(\tilde{a}) \; d\tilde{a} \geq \tilde{h}_{1,\varepsilon} + \tilde{h}_{2,\varepsilon} \tag{51}$$

We split the integral operator applied to U_ε in two parts

$$\begin{aligned} U_\varepsilon(t) - \int_0^t U_\varepsilon(\tilde{a}) K_\varepsilon(\tilde{a}) \; d\tilde{a} &= U_\varepsilon(t) - \frac{1}{\mu_0} \int_0^{t/\varepsilon} U_\varepsilon(t - \varepsilon a) \varrho(a) \; da \\ &= \underbrace{\frac{1}{\mu_0} \int_0^{+\infty} \left(U_\varepsilon(t) - U_\varepsilon(t - \varepsilon a) \right) \varrho(a) \; da}_{:= H_{1,\varepsilon}} + \underbrace{\frac{1}{\mu_0} \int_{t/\varepsilon}^{+\infty} U_\varepsilon(t - \varepsilon a) \varrho(a) \; da}_{:= H_{2,\varepsilon}} \end{aligned}$$

and we shall specify U_ε such that $H_{1,\varepsilon} \geq \tilde{H}_\varepsilon(t)$ and $H_{2,\varepsilon} \geq 0$. To this end we set

$$U_\varepsilon(t) := \varepsilon(K_0 + K_1 t), \quad \forall t \in \mathbb{R} \tag{52}$$

with constants K_0 and K_1 to be specified. One has obviously that

$$H_{1,\varepsilon}(t) = \frac{\varepsilon^2 K_1 \mu_1}{\mu_0} \geq \varepsilon^2 C_1 \geq \tilde{H}_\varepsilon(t)$$

a.e. on $\mathbb{R}_+$, provided that K_1 is chosen as

$$K_1 > \frac{\mu_0}{\mu_1} C_1$$

Using (52) and the change of variable $\tilde{a} = -t/\varepsilon + a$, we obtain that

$$H_{2,\varepsilon} = \int_0^{+\infty} \left(K_0 - \varepsilon K_1 \tilde{a} \right) \varrho(\tilde{a} + t/\varepsilon) \; d\tilde{a} = \left(K_0 - \varepsilon K_1 A_\varepsilon[\varrho](t) \right) \int_{\mathbb{R}_+} \varrho \left(\frac{t}{\varepsilon} + a \right) da$$

We are in the hypotheses of Lemma 14 : $A_\varepsilon[\varrho](t)$, the expectation of a given density ϱ with respect to the tail $a > t/\varepsilon$ is bounded by a positive constant $A_{\max}$

$$A_\varepsilon[\varrho](t) := \frac{\int_{\mathbb{R}_+} a \varrho(\frac{t}{\varepsilon} + a) \; da}{\int_{\mathbb{R}_+} \varrho(\frac{t}{\varepsilon} + a) \; da} \leq A_{\max}.$$

Therefore it suffices to chose $K_0 > \varepsilon K_1 A_{\max}$ in order to obtain that $H_{2,\varepsilon} \geq 0$. These computations show that U_ε is a super-solution. Then the comparison principle implies that, for all $0 \leq t \leq T$,

$$0 \leq \hat{Z}_\varepsilon(t) = \int_0^t |z_\varepsilon(s) - z_0(s)| \; ds \leq U_\varepsilon(t) = \varepsilon(K_0 + K_1 t) \to 0 \text{ as } \varepsilon \to 0,$$

hence $\hat{Z}_\varepsilon \to 0$ in $C([0,T])$, which ends the proof since $\|z_\varepsilon - z_0\|_{L^1(0,T)} \leq \left\| \hat{Z}_\varepsilon \right\|_{C([0,T])}$. □

7 A simple example

We construct *by hand* solutions of problems (1) and (3) when the load f is explicitly defined as

$$f(t) := \begin{cases} 1/2 & \text{if } 0 < t \leq \frac{1}{3}, \\ 1 & \text{if } \frac{1}{3} < t \leq \frac{2}{3}, \\ 3/2 & \text{if } \frac{2}{3} < t < 1, \end{cases} \tag{53}$$

and the kernel ϱ is a simple exponential (see more precise statements below). So defined f is of course of bounded variation on $(0, 1)$. The solution z_ε solving (1) and its limit z_0 show different regularities (see Figure 1) : the adhesive approximation is rougher than the limit solution. This is an interesting feature of our approach.

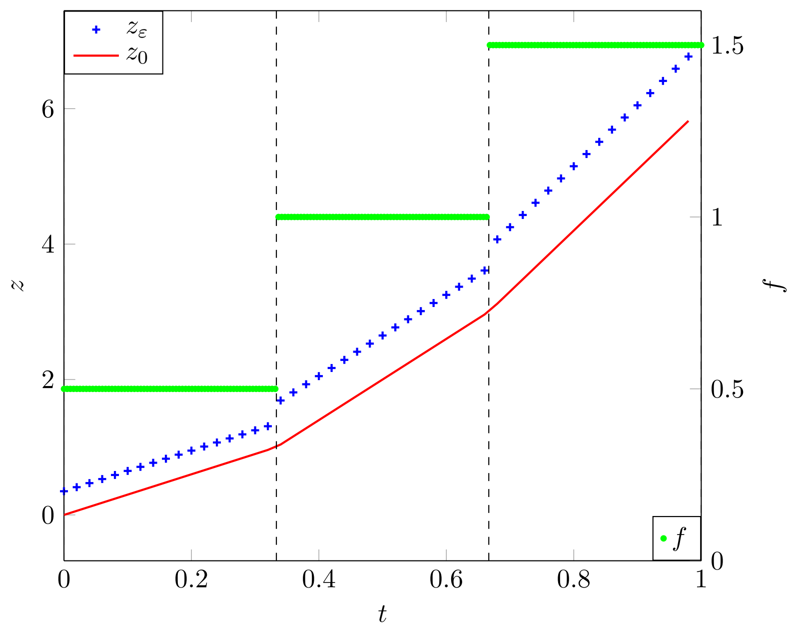

Figure 1: The solutions z_ε and z_0 as a function of time, for a fixed $\varepsilon = 10^{-1}$ on the left y-axis. The load f on the right y-axis.

Assumptions 4.

i) the load f is defined in (53),

ii) the on and off rates are constants defined as :

$$\zeta_\varepsilon = \zeta_0 = \zeta, \quad \beta_\varepsilon = \beta_0 = \beta.$$

iii) the initial condition is at equilibrium :

$$\rho_{I,\varepsilon} = \rho_0 = \frac{\beta\zeta}{\beta + \zeta} e^{-\zeta a}.$$

Lemma 15. Under Assumptions 4, one has that $\mu_{0,\varepsilon} = \mu_{0,0} = \beta/(\beta+\zeta)$, $\mu_{1,\varepsilon} = \mu_{1,0} = \beta/(\zeta(\beta+\zeta))$ and $\rho_0(a) = \mu_{0,0}\zeta e^{-\zeta a}$. Then the solution z_ε of (1) is $BPV([0,1])$ and it is explicitly given by

$$z_\varepsilon(t) = \int_0^t \frac{f}{\mu_{1,0}} ds + \varepsilon \frac{f(t)}{\mu_{0,0}} + \frac{1}{\mu_{0,0}} \int_0^{+\infty} z_p(-\varepsilon a)\rho_0 da$$

and hence,

$$z_\varepsilon(t) - z_0(t) = \varepsilon \frac{f(t)}{\mu_{0,0}} + \int_{-\infty}^0 z_p'(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds$$

with $z_0(t) = z_p(0) + \int_0^t f(s)ds/\mu_{1,0}$ is a continuous functions in $[0,1]$. Note that the last term is an ε order term according to Assumption 1, indeed it holds that

$$\left| \int_{-\infty}^0 z_p'(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) \, ds \right| \leq \frac{\varepsilon}{\zeta} \|z_p\|_{W^{1,\infty}(\mathbb{R}_-)}.$$

Proof. The specific choice of data and kernel allows to rephrase (1) as

$$z_\varepsilon(t) - \frac{\zeta}{\varepsilon} \int_0^t z_\varepsilon(s) \exp\left(-\frac{\zeta(t-s)}{\varepsilon}\right) ds = \varepsilon \frac{f(t)}{\mu_{0,0}} + \frac{\zeta}{\varepsilon} \int_{-\infty}^0 z_p(s) \exp\left(-\frac{\zeta(t-s)}{\varepsilon}\right) \, ds.$$

Next, setting

$$q_\varepsilon(t) = z_\varepsilon(t) \exp\left(\frac{\zeta t}{\varepsilon}\right), \quad t \geq 0.$$

Then we can rewrite (15) for all $t \geq 0$ as

$$q_\varepsilon(t) - \frac{\zeta}{\varepsilon} \int_0^t q_\varepsilon(s) ds = \varepsilon \exp\left(\frac{\zeta t}{\varepsilon}\right) \frac{f(t)}{\mu_{0,0}} + \frac{\zeta}{\varepsilon} \int_{-\infty}^0 z_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds. \tag{54}$$

By differentiating (54) in time and using (6), we prove that q_ε solve the equation

$$\begin{cases} \dot{q}_\varepsilon(t) - \dfrac{\zeta}{\varepsilon} \, q_\varepsilon(t) = \dfrac{\exp\left(\frac{\zeta t}{\varepsilon}\right)}{\mu_{0,0}} \left(\zeta f(t) + \varepsilon f'(t)\right), & t > 0, \\ q_\varepsilon(0^+) = \varepsilon f(0)/\mu_{0,0} + \dfrac{\zeta}{\varepsilon} \displaystyle\int_{-\infty}^0 z_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds \end{cases} \tag{55}$$

and therefore q_ε is explicitly given by

$$q_\varepsilon(t) = \exp\left(\frac{\zeta t}{\varepsilon}\right) \left(\varepsilon \frac{f(t)}{\mu_{0,0}} + \frac{\zeta}{\varepsilon} \int_{-\infty}^0 z_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds + \int_0^t \frac{f(s)}{\mu_{1,0}} \, ds \right)$$

which gives the formula of z_ε. Moreover, it is clear that z_ε is of bounded variation since f is it and z_0 is an absolutely continuous function. □

A Auxiliary proofs

In this appendix, the domain Ω is an open set of $\mathbb{R}^n$.

A.1 Proof of Theorem 1

As in the proof of [9, Theorem 14.9], the aim is to show that for every $\delta > 0$, there exists a sequence $\{f_\delta\}_\delta$ in $C^\infty(\Omega)$ such that

$$\int_\Omega |f - f_\delta| \, \mathrm{d}t < \delta \quad \text{and} \quad |Df_\delta|(\Omega) < \|Df\|(\Omega) + \delta.$$

Since the total variation $\|Df\|(\Omega)$ is bounded,

$$\lim_{j\to+\infty}\|Df\|(\Omega\setminus\{t,\mathrm{dist}(t,\partial\Omega)>1/j,|t|<j\})=0$$

then for fixed $\delta>0$, there exists a $j_0\in\mathbb{N}$ such that for all $j\geq j_0$,

$$\|Df\|(\Omega\setminus\{t,\mathrm{dist}(t,\partial\Omega)>1/j,|t|<j\})\leq\delta.$$

For $i\in\mathbb{N}$, we define the subdomain Ω_i of Ω by

$$\Omega_i:=\{t\in\Omega,\mathrm{dist}(t,\partial\Omega)>1/j_0+i,\ |t|<j_0+i\}$$

such that $\Omega_i\subset\subset\Omega_{i+1}$ and $\bigcup_{i=0}^{\infty}\Omega_i=\Omega$. Let $W_0=\Omega_1$ and $W_i=\Omega_{i+1}\setminus\bar{\Omega}_{i-1}$, where $\Omega_{-1}=\Omega_0:=\emptyset$, and let $\{\phi_i\}$ be a partition of the unity subordinate to the covering $\{W_i\}_{i\in\mathbb{N}}$

$$\phi_i\in C_0^{\infty}(W_i),\ 0\leq\phi_i\leq1,\ \sum_{i=0}^{\infty}\phi_i=1.$$

For $i\in\mathbb{N}$, the idea is to find $\delta_i>0$ so small that

$$\mathrm{supp}\ \chi_{\delta_i}*(\phi_i f)\subset W_i \tag{56}$$

which allows to write

$$\|\chi_{\delta_i}*(\phi_i f)-\phi_i f\|_{L^1(\Omega)}=\|\chi_{\delta_i}*(\phi_i f)-\phi_i f\|_{L^1(W_i)} \tag{57}$$

and then from the local approximation Theorem (see Theorem 2 p.125 in [4]),

$$\|\chi_{\delta_i}*(\phi_i f)-\phi_i f\|_{L^1(W_i)}<C\delta$$

and also by convenience, we can take $C=\frac{1}{2^i}$. Hence we conclude that

$$\int_{\Omega}|\chi_{\delta_i}*(\phi_i f)-\phi_i f|\ \mathrm{d}t<\delta\ 2^{-i} \tag{58}$$

$$\int_{\Omega}|\chi_{\delta_i}*(f\phi_i')-f\phi_i'|\ \mathrm{d}t<\delta\ 2^{-i}. \tag{59}$$

for a positive mollifiers χ_δ defined as

$$\chi_\delta(t):=\frac{1}{\delta}\chi(\frac{t}{\delta}),\qquad \chi(t)=\begin{cases}1 & \text{if } |t|<1\\ 0 & \text{if } |t|>2\end{cases}$$

Define

$$f_\delta=\sum_{i=0}^{\infty}\chi_{\delta_i}*(\phi_i f). \tag{60}$$

By the construction of $\{W_i\}$, we have $W_i\cap W_{i+1}\neq\emptyset$ and $W_i\cap W_{i+1}\cap W_{i+2}=\emptyset$ which give that for every $t\in\Omega$,

$$\#\{i\in\mathbb{N}:\chi_{\delta_i}*(\phi_i f)(t)\neq0\}\leq2$$

and since the finite sum of infinitely differentiable functions is infinitely differentiable, we conclude that $f_\delta\in C^{\infty}(\Omega)$ and

$$\int_{\Omega}|f_\delta-f|\mathrm{d}t\leq\sum_{i=1}^{\infty}\int_{\Omega}|\chi_{\delta_i}*(\phi_i f)-\phi_i f|\ \mathrm{d}t<\sum_{i=1}^{\infty}\frac{\delta}{2^i}\leq\delta,$$

since $f = \sum_{i=0}^{\infty} f\phi_i$. Thus $f_\delta \to f$ in $L^1(\Omega)$ as $\delta \to 0$. Using the theorem of Lower semi-continuity of variation measure (see chap 5, [4]), we have

$$\|Df\|(\Omega) \leq \liminf_{\delta \to 0} |Df_\delta|(\Omega). \tag{61}$$

it remains to show that

$$\limsup_{\delta \to 0} |Df_\delta|(\Omega) \leq \|Df\|(\Omega).$$

Let $\psi \in C_c^\infty(\Omega)$ be such that $|\psi|_\infty \leq 1$. Since $\phi_i f \in \mathrm{BV}(\Omega)$ for every $i \in \mathbb{N}$, we have by Lemma 14.10 in [9] that

$$\int_\Omega \chi_{\delta_i} * (\phi_i f)\ \psi'\ \mathrm{d}t = \int_\Omega \phi_i f\ (\chi_{\delta_i} * \psi)'\ \mathrm{d}t \tag{62}$$

let $\psi_{\delta_i} = \chi_{\delta_i} * \psi$, and using the fact that support of ψ is compact and that the partition of unity is locally finite, we have that

$$\begin{aligned}
\int_\Omega f_\delta\ \psi'\ \mathrm{d}t &= \sum_{i=1}^{+\infty} \int_\Omega \chi_{\delta_i} * (\phi_i f)\ \psi'\ \mathrm{d}t \\
&= \sum_{i=1}^{+\infty} \int_\Omega \phi_i f\ \psi'_{\delta_i}\ \mathrm{d}t \\
&= \sum_{i=1}^{+\infty} \int_\Omega f((\phi_i \psi_{\delta_i})' - \phi'_i \psi_{\delta_i})\ \mathrm{d}t := I_1 + I_2
\end{aligned}$$

since $supp(\phi_i \psi_{\delta_i}) \subset W_i$ and $|\phi_i \psi_{\delta_i}|_\infty \leq 1$, it follows that

$$\begin{aligned}
I_1 &= \int_\Omega f\ (\phi_i \psi_{\delta_i})'\mathrm{d}t + \sum_{i=2}^{+\infty} \int_\Omega f\ (\phi_i \psi_{\delta_i})'\mathrm{d}t \\
&\leq \|Df\|(\Omega) + \sum_{i=2}^{+\infty} \|Df\|(W_i) \\
&\leq \|Df\|(\Omega) + 3\|Df\|(\Omega \setminus \Omega_1) \\
&\leq \|Df\|(\Omega) + 3\delta
\end{aligned}$$

since each $t \in \Omega$ belongs at most two of the sets U_i. On the other hand, by Fubini's theorem we have

$$\begin{aligned}
I_2 &= -\sum_{i=1}^{+\infty} \int_\Omega \chi_{\delta_i} * (f\phi'_i)\psi\ \mathrm{d}t \\
&= -\sum_{i=1}^{+\infty} \int_\Omega (\chi_{\delta_i} * (f\phi'_i) - f\phi'_i)\,\psi\ \mathrm{d}t
\end{aligned}$$

by using the fact that $\sum_{i=1}^{+\infty} \phi'_i = 0$. We now use (59) and the fact that $|\psi|_\infty \leq 1$, to conclude that $I_2 \leq \delta$ and then we obtain that

$$\int_\Omega f_\delta\ \psi'\ \mathrm{d}t \leq \|Df\|(\Omega) + 3\delta.$$

By taking the supremum and passing to the limit when $\delta \to 0$, we obtain that

$$\limsup_{\delta \to 0} \|Df_\delta\|(\Omega) \leq \|Df\|(\Omega) \tag{63}$$

Finally, (61) and (63) together concludes the proof.

A.2 Proof of Lemma 5:

Let $f \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$. By using (60) and (56) we can write for all t in Ω,

$$\begin{aligned} |f_\delta(t)| &= \left| \sum_{i=0}^{\infty} \chi_{\delta_i} * (f\phi_i)(t) \right| \\ &\leq \sum_{i=0}^{\infty} \|f\|_{L^\infty} \int_{W_i} \phi_i(t-t')\chi_{\delta_i}(t') \, \mathrm{d}t' \\ &\leq \sum_{i=0}^{\infty} \|f\|_{L^\infty} \int_{W_i} \chi_{\delta_i}(t') \, \mathrm{d}t' \end{aligned}$$

since $0 \leq \phi_i \leq 1$. On the other hand, we have $W_i \cap W_{i-1} = \Omega_i \setminus \Omega_{i-1}$ and $W_{i-1} \cap W_i \cap W_{i+1} = \emptyset$ which implies that

$$|f_\delta(t)| \leq \|f\|_{L^\infty} \int_{W_{i-1}} \chi_{\delta_i}(t') \, \mathrm{d}t' + \|f\|_{L^\infty} \int_{W_i} \chi_{\delta_{i-1}}(t') \, \mathrm{d}t' \leq 2\|f\|_{L^\infty(\Omega)} \|\chi\|_{L^1(\mathbb{R})}$$

which ends the proof.

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