Numer. Math. (2004)
Digital Object Identifier (DOI) 10.1007/s00211-003-0514-5

Numerische
Mathematik

Kinetic approximation of a boundary value problem for conservation laws

Denise Aregba-Driollet[1]**,Vuk Milišić**[2]

[1] Université de Bordeaux 1, 351 cours de la Libération, F-33405 Talence, France; e-mail: aregba@math.u-bordeaux.fr
[2] Modelling and Scientific Computing Chair SB/IACS/CMCS Ecole Polytechnique Fédérale de Lausanne 1015 Lausanne, Switzlerland; e-mail: Vuk.Milisic@epfl.ch

Received January 17, 2002 / Revised version received November 5, 2003 /
Published online: ■ –

Summary. We design numerical schemes for systems of conservation laws with boundary conditions. These schemes are based on relaxation approximations taking the form of discrete BGK models with kinetic boundary conditions. The resulting schemes are Riemann solver free and easily extendable to higher order in time or in space. For scalar equations convergence is proved. We show numerical examples, including solutions of Euler equations.

Mathematics Subject Classification(2000): 65M06, 65M12, 76M20

1 Introduction

In this paper, we study a new class of numerical approximations of the initial-boundary value problem for the one-dimensional conservation law:

$$\partial_t u + \partial_x F(u) = 0, \tag{1}$$

where $u(t,x) \in I$, a domain of $\mathbb{R}^p$ ($p \geq 1$) for $x \geq 0, t \geq 0$, and $F \in C^1(I)$.

We denote u^0 the initial data:

$$u(0,x) = u^0(x), \quad x \geq 0, \tag{2}$$

and u_b the boundary data. It is well known that the condition

$$u(t,0) = u_b(t), \quad t \geq 0 \tag{3}$$

Correspondence to: D. Aregba-Driollet

2 1 1	0 5 1 4	B	Dispatch: 12/1/2004	Journal: Numer. Math.
Jour. No	Ms. No.		Total pages: 39	Not Used ☐
			Disk Received ☑	Corrupted ☐
			Disk Used ☑	Mismatch ☐

cannot be imposed in general. For example, consider the scalar equation with $F(u) = -u$. The problem (1)(2) owns a unique global solution $u(x,t) = u^0(x+t)$ in $]0,+\infty[\times]0,+\infty[$, so that (3) is not consistent. *A contrario*, if $F(u) = u$, condition (3) has to be imposed. More generally, one looks for a condition which is to be effective only in the inflow part of the boundary.

Several attempts have been done in this direction, in the scalar case as well as for systems. In the scalar case, Bardos, Le Roux and Nédélec use the viscosity approach and write the boundary condition as [3]:

(4)
$$\max\{\mathrm{sgn}(u(t,0) - u_b(t))(F(u(t,0)) - F(k)) : k \in I(u_b(t), u(t,0))\} = 0.$$

Here, $I(u_b(t), u(t,0))$ is the interval defined by $u_b(t)$ and $u(t,0)$. Functions with bounded variation have strong traces, so that (4) makes sense in this context. Existence and uniqueness of BV solutions of (1,2,4) is proved in [3].

It is easy to see that if (4) is satisfied and $u(t,0) \neq u_b(t)$, then $F'(u(t,0)) \leq 0$: the characteristic line is outgoing. Hence, the boundary condition applies as soon as $F'(u_b)$ is positive. In this article, we make the convergence analysis for the scalar case in this framework.

More recently, F. Otto gave a formulation which allows to prove existence and uniqueness for bounded measurable data. We refer the reader to the note [40] and the book [33].

For systems, one may generalize the viscosity method by requiring that

$$u(t,0) \in \mathcal{E}(u_b(t)) \tag{5}$$

where

$$\mathcal{E}(u_b(t)) = \left\{ \begin{array}{l} w\ ;\ G(w) - G(u_b) - \eta'(u_b)(F(w) - F(u_b)) \leq 0 \\ \forall(\eta, G) \text{ entropy, entropy flux pair} \end{array} \right\}$$

Another possibility is to consider the half Riemann problem in the quarter space $x > 0$, $t > 0$. Let $w(x/t; u_l, u_r)$ be the solution of the Riemann problem with left and right initial states u_l and u_r. We denote $\mathcal{R}(u_l)$ the set of all admissible states $u \in I$ such that there exists $u_r \in I$ such that $u = w(O^+; u_l, u_r)$. The boundary condition u_b is imposed in the following way:

$$u(t,0) \in \mathcal{R}(u_b(t)). \tag{6}$$

This condition is related to the use of Godunov scheme for solving the initial boundary value problem. In [16], F. Dubois and P. Le Floch introduce both (5) and (6), and compare them. For scalar or linear problems they are equivalent. In general, (6) is stronger than (5), as proved by A. Benabdallah and D. Serre in [5]. This result was completed recently by P.T. Kan, M.M. Santos and Z. Xin [25]. Moreover these authors give some important trace properties of solutions generated by Godunov and viscosity approximations, and show that Godunov scheme may produce numerical boundary layers, even for a scalar convex equation.

The numerical approximation of the initial boundary value problem for a conservation law has been studied by several authors: in [28], the author analyses Godunov and Lax-Friedrichs one dimensional schemes. Multi-dimensional discretizations for scalar equations have been developped and their convergence has been proved in various situations: the stream-line diffusion finite element method in [44], the finite volume method in [12], [6], [45]. See also the references in these papers.

The kinetic viewpoint gives naturally rise to boundary conditions, since inward and outward characteristics are easy to identify. One can see for example the reference [39] for an analysis of the scalar case with the kinetic equation of Perthame and Tadmor.

Our numerical schemes are based on a kinetic approximation of the problem (1). Consider the BGK-like system:

$$\partial_t f_k^\epsilon + \lambda_k \partial_x f_k^\epsilon = \frac{1}{\epsilon}(M_k(Pf^\epsilon) - f_k^\epsilon), \quad k \in \{1, \dots, N\} \tag{7}$$

where the λ_k are fixed velocities, P is defined by $Pf = \sum_k f_k$, and ϵ is a positive parameter. Each f_k^ϵ takes values in $\mathbb{R}^p$. In addition, we impose the initial and boundary data:

$$f_k^\epsilon(0, x) = M_k(u^0(x)) \tag{8}$$

and

$$f_k^\epsilon(t, 0) = M_k(u_b(t)) \quad \text{if} \quad \lambda_k > 0. \tag{9}$$

Functions M_k are continuous, piecewise C^1 Maxwellian functions depending on the macroscopic quantities $u^\epsilon = Pf^\epsilon$, the flux F and the velocities λ_k. The following compatibility conditions link systems (1) and (7):

$$\sum_{k=1}^{N} M_k(w) = w, \quad \sum_{k=1}^{N} \lambda_k M_k(w) = F(w) \tag{10}$$

for all $w \in I$. One example of model (7) satisfying conditions (10) is given in the next paragraph, and some others are described in Section 6. Denoting $v^\epsilon = \sum_{k=1}^{N} \lambda_k f_k^\epsilon$, system (7) gives:

$$\begin{cases} \partial_t u^\epsilon + \partial_x v^\epsilon & = 0, \\ \partial_t v^\epsilon + \sum_{k=1}^{N} \lambda_k^2 \partial_x f_k^\epsilon & = \frac{1}{\epsilon} \left(F(u^\epsilon) - v^\epsilon \right), \end{cases} \tag{11}$$

so that if f^ϵ converges (in a strong enough sense) to f when ϵ tends to zero, then u^ϵ converges to a solution of system (1).

Rigorous convergence results exist for the scalar case $p = 1$. In [46], W.C. Wang and Z. Xin study the relaxation model:

$$\begin{cases} \partial_t u^\epsilon + \partial_x v^\epsilon = 0, \\ \partial_t v^\epsilon + \lambda^2 \partial_x u^\epsilon = \frac{1}{\epsilon}(F(u^\epsilon) - v^\epsilon), \end{cases} \tag{12}$$

where λ is a positive constant, with boundary and initial conditions:

$$\begin{cases} u^\epsilon(t,0) = u_b(t), \\ u^\epsilon(0,x) = u^0(x), \\ v^\epsilon(0,x) = F(u^0(x)). \end{cases} \tag{13}$$

The system (12) can be put in a diagonal form (7) with $N = 2$, $\lambda_1 = -\lambda$, $\lambda_2 = \lambda$ and

$$M_1(u) = \frac{1}{2}\left(u - \frac{F(u)}{\lambda}\right), \quad M_2(u) = \frac{1}{2}\left(u + \frac{F(u)}{\lambda}\right). \tag{14}$$

It was introduced for numerical purposes by S. Jin and Z. Xin [24]. W.C. Wang and Z. Xin [46] show the existence and uniqueness of the solution of (12)(13). Convergence holds under strong restrictions: the initial boundary data must be a small perturbation of a constant state u^*, which is supposed to be non transonic (i.e. $F'(u^*) \neq 0$).

For system (12) the boundary condition $u^\epsilon(t,0) = u_b(t)$ can be written:

$$f_2^\epsilon(t,0) = M_2(u_b(t)) - \alpha\left[f_1^\epsilon(t,0) - M_1(u_b(t))\right]$$

with $\alpha = 1$. For all $\alpha \in [-1,1]$, the initial boundary value problem is well-posed for fixed $\epsilon > 0$, but convergence to a solution of (1)(2)(4) is not insured. The boundary condition (9) is obtained when $\alpha = 0$ and in this case R. Natalini and A. Terracina proved convergence without any restriction on the data nor on the flux function F, see [37]. It is also known that $\alpha \neq -1$ is a necessary condition for convergence, see [49], [32].

R. Natalini and A. Terracina's result of has been extended to the case of any set of velocities by V. Milisic [34].

The case of systems is an open problem. The recent work of F. Berthelin and F. Bouchut [7] concerning a kinetic approximation for isentropic gas dynamics is to be mentionned.

A necessary and crucial point for the kinetic approximations to be stable, as well as to be compatible with entropy conditions, is the monotonicity of the Maxwellian function in the following sense:

Definition 1.1 *The function M is Monotone Non Decreasing (MND) on I if the eigenvalues of $M_k'(u)$ are non negative for all k and all $u \in I$.*

Under some hypothesis which are trivial in the scalar case, it is easy to show that if M is MND on I, then the following condition is satisfied for all $m \in \{1, \dots, p\}$ and all $u \in I$:

$$\min_k \lambda_k \le a_m(u) \le \max_k \lambda_k \tag{15}$$

where the $a_m(u)$ are the eigenvalues of $F'(u)$. Such subcharacteristic conditions are well-known in the general setting of relaxation problems [47], [30]. In the scalar case the fact that M is MND is a sufficient condition for convergence, for the Cauchy problem (see [36]), as well as for the initial boundary value problem. For systems, F. Bouchut [8] investigates the BGK models which are compatible with a family of entropies for (1). He proves that if the system owns at least one strictly convex entropy, and if M is MND and has a symmetry property, then one can associate kinetic entropies to the entropies of (1). Also for systems, D. Serre [43] shows a convergence result *a la DiPerna* for the Cauchy problem: under the the same hypothesis as DiPerna [15], which are mainly that (1) is genuinely nonlinear and owns a convex positively invariant domain and a convex entropy, the relaxation approximation obtained with (12) converges to an entropy solution of (1) as soon as condition (15) is satisfied.

From the numerical viewpoint, we discretize the problem (7), (8), (9) in such a way that when ϵ tends to zero, we obtain a numerical scheme for the initial boundary value problem for the conservation law (1). Such kinetic schemes are known for the Cauchy problem. They were first introduced in the framework of the Boltzmann approach of hydrodynamic problems, see for example [42], [41]. We also mention the important paper of A. Harten, P.D. Lax, and B. van Leer [22] where a kinetic interpretation of flux splitting schemes is given. For general conservation laws, S. Jin and Z. Xin introduced a relaxation approximation in 1995 and constructed related numerical schemes for the Cauchy problem [24] . In one space dimension, Jin and Xin's system reduces to (12). Numerical schemes based on the system (7) were constructed, and their convergence was studied in [2] for the Cauchy problem. By a splitting technique, one first solves the transport part. Thanks to linearity and diagonality, this can be done easily. Note that this operator does not depend on ϵ. Then, a conservation property of the macroscopic variables during the collision allows to solve the singular source-term step with an explicit formula which has the correct asymptotic behaviour when ϵ tends to zero. A Riemann-solver free scheme is thus obtained, for a scalar equation ($u \in \mathbb{R}$), as for systems. Higher order precision in space as well as in time can be reached easily.

Here, our goal is to extend these techniques to the semi-bounded domain case. The numerical schemes are constructed for a general system of conservation laws, while stability and convergence are studied for the scalar case only. Due to the boundary condition, the stability estimates, especially those on the total variation, involve some new tools.

The paper is organized as follows: in Section 2, we explain the construction of our kinetic schemes. Then in Section 3, we show that in the scalar case numerical convergence holds to a weak solution of (1). In Section 4, first order scalar approximations are proved to converge to the entropy solution of the problem (1)(2)(4), that is in the sense of [3]. In Section 5, we show how to construct specific second order schemes for the boundary problem.

It is not a difficult task to extend the ideas of this paper, at least formally, to higher space dimensions, considering the multidimensional kinetic models constructed in [2]. In Section 6, we present some numerical experiments for a scalar equation and for Euler system, in one and two space dimensions. Various examples of kinetic models (7) are described and used in this last Section.

2 Kinetic schemes for the initial boundary value problem

2.1 Notations

First we introduce some useful notations. In this work we restrict ourselves to an uniform grid, that is

$$\mathbb{R}^+ = \bigcup_{i\in\mathbb{N}} [i\Delta x, (i+1)\Delta x[,$$

$$(0, T) = \bigcup_{0\le n\le L-1} [n\Delta t, (n+1)\Delta t[.$$

The coefficients Δt, Δx are time and space steps, and we denote :

$$x_{i-\frac{1}{2}} = i\Delta x,\ x_i = (i+\tfrac{1}{2})\Delta x,\ t^n = n\Delta t.$$

We look for a discrete solution under the form

$$u_\Delta(t, x) = \sum_{n=0}^{L-1} \sum_{i\in\mathbb{N}} u_i^n \mathbb{I}_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[\times[t^n, t^{n+1}[}(t, x)$$

where u_i^n is an approximation of

$$\frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(t^n, x)\, dx.$$

We define in the same way

$$u^n_\Delta(x) = \sum_{i\geq 0} u_i^n \mathbb{I}_{[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}[}(x).$$

The notations for f are similar. The data are approximated by

$$\begin{aligned} u_b^n = u_{-1}^n &= \tfrac{1}{\Delta t}\int_{t^n}^{t^{n+1}} u_b(t)dt, \ \ 0 \leq n \leq L-1, \\ u_i^0 &= \tfrac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u^0(x)dx, \qquad\qquad i \in \mathbb{N}, \end{aligned}$$

and we take for all k:

$$\begin{aligned} f_{b,k}^n = f_{-1,k}^n &= M_k(u_b^n), \ \ 0 \leq n \leq L-1, \\ f_{i,k}^0 &= M_k(u_i^0), \qquad\qquad i \in \mathbb{N}. \end{aligned}$$

2.2 First order in time discretizations

We discretize the problem (7)(8)(9) by carrying out an operator splitting, solving first the transport part, and afterwards the collision one.

Transport

We discretize on $[t^n, t^{n+1}]$ the set of transport equations:

$$(16)\qquad \begin{cases} \partial_t f_k + \lambda_k \partial_x f_k = 0,\ k = 1,\dots,N\,, \\ f(t^n, x) = f_\Delta^n(x), \\ f_k(t,0) = f_{b,k}^n \qquad\qquad if \ \ \lambda_k > 0. \end{cases}$$

The scheme is written in conservation form. For all $i \geq 0$:

$$(17)\qquad f_{i,k}^{n+\frac{1}{2}} = f_{i,k}^n - \lambda_k \tfrac{\Delta t}{\Delta x}\left(f_{i+\frac{1}{2},k}^n - f_{i-\frac{1}{2},k}^n\right),\ k \in \{1,\dots,N\}.$$

The numerical flux $\lambda_k f_{i+\frac{1}{2},k}^n$ is a Lipschitz continuous function $\lambda_k \mathcal{F}_k$ of $\{f_{i+j,k}^n,\ \ -1 \leq j \leq 2\}$ and satisfies the consistency condition

$$(18)\qquad \mathcal{F}(f,\dots,f) = f.$$

More precisely, we use schemes which can be written as a convex combination in Harten's formulation [21]:

$$(19)\qquad i \geq 0,\ \ \begin{cases} \lambda_k > 0 \ \ f_{i,k}^{n+\frac{1}{2}} = f_{i,k}^n(1 - D_{i-\frac{1}{2},k}^n) + D_{i-\frac{1}{2},k}^n f_{i-1,k}^n, \\ \lambda_k \leq 0 \ \ f_{i,k}^{n+\frac{1}{2}} = f_{i,k}^n(1 - D_{i+\frac{1}{2},k}^n) + D_{i+\frac{1}{2},k}^n f_{i+1,k}^n. \end{cases}$$

As a first order scheme the upwind approximation is chosen:

$$D^n_{i+\frac{1}{2},k} = |\lambda_k| \frac{\Delta t}{\Delta x} \ .$$

In that case, formula (19) is well defined for all the cells, including $i = 0$. At higher order, $D^n_{i+\frac{1}{2},k}$ is a nonlinear function of $\{f^n_{i+j}, \quad -1 \leq j \leq 2\}$. We postpone this study to Section 5. In all cases the time step restriction is:

$$\max_{1 \leq k \leq N} |\lambda_k| \frac{\Delta t}{\Delta x} \leq 1. \tag{20}$$

Collision

We now solve on $[t^n, t^{n+1}]$ and for all $i \geq 0$ the ODS:

$$\begin{cases} \partial_t f_k(t) = \frac{1}{\epsilon}(M_k(u) - f_k(t)) \ \ \forall k \in \{1, \ldots, N\}, \\ f(t^n) = f_i^{n+\frac{1}{2}}, \\ u = Pf = \sum_k f_k. \end{cases} \tag{21}$$

As we look for a scheme owning the correct relaxation limit, we have to be careful with the singularity with respect to ϵ. Actually, the compatibility relations (10) allow us to solve exactly the system:

$$u_i^{n+1} = u_i^{n+\frac{1}{2}} \tag{22}$$

and

$$f_{i,k}(t^{n+1}) = (1 - e^{-\frac{\Delta t}{\epsilon}}) M_k(u_i^{n+\frac{1}{2}}) + e^{-\frac{\Delta t}{\epsilon}} f_{i,k}^{n+\frac{1}{2}}, \quad i \geq 0. \tag{23}$$

Thus, when ϵ tends to zero, the limit exists and this collisional step reduces to a projection on equilibrium state:

$$f_{i,k}^{n+1} = M_k(u_i^{n+\frac{1}{2}}) = M_k(u_i^{n+1}), \quad i \geq 0. \tag{24}$$

We underline that the collision step is solved at the border cell in the same way as it is inside the grid. Thus, we have defined a whole time step, and we can write the complete scheme in macroscopic variables:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}}), \quad i \geq 0 \tag{25}$$

where

$$g^n_{i+\frac{1}{2}} = \sum_{k=1}^{N} \lambda_k f^n_{i+\frac{1}{2},k}, \quad i \geq 0.$$

This scheme is consistent since

$$g^n_{i+\frac{1}{2}} = \mathcal{G}(u^n_{i-1}, \ldots, u^n_{i+2}) = \sum_{k=1}^{N} \lambda_k \mathcal{F}_k(M_k(u^n_{i-1}), \ldots, M_k(u^n_{i+2}))$$

and by relations (10) and (18):

$$\mathcal{G}(u, \ldots, u) = F(u).$$

When transport is approximated by the upwind first order scheme, the previous fluxes can be written as

$$g^n_{i+\frac{1}{2}} = \sum_{\lambda_k > 0} \lambda_k M_k(u^n_i) + \sum_{\lambda_k < 0} \lambda_k M_k(u^n_{i+1}),$$

while the flux on the boundary interface is

$$g^n_{-\frac{1}{2}} = \sum_{\lambda_k > 0} \lambda_k M_k(u^n_b) + \sum_{\lambda_k < 0} \lambda_k M_k(u^n_0).$$

Note that there is no ghost cell, and that on the whole grid we apply the same scheme. The time step restriction is the one given by the transport part of the scheme, namely condition (20). As we always suppose that the Maxwellian functions are MND, the subcharacteristic condition (15) is satisfied. Hence, we have also the usual condition:

$$\max_{1 \le m \le p} |a_m(u)| \frac{\Delta t}{\Delta x} \le 1.$$

2.3 Higher order in time

To construct higher order in time approximations we remark that the we have obtained numerical flux functions g such that formula (25) may be viewed as a first order discretization of the differential equation:

$$u'_i = u_i - \frac{\overline{g_{i+\frac{1}{2}}} - \overline{g_{i-\frac{1}{2}}}}{\Delta x}\ . \tag{26}$$

Here $\overline{g_{i+\frac{1}{2}}}$ is the limit of $g_{i+\frac{1}{2}}$ when $\Delta t \to 0$ and Δx is fixed. This limit exists if one uses for (16) an upwind first order scheme or a higher order MUSCL reconstruction-transport-projection scheme, see the examples below. Therefore, one may discretize (26) with any high order (for example Runge-Kutta) method. The stability condition remains the same. As usual the order in time has to be related with order in space.

3 Stability results in the scalar case

For Sections 3 and 4, we restrict ourselves to the convergence of kinetic numerical schemes towards the unique entropy solution of (1) where u is a scalar, and F a flux in $C^1(\mathbb{R}; \mathbb{R})$. We first present some useful preliminary results.

3.1 Preliminaries

Definition 3.1 *A function $u \in BV(\mathbb{R}^+\times]0,T[)$ is a weak entropy solution of problem (1)(2)(3) if for all $c \in \mathbb{R}$ and all non negative test function $\phi \in C^1([0,T[\times[0,+\infty[)$ with compact support, it satisfies the following inequality :*

$$\begin{aligned}(27)\qquad &\int_{\mathbb{R}^+}\int_0^T |u-c|\partial_t\phi + \mathrm{sgn}(u-c)(F(u)-F(c))\partial_x\phi \, dt\, dx \\ &\int_{\mathbb{R}^+} |u^0(x)-c|\phi(0,x)\, dx + \int_0^T \mathrm{sgn}(u_b(t)-c)(F(u(t,0)) \\ &-F(c))\phi(t,0)\, dt \geq 0.\end{aligned}$$

This definition is given in [3], where uniqueness of such a solution is proved by using Kružkov's techniques. A weak entropy solution satisfies condition (4).

Now we present a very useful tool, specific to kinetic approximation, see [8] for more general formulation. The following theorem links the entropy functions of (1) and the existence of kinetic entropy functions associated to (7). Note that all results use only an invariant interval I in the phase space ($u \in I$), this is coherent with the boundary problem.

Theorem 3.1 *Let I be an invariant interval for (1) and $I_k = M_k(I)$. The two following properties are equivalent :*

*i) The Maxwellian function is monotone non decreasing (*MND*) on I.*
ii) For a given entropy pair (S,G) of (1), there exist kinetic entropies S_k defined on I_k for $k=1,\dots,N$ such that :
(E1) for all k, S_k is convex,
(E2) for $u \in I$, $\sum_k S_k(M_k(u)) = S(u) + C$ and C is a constant.
(E3) if $u_f = \sum_k f_k \in I$, then :

$$\sum_k S_k(M_k(u_f)) \leq \sum_k S_k(f_k).$$

For example, if S is a Kružkov entropy $S(u) = |u-c|$, then we can define $S_k(f_k) = |f_k - M_k(c)|$. If M is MND, it is clear that properties (E1) and (E2) are satisfied. So is (E3):

$$\begin{aligned}\sum_{k=1}^N S_k(M_k(u_f)) &= |u_f - c| \\ &= \left|\sum_{k=1}^N (f_k - M_k(c))\right| \\ &\leq \sum_{k=1}^N S_k(f_k).\end{aligned}$$

3.2 Assumptions

Our hypothesis are the following :

* On the data, we suppose that

$$(H1)\begin{cases} u^0 \in L^\infty(\mathbb{R}_+) \cap BV(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \\ \\ u_b \in L^\infty(0,T) \cap BV(0,T) \cap L^1(0,T) \end{cases}$$

* On the Maxwellian function, we need monotonicity (MND), that is for all k:

$$\text{(H2)} \quad M_k'(u) \geq 0 \;\; \forall u \in I, \quad I = [-\mu_\infty, \mu_\infty]$$

with

$$\mu_\infty = \max(||u_b||_{L^\infty(\mathbb{R}_t^+)}, ||u^0||_{L^\infty(\mathbb{R}_x^+)})$$

* On the transport part of the scheme we impose that

$$\text{(H3)} \quad \forall i, n, k \quad D^n_{i-\frac{1}{2},k} \in [0,1]$$

3.3 L^∞ bounds

Lemma 3.1 *Under the assumptions (H1)-(H3), the kinetic schemes (25) are L^∞ stable: for all n and all $i \geq 0$:*

$$u_i^n \in I.$$

Proof. Note that by (H2), the boundary condition can be estimated for every n as

$$M_k(-\mu_\infty) \leq f^n_{b,k} \leq M_k(\mu_\infty).$$

Results are shown by induction. Suppose that for all $i \geq 0$:

$$-\mu_\infty \leq u_i^n \leq \mu_\infty.$$

Then

$$M_k(-\mu_\infty) \leq f^n_{i,k} \leq M_k(\mu_\infty).$$

Thus, using the convex formulation on transport:

$$\begin{aligned} &(1 - D^n_{i-\frac{1}{2},k})M_k(-\mu_\infty) \\ &\quad + D^n_{i-\frac{1}{2},k} M_k(-\mu_\infty) \leq (1 - D^n_{i-\frac{1}{2},k}) f^n_{i,k} + D^n_{i-\frac{1}{2},k} f^n_{i-1,k} \leq \\ &\qquad\qquad \leq (1 - D^n_{i-\frac{1}{2},k}) M_k(\mu_\infty) + D^n_{i-\frac{1}{2},k} M_k(\mu_\infty). \end{aligned}$$

for the positive velocity components, for example, which gives

$$M_k(-\mu_\infty) \le f_{i,k}^{n+\frac{1}{2}} \le M_k(\mu_\infty).$$

The first remark enables us to claim the same property on the first cell as well. Because $u_i^{n+\frac{1}{2}} = \sum_k f_{i,k}^{n+\frac{1}{2}}$ we have

$$-\mu_\infty \le u_i^{n+\frac{1}{2}} = u_i^{n+1} \le \mu_\infty$$

which ends the proof. □

3.4 BV bounds

This part is the most difficult to establish, because it is specific to the boundary problem. In fact, when applying the Green-Gauss formula for the Cauchy problem there is no border term, while for the initial-boundary value problem there is, and it needs to be controlled. For our scheme, the estimates do not involve the same techniques as in the continuous case (developed in [34]). In fact in the discrete case, the BV bounds are obtained thanks to an "entropy-like" relationship, as in the continuous case one uses the kinetic equations.

In the following proposition we give an estimate of the total variation in space of the numerical solution. In contrast with what is done in [37], the total variation in time on the boundary is not needed.

Proposition 3.1 *Under hypothesis (H1)–(H3), we have*

$$TV(u_\Delta^{n+1}) = TV(f_\Delta^{n+1}) \le TV(u_\Delta^0) + TV(u_{b,\Delta}) + |u_0^0 - u_b^0|$$

$$\le TV(u^0) + TV(u_b) + 2\mu_\infty$$

where $TV(f_\Delta^{n+1}) = \sum_k TV(f_{k,\Delta}^{n+1})$.

Proof. 1 Collision

Using (H2), it is easy to show that

$$TV(u_\Delta^{n+1}) = TV(f_\Delta^{n+1}) \le TV(f_\Delta^{n+\frac{1}{2}}).$$

2 Transport

. Consider $\lambda_k > 0$. For the first cell we have :

$$|f_{1,k}^{n+\frac{1}{2}} - f_{0,k}^{n+\frac{1}{2}}| \le (1 - D^n_{-\frac{1}{2},k})|f_{1,k}^n - f_{0,k}^n| + D^n_{-\frac{1}{2},k}|f_{0,k}^n - M_k(u_b^n)|.$$

For the whole grid, we then have

$$\begin{aligned}\sum_{i=1}^{\infty}|f_{i,k}^{n+\frac{1}{2}}-f_{i-1,k}^{n+\frac{1}{2}}| &\le \sum_{i=1}^{\infty}(1-D_{i-\frac{1}{2},k}^{n})|f_{i,k}^{n}-f_{i-1,k}^{n}|\\ &+\sum_{i=1}^{\infty}D_{i-\frac{1}{2},k}^{n}|f_{i,k}^{n}-f_{i-1,k}^{n}|\\ &+D_{-\frac{1}{2},k}^{n}|f_{0,k}^{n}-M_k(u_b^n)|\\ &\le \sum_{i=1}^{\infty}|f_{i,k}^{n}-f_{i-1,k}^{n}|+D_{-\frac{1}{2},k}^{n}|f_{0,k}^{n}-M_k(u_b^n)|.\end{aligned}$$

. Consider $\lambda_k < 0$. In the same way:

$$\begin{aligned}\sum_{i=1}^{\infty}|f_{i,k}^{n+\frac{1}{2}}-f_{i-1,k}^{n+\frac{1}{2}}| &\le \sum_{i=1}^{\infty}(1-D_{-\frac{1}{2},k}^{n})|f_{i,k}^{n}-f_{i-1,k}^{n}|\\ &+\sum_{i=2}^{\infty}D_{-\frac{1}{2},k}^{n}|f_{i,k}^{n}-f_{i-1,k}^{n}|\\ &\le \sum_{i=1}^{\infty}|f_{i,k}^{n}-f_{i-1,k}^{n}|-D_{\frac{1}{2},k}^{n}|f_{1,k}^{n}-f_{0,k}^{n}|.\end{aligned}$$

. For $\lambda_k = 0$:

$$\sum_{i=1}^{\infty}|f_{i,k}^{n+\frac{1}{2}}-f_{i-1,k}^{n+\frac{1}{2}}|=\sum_{i=1}^{\infty}|f_{i,k}^{n}-f_{i-1,k}^{n}|.$$

The total variation of the transport part can thus be written as

$$\begin{aligned}\sum_k TV(f_k^{n+\frac{1}{2}}) \le \sum_k TV(f_k^n)+\sum_{\lambda_k>0}D_{-\frac{1}{2},k}^{n}|f_{0,k}^{n}-M_k(u_b^n)|\\ -\sum_{\lambda_k<0}D_{\frac{1}{2},k}^{n}|f_{1,k}^{n}-f_{0,k}^{n}|.\end{aligned}$$

For the whole step we have,

$$\begin{aligned}\sum_k TV(f_k^{n+1}) \le \sum_k TV(f_k^n)+\sum_{\lambda_k>0}D_{-\frac{1}{2},k}^{n}|f_{0,k}^{n}-M_k(u_b^n)|\\ -\sum_{\lambda_k<0}D_{\frac{1}{2},k}^{n}|f_{1,k}^{n}-f_{0,k}^{n}|.\end{aligned}\tag{28}$$

In order to control the two last terms in this inequality, we need the following key argument:

Lemma 3.2 *Under the same hypothesis, we have*

$$\begin{aligned}\sum_{\lambda_k>0} D^n_{-\frac{1}{2},k}|f^n_{0,k} - M_k(u^n_b)| &- \sum_{\lambda_k<0} D^n_{\frac{1}{2},k}|f^n_{1,k} - f^n_{0,k}| \\ &\leq \sum_k |f^n_{0,k} - M_k(u^{n-1}_b)| \\ &- \sum_k |f^{n+1}_{0,k} - M_k(u^n_b)| + |u^n_b - u^{n-1}_b|\end{aligned}$$

Proof. In a standard way, we have

$$\begin{aligned}\sum_k |f^{n+1}_{0,k} - M_k(c)| = \sum_k |M_k(u_0^{n+\frac{1}{2}}) - M_k(c)| &= |u_0^{n+\frac{1}{2}} - c| \\ \leq \sum_k |f^{n+\frac{1}{2}}_{0,k} - M_k(c)|.\end{aligned}$$

Now taking $c = u^n_b$, we have that

$$\sum_k |f^{n+1}_{0,k} - M_k(u^n_b)| \leq \sum_k |f^{n+\frac{1}{2}}_{0,k} - M_k(u^n_b)|.$$

For the transport part, we have

$$\begin{aligned}\sum_k |f^{n+\frac{1}{2}}_{0,k} - M_k(u^n_b)| &\leq \sum_{\lambda_k>0} |(1 - D^n_{-\frac{1}{2},k}) f^n_{0,k} + D^n_{-\frac{1}{2},k} M_k(u^n_b) - M_k(u^n_b)| \\ &+ \sum_{\lambda_k\leq 0} |(1 - D^n_{\frac{1}{2},k}) f^n_{0,k} + D^n_{\frac{1}{2},k} f^n_{1,k} - M_k(u^n_b)| \\ &\leq \sum_{\lambda_k>0} (1 - D^n_{-\frac{1}{2},k}) |f^n_{0,k} - M_k(u^n_b)| \\ &+ \sum_{\lambda_k\leq 0} \Big[D_{\frac{1}{2},k} |f^n_{0,k} - f^n_{1,k}| + |f^n_{0,k} - M_k(u^n_b)| \Big] \\ &\leq \sum_k |f^n_{0,k} - M_k(u^n_b)| \\ &+ \sum_{\lambda_k<0} D_{\frac{1}{2},k} |f^n_{0,k} - f^n_{1,k}| - \sum_{\lambda_k>0} D_{-\frac{1}{2},k} |f^n_{0,k} - M_k(u^n_b)| \\ &\leq \sum_k |f^n_{0,k} - M_k(u^{n-1}_b)| + \sum_k |M_k(u^n_b) - M_k(u^{n-1}_b)| \\ &+ \sum_{\lambda_k<0} D_{\frac{1}{2},k} |f^n_{0,k} - f^n_{1,k}| - \sum_{\lambda_k>0} D_{-\frac{1}{2},k} |f^n_{0,k} - M_k(u^n_b)|,\end{aligned}$$

which ends the proof. □

Thanks to the last result, (28) becomes

$$\begin{aligned} TV(f_\Delta^{n+1}) \le TV(f_\Delta^n) &+ \sum_k |f_{0,k}^n - M_k(u_b^{n-1})| \\ &- \sum_k |f_{0,k}^{n+1} - M_k(u_b^n)| + |u_b^n - u_b^{n-1}| \end{aligned}$$

which by induction can be written as:

$$\begin{aligned} TV(f_\Delta^{n+1}) &\le TV(f_\Delta^0) + \sum_k |f_{0,k}^0 - M_k(u_b^0)| + TV(u_{b,\Delta}) \\ &\le TV(u_\Delta^0) + |u_{0,\Delta}^0 - u_{b,\Delta}^0| + TV(u_{b,\Delta}). \end{aligned}$$

We projected the initial condition on the equilibrium manifold, which gives the desired result. □

3.5 $L^1(\mathbb{R}^+)$ bounds

We use now the TV stability to establish the L^1 estimates.

Lemma 3.3 *Under the assumptions (H1)-(H3), the kinetic schemes (25) are L^1 stable.*

$$||u_\Delta^{n+1}||_{L^1(\mathbb{R}^+)} = ||f_\Delta^{n+1}||_{L^1(\mathbb{R}^+)} \le ||u^0||_{L^1(\mathbb{R}^+)} + ||u_b||_{L^1(0,T)} + T\ CK$$

where $||f_\Delta^{n+1}||_{L^1(\mathbb{R}^+)} = \sum_k ||f_{k,\Delta}^{n+1}||_{L^1(\mathbb{R}^+)}$, *$C$ is proportional to* $\sup_k |\lambda_k|$, *and*

$$K = TV(u^0, \mathbb{R}^+) + TV(u_b, (0,T)) + 2\mu_\infty. \tag{29}$$

Proof. 1 Collision

By (H2) we have

$$||u_\Delta^{n+1}||_{L^1(\mathbb{R}^+)} = ||f_\Delta^{n+1}||_{L^1(\mathbb{R}^+)} \le ||f_\Delta^{n+\frac{1}{2}}||_{L^1(\mathbb{R}^+)}.$$

2 Transport part

For positive velocity components, we can see that

$$|f_{i,k}^{n+\frac{1}{2}}| \le |f_{i,k}^n| + D_{i-\frac{1}{2},k}^n |f_{i,k}^n - f_{i-1,k}^n|$$

for all $i \ge 1$, while on the edge, we have

$$|f_{0,k}^{n+\frac{1}{2}}| \le (1 - D_{-\frac{1}{2},k}^n)|f_{i,k}^n| + D_{-\frac{1}{2},k}|M_k(u_b^n)|.$$

So that after summing over i, one has

$$\sum_{i\geq 0} |f_{i,k}^{n+\frac{1}{2}}| \leq \sum_{i\geq 0} |f_{i,k}^n| + TV(f_{i,k}^n) + |M_k(u_b^n)|.$$

For the non entering characteristics, we have no boundary terms, so that for every $i \geq 0$ we have

$$|f_{i,k}^{n+\frac{1}{2}}| \leq |f_{i,k}^n| + D_{i+\frac{1}{2},k}^n |f_{i,k}^n - f_{i+1,k}^n|,$$

which after summing over the whole grid gives

$$\sum_{i\geq 0} |f_{i,k}^{n+\frac{1}{2}}| \leq \sum_{i\geq 0} |f_{i,k}^n| + TV(f_{k,\Delta}^n).$$

For all components we can write

$$\begin{aligned}\sum_k ||f_{k,\Delta}^{n+\frac{1}{2}}||_{L^1(\mathbb{R}^+)} \leq \sum_k ||f_{k,\Delta}^n||_{L^1(\mathbb{R}^+)} + \Delta x \sum_k TV(f_{k,\Delta}^n) \\ + \Delta x \sum_{\lambda_k > 0} |M_k(u_b^n)|\end{aligned}$$

which gives, on u

$$\begin{aligned}||u_\Delta^{n+1}||_{L^1(\mathbb{R}^+)} &\leq ||u_\Delta^n||_{L^1(\mathbb{R}^+)} + \Delta x TV(u_\Delta^n, \mathbb{R}^+) + \Delta x \sum_{\lambda_k > 0} |M_k(u_b^n)| \\ &\leq ||u^0||_{L^1(\mathbb{R}^+)} + n\Delta x K + ||u_b||_{L^1(0,T)}\end{aligned}$$

where K is defined in (29). We conclude by noting that Δt and Δx satisfy the CFL condition (20) and their ratio is fixed.

□

3.6 Convergence results

Lemma 3.4 *Let u_Δ be the numerical solution given by (25). If the assumptions (H1)-(H3) are satisfied, then the following time estimate holds for u_Δ:*

$$||u_\Delta^{n+1} - u_\Delta^n||_{L^1(\mathbb{R}_+)} \leq CK\Delta x$$

where K is defined in (29).

Proof. For the transport part, we take for example the positive velocity components, for $i \geq 1$:

$$\begin{aligned} |f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^n| &\leq D_{i-\frac{1}{2}}^n |f_{i,k}^n - f_{i-1,k}^n| \\ &\leq |f_{i,k}^n - f_{i-1,k}^n| \end{aligned}$$

whereas for the boundary cell one has :

$$|f_{0,k}^{n+\frac{1}{2}} - f_{0,k}^n| \leq |f_{0,k}^n - M_k(u_b^n)| \leq 2M_k(\mu_\infty).$$

Summing over $i \geq 1$ we obtain:

$$\sum_{i \geq 0} |f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^n| \leq TV(f_k^n, \mathbb{R}^+) + 2M_k(\mu_\infty).$$

The same computation on other components leads to:

$$\begin{aligned} |f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^n| &\leq D_{i+\frac{1}{2}}^n |f_{i,k}^n - f_{i+1,k}^n| \\ &\leq |f_{i,k}^n - f_{i+1,k}^n| \end{aligned}$$

which gives finally,

$$\begin{aligned} ||u_\Delta^{n+1} - u_\Delta^n||_{L^1(\mathbb{R}^+)} &= \Delta x \sum_i |u^{n+1} - u_i^n| = \Delta x \sum_i |u^{n+\frac{1}{2}} - u_i^n| \\ &\leq \Delta x (\sum_k TV(f_k^n, \mathbb{R}^+) + 2 \sum_{\lambda_k > 0} M_k(\mu_\infty)) \\ &\leq \Delta x (TV(u_\Delta^n, \mathbb{R}^+) + 2\mu_\infty). \end{aligned}$$

□

As a corollary, we have the equicontinuity in time :

Lemma 3.5 *Under the assumptions (H1)-(H3):*

(30) $\quad \forall t, t' \in [0, T], \quad ||u_\Delta(t) - u_\Delta(t')||_{L^1(\mathbb{R}_+)} \leq CK(\Delta t + |t - t'|)$

where K is defined in (29).

We are now able to prove convergence of subsequences of $(u_\Delta)_\Delta$ towards weak solutions of (1). By continuity with respect to time, the limits satisfy the initial condition (2). At this point we have no estimates allowing to prove that entropy solutions are obtained and the limits of the scheme are not necessarily unique. As the boundary condition is known, *via* formulation (4), only for entropy solutions, it is not always satisfied by a limit of the scheme. This result is completed in the next Section, where convergence to weak entropy solutions of the problem (1)(2)(4) is proved for a class of first order approximations.

Theorem 3.2 *Suppose that hypothesis (H1)-(H3) hold and let $T > 0$ be fixed. Consider a sequence $\Delta x_j \to 0$, $(j \to +\infty)$ with $\frac{\Delta t_j}{\Delta x_j}$ kept constant. There exists a subsequence of $(u_{\Delta_j})_{j\geq 0}$ which converges in $\mathrm{L}^\infty(0,T;\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+))$ to a solution u of (1)(2), and $u \in C^0([0,T],\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+)) \cap \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^+))$.*

Proof. We follow exactly the method of [13] and just give a sketch of the proof: for all $t \geq 0$, $\{u_\Delta(t), \quad \Delta t > 0\}$ is bounded in $\mathrm{L}^1 \cap \mathrm{L}^\infty$. Moreover, by Proposition 3.1, we can apply Fréchet-Kolmogorov theorem and obtain a relatively compact set of ${\mathrm{L}^1}_{\mathrm{loc}}$. Using the equicontinuity property (30) as in the proof of Ascoli-Arzela's theorem we obtain convergence in $\mathrm{L}^\infty(0,T;\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+))$ and a.e. Every limit is a solution of the problem by Lax-Wendroff's theorem. □

4 First order approximation (FO) and entropy solution

In this section, we restrict ourselves to the first order in space approximation, that is

$$D^n_{i-\frac{1}{2},k} = \xi_k = |\lambda_k|\frac{\Delta t}{\Delta x}. \tag{31}$$

In that case, assumption (H3) is equivalent to the time step restriction (20) so that Theorem 3.2 can be written as:

Theorem 4.1 *Suppose that hypothesis (H1)-(H2) are satisfied together with conditions (31),(20). Let $T > 0$ be fixed. Consider a sequence $\Delta x_j \to 0$, $(j \to +\infty)$ with $\frac{\Delta t_j}{\Delta x_j}$ kept constant.*

There exists a subsequence of $(u_{\Delta_j})_{j\geq 0}$ which converges in $\mathrm{L}^\infty(0,T;\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+))$ to a solution u of (1)(2), and $u \in C^0([0,T],\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+)) \cap \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^+))$.

Using the upwind scheme to discretize the transport equations (16) has two consequences: we are able to estimate the trace of the numerical solution and we have discrete entropy inequalities. These are the tools allowing us to prove that the scheme converges towards the unique weak entropy solution of the initial boundary value problem (1)(2)(4).

4.1 Estimates on the trace

To show the consistency of the trace with the boundary condition (see [3]), we need the following estimates on the trace of the solution given by the first order scheme.

Lemma 4.1 *Suppose that hypothesis (H1)-(H2) are satisfied together with conditions (31),(20). Let $T > 0$ be fixed and consider $n \geq 1$ such that $n\Delta t \leq T$. One has:*

$$\sum_{p=1}^{n}\sum_{\lambda_k<0} \xi_k |M_k(u_0^p) - M_k(u_0^{p-1})| \leq TV(u_{b,\Delta},(0,T)) + TV(u_\Delta^0,\mathbb{R}^+) + |u_0^0 - u_b^0|$$

$$\leq TV(u_b,(0,T)) + TV(u^0,\mathbb{R}^+) + 2\mu_\infty.$$

Proof. Consider $\lambda_k > 0$. We have for all $i \geq 0$:

$$|f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}| \leq (1-\xi_k)|f_{i,k}^n - f_{i,k}^{n-1}| + \xi_k |f_{i-1,k}^n - f_{i-1,k}^{n-1}|$$

which gives after summing over all cells of the grid:

$$\sum_{i=0}^{\infty} |f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}| \leq \sum_{i=0}^{\infty} |f_{i,k}^n - f_{i,k}^{n-1}| + \xi_k |M_k(u_b^n) - M_k(u_b^{n-1})|$$

. For $\lambda_k < 0$ and $i \geq 0$:

$$|f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}| \leq (1-\xi_k)|f_{i,k}^n - f_{i,k}^{n-1}| + \xi_k |f_{i+1,k}^n - f_{i+1,k}^{n-1}|$$

so that

$$\sum_{i=0}^{\infty} |f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}| \leq \sum_{i=0}^{\infty} |f_{i,k}^n - f_{i,k}^{n-1}| - \xi_k |M_k(u_0^n) - M_k(u_0^{n-1})|$$

. For $\lambda_k = 0$ and $i \geq 0$:

$$|f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}| = |f_{i,k}^n - f_{i,k}^{n-1}|.$$

The whole transport step yields :

$$\sum_{i,k} |f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}| \leq \sum_{i,k} |f_{i,k}^n - f_{i,k}^{n-1}| + \sum_{\lambda_k>0} \xi_k |M_k(u_b^n) - M_k(u_b^{n-1})| - \sum_{\lambda_k<0} \xi_k |M_k(u_0^n) - M_k(u_0^{n-1})|.$$

For the projection step, for $i \geq 0$:

$$\sum_k |f_{i,k}^{n+1} - f_{i,k}^n| \leq \sum_k |f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}|.$$

That gives

$$\sum_{\lambda_k<0} \xi_k |M_k(u_0^n) - M_k(u_0^{n-1})| \leq |u_b^n - u_b^{n-1}| + \sum_{i,k} |f_{i,k}^n - f_{i,k}^{n-1}| - \sum_k |f_{i,k}^{n+1} - f_{i,k}^n|.$$

By summation one gets :

$$\sum_{p=1}^n \sum_{\lambda_k<0} \xi_k |M_k(u_0^p) - M_k(u_0^{p-1})| \leq TV(u_b, (0,T)) + \sum_{i,k} |f_{i,k}^1 - f_{i,k}^0|$$

with

$$\sum_{i,k} |f_{i,k}^1 - f_{i,k}^0| = \sum_{i,k} |M_k(u_i^{\frac{1}{2}}) - M_k(u_i^0)| = \sum_i |u_i^{\frac{1}{2}} - u_i^0|$$

and this can be estimated by

$$\sum_i |u_i^{\frac{1}{2}} - u_i^0| \leq TV(u^0, \mathbb{R}^+) + |u_0^0 - u_b^0|$$

□

Let us define for all k such that $\lambda_k < 0$:

$$\omega_{k,\Delta}(t) = \omega_k^n = M_k(u_0^n) \quad \text{if} \quad t \in [t_n, t_{n+1}[.$$

Corollary 4.1 *Under the assumptions of Lemma 4.1,* $\{\omega_{k,\Delta}, \Delta x > 0\}$ *is bounded in* $BV(0,T)$ *and for all* $\Delta x_j \to_{j\to+\infty} 0$*, we can extract an* $L^1(0,T)$ *convergent subsequence of* $(\omega_{k,\Delta_j})_{j\geq 0}$.

4.2 Discrete entropy inequalities

In order to apply the usual tools of the Lax-Wendroff theorem, we use the standard discrete entropy inequalities already established for the Cauchy problem.

Lemma 4.2 *Suppose that hypothesis (H1)-(H2) are satisfied together with conditions (31),(20). Let (S, G) be a given entropy pair of (1). Let $T > 0$ be fixed and consider $n \geq 0$ such that $(n+1)\Delta t \leq T$. For all $i \geq 0$, we have :*

$$S(u_i^{n+1}) - S(u_i^n) + \frac{\Delta t}{\Delta x}[G^n_{i+\frac{1}{2}} - G^n_{i-\frac{1}{2}}] \leq 0 \tag{32}$$

where

$$G^n_{i+\frac{1}{2}} = \sum_{\lambda_k>0} \lambda_k S_k(M_k(u_i^n)) + \sum_{\lambda_k<0} \lambda_k S_k(M_k(u_{i+1}^n))$$

while the boundary flux reads

$$G^n_{-\frac{1}{2}} = \sum_{\lambda_k>0} \lambda_k S_k(M_k(u_b^n)) + \sum_{\lambda_k<0} \lambda_k S_k(M_k(u_0^n)).$$

Here the functions S_k are the ones given by Theorem 3.1.

Proof. We use the transport-projection formulation as in (19) and in (24). Then it is enough to use the transport convex form and to apply a convex entropy.

$$\begin{aligned} &\lambda_k > 0 : S_k(f_{i,k}^{n+\frac{1}{2}}) \leq S_k(f_{i,k}^n)(1-\xi_k) + \xi_k S_k(f_{i-1,k}^n) \\ &\lambda_k \leq 0 : S_k(f_{i,k}^{n+\frac{1}{2}}) \leq S_k(f_{i,k}^n)(1-\xi_k) + \xi_k S_k(f_{i+1,k}^n) \end{aligned} \tag{33}$$

Summing up over k :

$$\begin{aligned} \sum_k S_k(f_{i,k}^{n+\frac{1}{2}}) - \sum_k S_k(f_{i,k}^n) \leq &- \sum_{\lambda_k>0} \xi_k(S_k(f_{i,k}^n) - S_k(f_{i-1,k}^n)) \\ &+ \sum_{\lambda_k<0} \xi_k(S_k(f_{i+1,k}^n) - S_k(f_{i,k}^n)). \end{aligned} \tag{34}$$

As $f_{i,k}^n = M_k(u_i^n)$, by Theorem 3.1:

$$\sum_k S_k(f_i^n) = S(u_i^n) + C.$$

Now, $u_i^{n+1} = \sum_k f_{i,k}^{n+\frac{1}{2}}$, so that the same theorem gives:

$$\begin{aligned} S(u_i^{n+1}) - S(u_i^n) \leq &- \sum_{\lambda_k>0} \xi_k(S_k(f_{i,k}^n) - S_k(f_{i-1,k}^n)) \\ &+ \sum_{\lambda_k<0} \xi_k(S_k(f_{i+1,k}^n) - S_k(f_{i,k}^n)). \end{aligned}$$

We obtain the desired discrete entropy inequality by recalling that $\xi_k = |\lambda_k| \frac{\Delta t}{\Delta x}$. □

4.3 Passing to the limit towards a continuous inequality

We now claim our main convergence result:

Theorem 4.2 *Suppose that hypothesis (H1)-(H2) are satisfied together with conditions (31),(20). Let $T > 0$ be fixed. When the discretization step Δx goes to 0 and $\frac{\Delta t}{\Delta x}$ is constant, the solution given by the first order scheme (25) converges in $\mathrm{L}^\infty(0,T;\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+))$ towards the unique weak entropy solution of problem (1)(2)(4).*

We recall that according to Definition 3.1, the solution is characterized by the inequality (27). With the help of Lemmas 4.1 and 4.2, we first establish a continuous entropy inequality where kinetic entropies are involved. We find the same result as in [37].

Lemma 4.3 *We make the same assumptions as in Theorem 4.2. Let (S,G) be a given entropy pair of (1). Let $T > 0$ be fixed. Consider a sequence $\Delta x_j \to 0$, $(j \to +\infty)$ with $\frac{\Delta t_j}{\Delta x_j}$ kept constant. There exists a subsequence of $(u_{\Delta_j})_{j\geq 0}$ which converges to a weak solution of (1)(2) satisfying :*

$$
\begin{aligned}
&\int_{(0,T)\times\mathbb{R}_+} S(u)\partial_t\phi + G(u)\partial_x\phi \; dt\; dx \\
&+ \sum_{\lambda_k>0} \lambda_k \int_{(0,T)} S_k(M_k(u_b(t)))\phi(t,0)\; dt \\
&+ \sum_{\lambda_k<0} \lambda_k \int_{(0,T)} S_k(\omega_k)\phi(t,0)\; dt + \int_{\mathbb{R}_+} S(u^0(x))\phi(0,x)\; dx \geq 0
\end{aligned}
$$

for all $\phi \in C^1_0([0,T[\times[0,+\infty[)$. Here ω_k is a limit of $(\omega_{k,\Delta_j})_{j\geq 0}$ obtained by Corollary 4.1.

Proof. We call ϕ_i^n the piecewise constant approximation of a positive test function of $C^1_0([0,T[\times[0,+\infty[)$.

$$
\phi_i^n = \frac{1}{\Delta x \Delta t} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{t^n}^{t^{n+1}} \phi(t,x)\; dt\; dx \quad \text{with} \quad \phi \in C^1_0([0,+\infty[\times[0,T))
$$

We multiply the inequalities (32) obtained above by ϕ_i^n and we sum all them over the space-time domain $\mathbb{R}_+ \times (0,T)$:

$$
\sum_{i=0}^{\infty}\sum_{n=0}^{L}(S(u_i^{n+1}) - S(u_i^n))\phi_i^n + \frac{\Delta t}{\Delta x}\sum_{i=0}^{\infty}\sum_{n=1}^{L}(G^n_{i+\frac{1}{2}} - G^n_{i-\frac{1}{2}})\phi_i^n \leq 0
$$

Integration by parts gives in the first term the following quantities :

$$-\Delta x \sum_{i=0}^{\infty} S(u_i^{L+1})\phi_i^L + \Delta x \sum_{i=0}^{\infty}\sum_{n=1}^{L} S(u_i^n)(\phi_i^n - \phi_i^{n-1}) + \Delta x \sum_{i=0}^{\infty} S(u_i^0)\phi_i^0$$
$$+\Delta t \sum_{n=0}^{L}\sum_{i=0}^{\infty} G_{i+\frac{1}{2}}(\phi_{i+1}^n - \phi_i^n) + \Delta t \sum_{n=0}^{L} G^n_{-\frac{1}{2}}\phi_0^n \geq 0$$

The first term vanishes for Δx small enough, and the previous expression becomes :

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_0^T S(u_\Delta)\frac{\phi_\Delta(t,x) - \phi_\Delta(t-\Delta t, x)}{\Delta t}\, dt\, dx \\
&+ \int_{\mathbb{R}^+} S(u_\Delta(0,x))\phi_\Delta(0,x)\, dx \\
&+ \int_0^T \int_{\mathbb{R}^+} G_\Delta(t,x)\frac{(\phi_\Delta(t,x+\Delta x) - \phi_\Delta(t,x))}{\Delta x} dx\, dt \\
&+ \int_0^T G_\Delta(t, \frac{\Delta x}{2})\phi_\Delta(t, \frac{\Delta x}{2})\, dt \geq 0
\end{aligned}$$

where

$$G_\Delta(t,x) = \sum_{\lambda_k>0} \lambda_k S_k(M_k(u_\Delta(t,x))) + \sum_{\lambda_k<0} \lambda_k S_k(M_k(u_\Delta(t,x+\Delta x)))$$

By the stability arguments established before, and regularity of the test functions and their approximations the previous expression tends to :

$$\begin{aligned}
\int_0^\infty \int_0^T S(u)\partial_t\phi\, dt\, dx + \int_0^\infty S(u(0,x))\phi(0,x)\, dx& \\
+ \int_0^T \int_0^\infty G(t,x)\partial_x\phi dx\, dt& \\
+ \int_0^T G_b(t,0)\phi(t,0)\, dt \geq 0&
\end{aligned}$$

where

$$G(t,x) = \sum_k \lambda_k S_k(M_k(u(t,x)))$$

and

$$G_b(t,0) = \sum_{\lambda_k>0} \lambda_k S_k(M_k(u_b(t))) + \sum_{\lambda_k<0} \lambda_k S_k(\omega_k).$$

The last term is obtained thanks to Lemma 4.1, because estimates are uniform in Δx. □

Here we recall a lemma used in [34]:

Lemma 4.4 *Suppose that the Maxwellian functions are* MND. *If we call* $S_k(f_k) = |f_k - M_k(c)|$, *the microscopic entropy functions associated to the Kružkov entropy for (1) we have*

$$
\begin{aligned}
&\textstyle\sum_{\lambda_k>0} \lambda_k |M_k(u_b) - M_k(c)| + \sum_{\lambda_k<0} \lambda_k |f_k - M_k(c)| \\
&\le \operatorname{sgn}(u_b - c)[\textstyle\sum_{\lambda_k>0} \lambda_k M_k(u_b) + \sum_{\lambda_k<0} \lambda_k f_k - F(c)]
\end{aligned}
\tag{35}
$$

Proof of Theorem 4.2 Now we apply the Lemma 4.3 to the Kružkov entropy functions and using (35), we have the following estimate

$$
\begin{aligned}
&\int_{(0,T)\times\mathbb{R}_+} |u - c|\partial_t\phi + \operatorname{sgn}(u - c)(F(u) - F(c))\partial_x\phi \; dt \; dx \\
&+ \int_{\mathbb{R}_+} |u^0(x) - c|\phi \; dx \\
&+ \int_{(0,T)} \operatorname{sgn}(u_b - c)\left[\sum_{\lambda_k>0} \lambda_k M_k(u_b) + \sum_{\lambda_k<0} \lambda_k \omega_k - F(c)\right]\phi(t,0) \; dt \ge 0.
\end{aligned}
\tag{36}
$$

Taking a special test function ϕ, such that

$$
\phi(t,x) = \rho(t)\max\{0, 1 - \frac{x}{\eta}\}, \qquad \eta \in \mathbb{R}^+ ,
$$

where ρ is a positive test function in $C_0^\infty(0,T)$, one gets :

$$
\begin{aligned}
&\int_0^T \operatorname{sgn}(u(t,0) - c)(F(u(t,0)) - F(c))\rho(t)dt \\
&\le \int_0^T \operatorname{sgn}(u_b(t) - c)\left[\sum_{\lambda_k>0} \lambda_k M_k(u_b) + \sum_{\lambda_k<0} \lambda_k \omega_k(t) - F(c)\right]\rho(t)dt,
\end{aligned}
$$

letting η tend to zero. As this is true for every positive test function $\rho(t)$, it is true almost everywhere so that one has :

$$
\begin{aligned}
&\operatorname{sgn}(u(t,0) - c)(F(u(t,0)) - F(c)) \\
&\le \operatorname{sgn}(u_b(t) - c)\left[\sum_{\lambda_k>0} \lambda_k M_k(u_b) + \sum_{\lambda_k<0} \lambda_k \omega_k(t) - F(c)\right].
\end{aligned}
$$

Taking $c > \sup(u(t,0), u_b(t))$ and $c < \inf(u(t,0), u_b(t))$ it yields :

$$
\sum_{\lambda_k<0} \lambda_k \omega_k(t) = F(u(t,0)) - \sum_{\lambda_k>0} \lambda_k M_k(u_b(t)),
$$

we replace $\sum_{\lambda_k<0} \lambda_k \omega_k(t)$ by its value in (36) and we get the inequality (27). Consequently, the extracted subsequence converges towards the unique entropy solution of the problem (1)(2)(4). Hence, the whole sequence also converges towards this solution and the proof is complete. □

5 Stability of second order schemes

In this part, we focus our attention on second order discretizations constructed by MUSCL methods on the transport part. In general, such schemes do not satisfy discrete entropy inequalities so that convergence to the entropy solution of the problem is hopeless. Nevertheless, we can obtain convergence to a weak solution of (1)(2) by satisfying the requirements of Section 3, namely the assumption (H3). A particular interest is given to the boundary coefficients.

5.1 Second order convection

We look for coefficients $D^n_{i+\frac{1}{2},k}$ ($i \geq -1$ if $\lambda_k > 0, i \geq 0$ otherwise) such that (19) is a second order approximation of problem (16). We use a reconstruction-transport-projection method. Given the $f^n_{i,k}$, we construct a piecewise linear function

$$\overline{f^n_{i,k}}(x) = f^n_{i,k} + (x - x_i)\sigma^n_{i,k}, \quad x \in]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[, \quad i \geq 0.$$

The $\sigma^n_{i,k}$ are limited slopes, see Subsection 5.2 below. For $\lambda_k > 0$, let $\overline{f^n_{b,k}}(t)$ be a reconstructed boundary function.

Then we solve exactly the transport equations on $[t^n, t^{n+1}]$ with data $\overline{f^n_{i,k}}$ and $\overline{f^n_{b,k}}$ and we project the result on the set of functions which are constant on each cell.

If the velocity λ_k is positive, we obtain for all $i \geq 1$:

$$(37) \qquad f^{n+\frac{1}{2}}_{i,k} = (1 - \xi_k) f^n_{i,k} + \xi_k f^n_{i-1,k} - \Delta x \, \frac{\xi_k(1-\xi_k)}{2}(\sigma^n_{i,k} - \sigma^n_{i-1,k}).$$

For $i = 0$, we find:

$$f^{n+\frac{1}{2}}_{0,k} = (1 - \xi_k) f^n_{0,k} - \Delta x \, \frac{\xi_k(1-\xi_k)}{2}\sigma^n_{0,k} + \frac{\lambda_k}{\Delta x} \int_{t^n}^{t^{n+1}} \overline{f^n_{b,k}}(t)dx.$$

Therefore, it is not useful to approximate the boundary data by a piecewise linear function under the form $\overline{f^n_{b,k}}(t) = f^n_{-1,k} + \sigma^n_{-1,k}\left(t - \dfrac{t^n + t^{n+1}}{2}\right)$: it has the same average as the constant $f^n_{-1,k}$. Following this argument we take for the boundary:

$$\sigma^n_{-1,k} = 0.$$

With this notation, the formula (37) is also true for $i = 0$. Note that this does not reduce the boundary cell to first order approximation because on the other interface the slope σ^n_0 is still active.

For non positive velocities, we have for all $i \geq 0$:

$$f^{n+\frac{1}{2}}_{i,k} = (1-\xi_k) f^n_{i,k} + \xi_k f^n_{i+1,k} - \Delta x\, \frac{\xi_k(1-\xi_k)}{2} (\sigma^n_{i+1,k} - \sigma^n_{i,k}). \tag{38}$$

Comparing with (19), we can write $D^n_{i+\frac{1}{2},k}$ explicitly:

$$D^n_{i+\frac{1}{2},k} = \xi_k \left[1 + \mathrm{sgn}(\lambda_k)\, \Delta x \frac{(1-\xi_k)}{2} \frac{\sigma^n_{i+1,k} - \sigma^n_{i,k}}{\Delta f^n_{i+\frac{1}{2},k}} \right] \tag{39}$$

with the convention that if $\Delta f^n_{i+\frac{1}{2},k} = 0$ then $D^n_{i+\frac{1}{2},k} = \xi_k$. If λ_k is positive (39) is available for $i \geq -1$, otherwise it is available for $i \geq 0$.

5.2 Stability and convergence properties

The condition (H3) is satisfied if the slopes $\sigma^n_{i,k}$ are chosen in the following way. If $i \geq 1$:

$$\sigma^n_{i,k} = \mathrm{minmod}\left(X_{1,k,i} \frac{\Delta f^n_{i+\frac{1}{2},k}}{\Delta x}, X_{2,k,i} \frac{\Delta f^n_{i-\frac{1}{2},k}}{\Delta x} \right).$$

where

$$\mathrm{minmod}(a,b) = \min(|a|,|b|) \frac{\mathrm{sgn}(a) + \mathrm{sgn}(b)}{2} \quad \text{and} \quad \Delta f^n_{i+\frac{1}{2},k} = f^n_{i+1,k} - f^n_{i,k}.$$

The two coefficients $X_{1,k,i}$, $X_{2,k,i}$ amplifying the gradients can be chosen to make the upwind scheme closer to the central scheme, without loss of stability, see [14].

It remains to determine $\sigma^n_{0,k}$ for all $k \in \{1, \dots, N\}$. If λ_k is positive, we have the imposed boundary condition at the edge of the cell and $f^n_{0,k}$ at the center, so that we can consider that the slope $\sigma^n_{0,k}$ can be defined as

$$\sigma^n_{0,k} = \mathrm{minmod}\left(X_{1,k,0} \frac{f^n_{1,k} - f^n_{0,k}}{\Delta x}, 2X_{2,k,0} \frac{f^n_{0,k} - M_k(u^n_b)}{\Delta x} \right). \tag{40}$$

For the others characteristics, we have only the values inside the spatial domain, that is $f^n_{1,k} - f^n_{0,k}$, so that we propose the following slope

$$\sigma^n_{0,k} = X_{1,k,0}(f^n_{1,k} - f^n_{0,k})$$

Proposition 5.1 *Suppose that the condition (20) is satisfied. If for $\lambda_k > 0$:*

$$0 \le X_{2,k,0} \le \frac{1}{\xi_k} \quad and \quad for \ \ all \ \ i \ge 0:$$
$$0 \le X_{1,k,i} \le \frac{2}{1-\xi_k}\ , 0 \le X_{2,k,i+1} \le \frac{2}{\xi_k}\ ,$$

and if for $\lambda_k < 0$:

$$for \ \ all \ \ i \ge 0 : 0 \le X_{1,k,i} \le \tfrac{2}{\xi_k}\ , 0 \le X_{2,k,i+1} \le \frac{2}{1-\xi_k},$$

then for any i and k:

$$D^n_{i+\frac{1}{2},k} \in [0,1].$$

Proof. By classical arguments the result is true for $i \ge 1$ and all k, see [14]. Let us study the cells near the boundary.

1) $\lambda_k < 0, i = 0$.
   In this case, the Harten's convex coefficients are :
$$D^n_{\frac{1}{2},k} = \xi_k \left[1 - \frac{(1-\xi_k)}{2}\left(\text{minmod}(X_{1,k,1}\frac{f^n_{2,k} - f^n_{1,k}}{f^n_{1,k} - f^n_{0,k}}, X_{2,k,1}) - X_{1,k,0}\right)\right]$$
   If $X_{1,k,0} \le \frac{2}{\xi_k}$ we have $D^n_{\frac{1}{2},k} \le 1$, and if $X_{2,k,1} \le \frac{2}{(1-\xi_k)}$, $D^n_{\frac{1}{2},k}$ is greater or equal to 0.
2) $\lambda_k > 0, i = -1$.
   As $\sigma^n_{-1,k} = 0$ and $\sigma^n_{0,k}$ is defined by (40) we have :
$$D^n_{-\frac{1}{2},k} = \xi_k \left[1 + \frac{(1-\xi_k)}{2}\left(\text{minmod}(\frac{X_{1,k,0}(f^n_{1,k} - f^n_{0,k})}{f^n_0 - M_k(u^n_b)}, 2X_{2,k,0})\right)\right]$$
   and obviously $D^n_{-\frac{1}{2},k}$ is positive, while $D^n_{-\frac{1}{2},k} \le 1$ if $X_{2,k,0} \le \frac{1}{\xi_k}$.
3) $\lambda_k > 0, i = 0$.
   Using similar arguments we obtain that $D^n_{\frac{1}{2},k} \in [0,1]$ if
$$0 \le X_{1,k,0} \le \frac{2}{1-\xi_k} \quad \text{and} \quad 0 \le X_{2,k,1} \le \frac{2}{\xi_k}\ .$$

□

Consequently, we have convergence to weak solutions, and by Theorem 3.2:

Theorem 5.1 *Suppose that hypothesis (H1)-(H2) are satisfied together with the conditions of Proposition 5.1. Let $T > 0$ be fixed. Consider a sequence $\Delta x_j \to 0$, $(j \to +\infty)$ with $\frac{\Delta t_j}{\Delta x_j}$ kept constant.*

There exists a subsequence of $(u_{\Delta_j})_{j \ge 0}$ which converges in $L^\infty(0,T; L^1_{loc}(\mathbb{R}^+))$ to a solution u of (1)(2), and $u \in C^0([0,T], L^1_{loc}(\mathbb{R}^+)) \cap L^\infty(0,T; L^\infty(\mathbb{R}^+))$.

6 Numerical experiments

6.1 The scalar case

First we study the equation (1) in the scalar case with

$$F(u) = \frac{u^3}{3}$$

which admits sonic points. From the work of Z. Xin and W.C. Wang [46], it results that stability and uniqueness were obtained under the strong constraint that the data must be a small perturbation of a constant nontransonic state. Here we show first that numericaly, the scheme associated to a boundary condition of [46] still gives significant results. Moreover we show that it is possible to provide, thanks to our new formulation, a three velocities scheme that enhances the precision of the solution, even using first order schemes.

In order to test first and second order schemes we take an oscillating boundary condition, while the initial condition is a constant:

$$\begin{cases} u_b(t) = \sin(6t) & t \in \mathbb{R}^+, \\ u^0(x) = 0 & x \in \mathbb{R}^+, \end{cases}$$

Actually, the exact solution can not be obtained easily. We can also remark that the sonic point is frequently crossed.

6.1.1 Kinetic approximations

- **Two velocities model**
  We approximate (1) by the relaxation system (12) which reads

  $$\partial_t f_\pm \pm \lambda \partial_x f_\pm = \frac{1}{\epsilon}(M_\pm(u) - f_\pm)$$

  with

  $$M_\pm(u) = \frac{1}{2}(u \pm \frac{F(u)}{\lambda}).$$

  This model discretized at the first order gives a Lax Friedrichs type scheme. The Maxwellian functions are MND if and only if the subcharacteristic condition is satisfied:

  $$\lambda > \sup_{u \in I} |F'(u)|. \tag{41}$$

  In fact, we reset the value of λ at each time step, by taking the maximum of $|F'(u^n_\Delta)|$ on the whole grid.

As explained in the introduction, there is a freedom degree while imposing the boundary condition that reads

$$f_{b,+}^n = M_+(u_b^n) - \alpha(f_{0,-}^n - M_-(u_b^n)).$$

We show tests for two particular values of α. For $\alpha = 0$ we have the equilibrium on f_+ that was studied above (called NT on the figures). The boundary condition studied in [46, 10] is obtained for $\alpha = 1$ (called XW). All numerical tests show that schemes for those two values provide different approximation for a fixed discretization step Δx. However, they both tend to the same solution as Δx goes to 0.

- **Three velocities model**
  In order to improve the approximation, we now use the following kinetic system, already proposed in [2]:

$$\begin{cases} \partial_t f_+^\epsilon + \lambda \partial_x f_+^\epsilon = \frac{1}{\epsilon}(M_+(u) - f_+^\epsilon) \\ \\ \partial_t f_0^\epsilon \qquad\qquad\quad = \frac{1}{\epsilon}(M_0(u) - f_0^\epsilon) \\ \\ \partial_t f_-^\epsilon - \lambda \partial_x f_-^\epsilon = \frac{1}{\epsilon}(M_-(u) - f_-^\epsilon) \end{cases}$$

with $\lambda > 0$ and

$$\begin{cases} M_+(u) = (\int_0^u [F'(s)]_+ \, ds)/\lambda, \\ \\ M_-(u) = -(\int_0^u [F'(s)]_- \, ds)/\lambda, \\ \\ M_0(u) = u - (\int_0^u |F'(s)| ds)/\lambda, \end{cases}$$

where $[\cdot]_\pm$ is the positive (resp. negative) part. We impose at the boundary only :

$$f_+(t, 0) = M_+(u_b(t)).$$

Here, M is MND if and only if the condition (41) is satisfied, as for the relaxation model.
At first order, this model gives an Engquist-Osher type scheme. On the pictures this scheme is called $3V_eq$. Note that as $F'(u) = u^2$, $M_-(u) = 0$ and the scheme is purely upwind.

6.1.2 First order schemes and results We display the different schemes (NT, XW and the three velocity model $3V_eq$) on grids with 200 and 500 points in Figure 1. In Figure 2 we increase the number of points up to 2000. One can observe that the convergence rate for relaxation schemes (XW and NT) is much slower than the three velocity one ($3V_eq$). This is due to the better upwinding contained in the three velocities model.

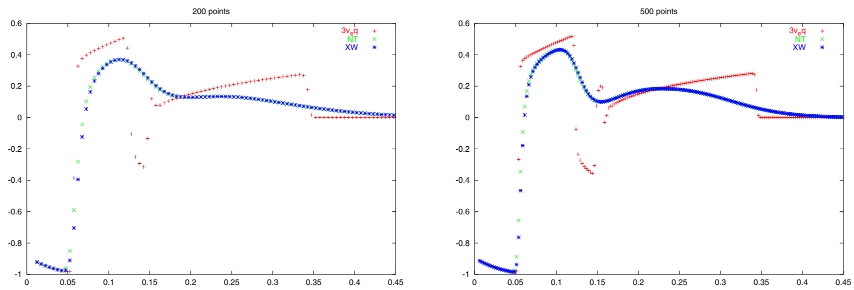

Fig. 1. First order schemes for the cubic flux: 200 & 500 points, $t = 4$, $CFL = 0.7$

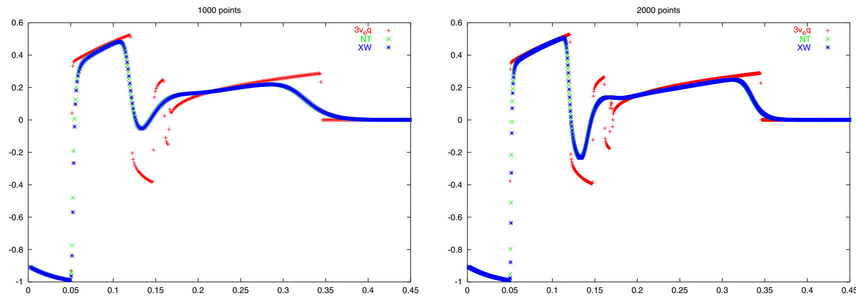

Fig. 2. First order schemes: 1000 & 2000 points, $t = 4$, $CFL = 0.7$

6.1.3 Second order schemes and slope limiters For the same data, we present numerical results for the second order schemes. We use two types of slope limiters. The first one is the well-known min mod limiter that reads

$$\sigma(a, b) = \min\operatorname{mod}(a, b) = \frac{\operatorname{sgn}(a) + \operatorname{sgn}(b)}{2} \min(|a|, |b|)$$

while the second is optimized to reduce the numerical dissipation of the transport scheme, and can be written :

	$\sigma(a, b)$
$\lambda_k > 0$	$2 \min\operatorname{mod}\left(\frac{a}{(1-\xi)}, \frac{b}{\xi}\right)$
$\lambda_k < 0$	$2 \min\operatorname{mod}\left(\frac{a}{\xi}, \frac{b}{(1-\xi)}\right)$

where $\xi = \frac{\lambda \Delta t}{\Delta x}$. This limiter was introduced by F. Lagoutière and B. Després in [16], and gives great precision as we can see on Figure 3 below. We notice that a fast oscillating structure appears for $x \in [0.1, 0.2]$ around the transonic state $u^* = 0$, it is a compression effect of slow waves by waves of greater amplitude. We can also remark that the structure becomes more and more complicated as Δx goes to 0.

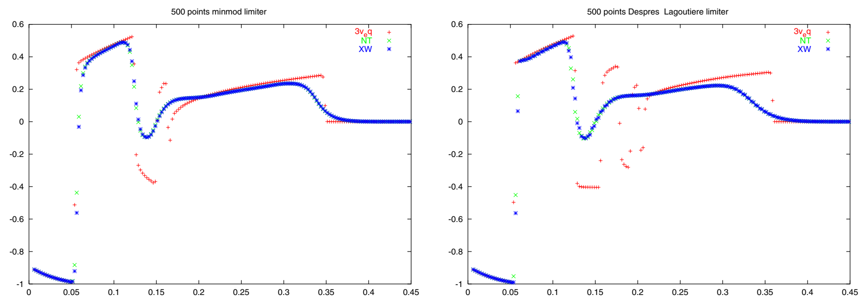

Fig. 3. Second order schemes for the cubic flux: 500 points, $t = 4$, $CFL = 0.7$

6.2 Euler's equations

We approximate a weak solution u of Euler's system of gas dynamics, in one and two dimensions. This system has the abstract form

$$\begin{cases} \partial_t u + \sum_{d=1}^{D} \partial_{x_d} F_d(u) = 0 \; \forall (t,x) \in [0,T] \times \Omega \\ u(0,x) = u^0(x) \qquad\qquad\qquad \forall x \in \Omega \end{cases}$$

and we would like to impose the following boundary condition where it is possible :

$$u(t,x) = u_b(t,x) \;\; \forall x \in \partial\Omega, t \geq 0$$

$u(t,x) \in \mathbb{R}^p$, and $p = 3$ if $x \in \mathbb{R}$, $p = 4$ if $x \in \mathbb{R}^2$. The system is hyperbolic (symmetrizable) and the fluxes F_d are smooth functions. In [2], the kinetic models are presented in a multidimensional version to which we add here a boundary condition:

$$\begin{cases} \partial_t f_k + \overrightarrow{\lambda_k} \cdot \overrightarrow{\nabla} f_k = \frac{1}{\epsilon}(M_k(u) - f_k) \; \forall (t,x) \in [0,T] \times \Omega; \quad k \in \{1,\dots,N\} \\ f_k(0,x) = f_k^0(x) \qquad\qquad \forall x \in \Omega \\ f_k(t,x) = M_k(u_b(t,x)) \qquad \forall x \in \partial\Omega, t \geq 0 \quad \forall k \text{ s.t. } \overrightarrow{\nu} \cdot \overrightarrow{\lambda_k} < 0 \end{cases}$$

Here $\overrightarrow{\nu}$ is the outgoing normal vector, defined for every (t,x) in $[0,T] \times \partial\Omega$, while $\overrightarrow{\lambda_k}$ are vectors of $\mathbb{R}^D$, and M_k are Lipschitz continuous functions defined on $\mathbb{R}^p$ with values on $\mathbb{R}^p$. The compatibility conditions (10) are extended as: for $u \in I$, some fixed rectangle of $\mathbb{R}^p$:

$$\begin{cases} \sum_k M_k(u) = u, \\ \sum_k (\lambda_k)_d M_k(u) = F_d(u) \; d = \{1,\dots,D\}. \end{cases}$$

As in the one-dimensional case, we impose that M is MND in the sense of Definition 1.1.

6.2.1 Euler's equations in one space dimension We consider the one dimensional Euler system, on the bounded domain [0, 1]:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 & \forall x \in [0,1], \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p) = 0, \\ \partial_t E + \partial_x((E+p)v) = 0 \end{cases} \tag{42}$$

where ρ, v, p and E are respectively the density, velocity, pressure and total energy of a perfect gas:

$$p = (\gamma - 1)(E - \frac{1}{2}\rho v^2)\ . \tag{43}$$

with $\gamma = 1.4$.

$$u = \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix} \text{ and } F(u) = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ (E+p)v \end{pmatrix}.$$

We use the relaxation model (12), that is the two velocities model described in Subsection 6.1.1. Remark that the model is formally the same for a scalar equation or for a system. The only difference is that functions $f_\pm$ and $M_\pm$ take now values in $\mathbb{R}^3$ instead of being scalar. The Maxwellian functions $M_\pm$ are MND if

$$\sigma(M'_\pm) > 0$$

wich reads on λ

$$\lambda > \max_{u \in I} |\sigma(F'(u))|. \tag{44}$$

We have tested our schemes with the two interacting blast waves benchmark. The initial data is built with three constant states of a gamma-law gas. The domain is bounded by two walls separated by a distance of unity. The density is uniform, while the pressure is :

$$p = \begin{cases} 1000, & x < \frac{1}{10}, \\ \\ 0.01, & x \in [\frac{1}{10}, \frac{9}{10}], \\ \\ 100, & x > \frac{9}{10}\ . \end{cases}$$

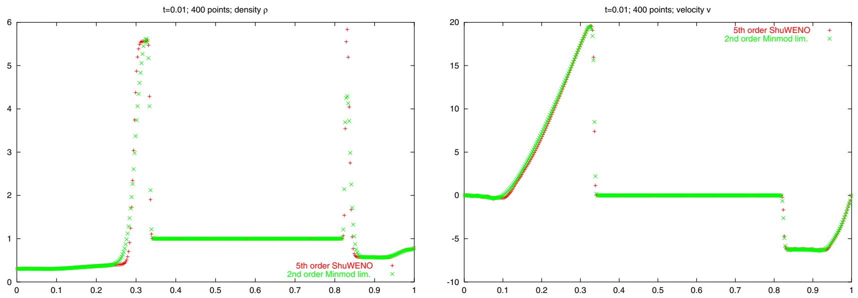

Fig. 4. Density ρ and velocity u at t=0.01

To explain how we modelize reflecting walls with the kinetic model, let u_0^n be the vector value in the first cell of the grid. On the entering characteristics we apply at $x = 0$ for example

$$f_{+,b}^n = M_+ \begin{pmatrix} \rho_0^n \\ -\rho_0^n v_0^n \\ E_0^n \end{pmatrix}.$$

as at $x = 1$, we impose

$$f_{-,b}^n = M_- \begin{pmatrix} \rho_J^n \\ -\rho_J^n v_J^n \\ E_J^n \end{pmatrix}.$$

where J is the last cell of the grid. In fact, we inject a reflected state inside the domain.

The grid is uniform, and contains 400 points, the CFL number is set to 0.35. We use the two velocities model (that is again $M_\pm(u) = \frac{1}{2}(u \pm \frac{F(u)}{\lambda})$). To discretize the transport port, we use a fifth order WENO scheme (S.W. Shu, personal communication), that we compare with a second order one that uses standard minmod limiters. Both schemes are integrated in time up to the third order, and the MND condition (44) yields only locally inside the dependence domain, (if we want to compute u_i^{n+1} we satisfy (44) only in a neighbourhood of x_i).

In the regions of huge variation the fifth order scheme brings more precision. Especially, the collapse of the two sets of waves (see fig 4) is described with more accuracy. Both solution are relevant.

6.3 A Mach 3 Wind Tunel With a Step

This benchmark used in several papers (see [48]), begins with uniform Mach 3 flow in a wind tunnel containing a step. The wind tunnel is 1 length units

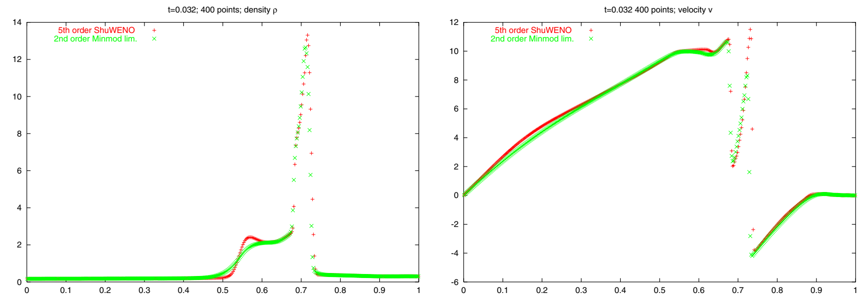

Fig. 5. ρ and u at t=0.032

wide and 3 lengths units long. The step is 0.2 length units high and is located 0.6 length units long. At the left, we imposed a supersonic inflow boundary condition, while at the right side we imposed an outgoing condition. Initially the wind tunnel is filled with a gamma-law gas, with $\gamma = 1.4$, which everywhere has

$$\begin{cases} \rho = 1.4 \\ u = 3.0 \\ v = 0.0 \\ E = \frac{1}{2}\rho(u^2 + v^2) + \frac{p}{\gamma - 1} \quad \text{with } p = 1.0 \end{cases}$$

On the boundary we impose reflecting boundary condition , that is

$$\overrightarrow{\nu} \cdot \overrightarrow{u} = 0$$

where $\overrightarrow{\nu}$ is the outgoing normal vector for points of the boundary. We let evolve the other components, as in the 1D case. To solve the 2D Euler equations we use the following 4 velocities model

– As velocity vectors we use

$$\overrightarrow{\lambda_1} = \lambda_x \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{\lambda_2} = \lambda_y \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\overrightarrow{\lambda_3} = \lambda_x \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{\lambda_4} = \lambda_y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

– while the corresponding Maxwellian functions are

$$M_1(u) = \tfrac{1}{4}[u + \tfrac{2F}{\lambda_x}]$$

$$M_2(u) = \tfrac{1}{4}[u - \tfrac{2G}{\lambda_y}]$$

$$M_3(u) = \tfrac{1}{4}[u - \tfrac{2F}{\lambda_x}]$$

$$M_4(u) = \tfrac{1}{4}[u + \tfrac{2G}{\lambda_y}]$$

F and G are the x (resp. y) fluxes given by the Euler system.

The stability condition reads

$$\lambda_x > \sup_{u\in I} |\sigma(F'(u))|$$

$$\lambda_y > \sup_{u\in I} |\sigma(G'(u))|$$

where $\sigma(\cdot)$ is the spectrum of the matrices.

The presented results are obtained with a second order approximation in space and first order in time. We can see that a strong shock appears on the bottom boundary, but its effects on the solution are rather small. The approximation of the strong shock reflecting on the walls is quite efficient.

6.4 Double Mach Reflection of a Strong Shock

This test problem describes the reflection of a planar Mach shock in air, hitting a wedge. The setup is a Mach 10 shock which initially makes 60° angle with the reflecting wall. Ahead of the shock the undisturbed air has density 1.4 and pressure 1.0. The computational domain is $[0, 4] \times [0, 1]$ and the reflecting wall lies at the bottom of the domain starting at $x = 1/6$. See [48] for a more detailed description of the setup.

The kinetic model is a generalization of the previous one:

$$\lambda_{jd} = \delta_{jd}\lambda_{md}, \quad \lambda_{2+j,d} = \delta_{jd}\lambda_{pd}, \quad j = 1, 2, \quad d = 1, 2,$$

where δ_{jd} is the Kroenecker symbol and $\lambda_{md} < \lambda_{pd}$. The Maxwellian functions are:

$$\begin{cases} M_d(u) &= \dfrac{1}{\lambda_{pd} - \lambda_{md}} \left(\dfrac{\lambda_{pd}}{2} u - F_d(u) \right), \\ M_{2+d}(u) &= \dfrac{1}{\lambda_{pd} - \lambda_{md}} \left(-\dfrac{\lambda_{md}}{2} u + F_d(u) \right). \end{cases} \quad d = 1, 2.$$

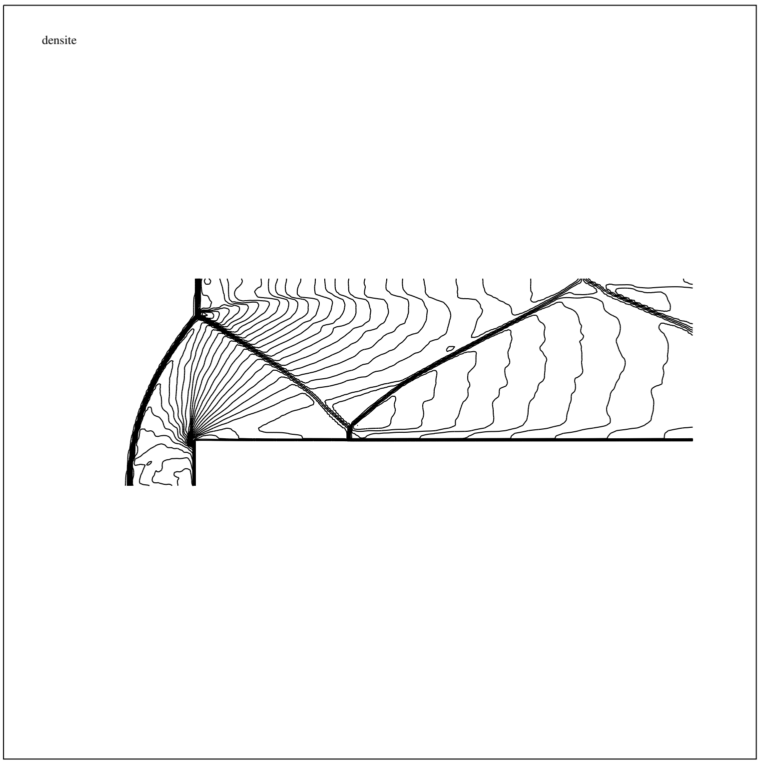

Fig. 6. Density with Despres & Lagoutiere limiters for CFL=0.2, at $t = 3.5$, and $\Delta x = \Delta y = \frac{1}{80}$

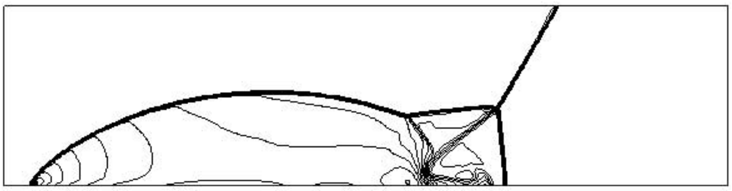

Fig. 7. Density at $t = 0.2$

If $\lambda_{md} = -\lambda_{pd}$ we recover the previous model. In general the function M is monotone if, for all $u \in I$,

$$\lambda_{md} \le a_{ld}(u) \le \lambda_{pd}, \quad d = 1, 2, \quad l = 1, 4$$

where the $a_{ld}(u)$ are the eigenvalues of $F'_d(u)$.

Figure 7 shows the density at time $t = 0.2$, for a CFL number 0.3 and for $\Delta x = \Delta y = 1/120$. The solution is displayed with 30 isolines, and is obtained with a second order scheme in time as in space. This scheme uses Van Leer's limiters for the space second order approximation.

The double Mach reflection and the produced jet are clearly discernible on the figure. The weak shock generated at the kink in the main reflected shock and the contact discontinuity emerging from the three-shock interaction are fairly broad.

Acknowledgements. We would like to thank Francesca Guarguaglini, Roberto Natalini, and Andrea Terracina for usefull discussions, and interesting advices. The research was partially supported by TMR hyperbolic project HCL # ERBFMRXCT960033.

References

[1] Amadori, D.: Initial-boundary value problems for nonlinear systems of conservation laws. Nonlinear Differ. Equ. Appl. **4**, 1–42 (1997)
[2] Aregba-Driollet, D., Natalini, R.: Discrete kinetic schemes for multidimensional conservation laws. SIAM J. Num. Anal. **37**(6), 1973–2004 (2000)
[3] Bardos, C., le Roux, A. Y., Nédélec, J.-C.: First order quasi-linear equations with boundary conditions. Comm. Partial Differential Equations. **4**(9) 1017–1034 (1979)
[4] Benabdallah, A.: The p-system on an interval. C. R. Acad. Sci. Paris Sér. I Math. **303**(4), 123–126 (1986)
[5] Benabdallah, A., Serre, D.: Problèmes aux limites pour des systèmes hyperboliques non linéaires de deux équations à une dimension d'espace. C. R. Acad. Sci. Paris Sér. I Math.. **305**(15), 677–680 (1987)
[6] Benharbit, S., Chalabi, A., Vila, J.-P.: Numerical viscosity and convergence of finite volume methods for conservation laws with boundary conditions. SIAM J. Numer. Anal. **32**(3), 775–796 (1995)
[7] Berthelin, F., Bouchut, F.: Weak entropy boundary conditions for isentropic gas dynamics via kinetic relaxation. J. Diff. Eq. To appear, 2002
[8] Bouchut, F.: Construction of BGK models with a family of kinetic entropies for a given system of conservation laws. J. Statist. Phys. **95**(1-2), 113–170 (1999)
[9] Bouchut, F., Bourdarias, C., Perthame, B.: A MUSCL method satisfying all the numerical entropy inequalities. Math. Comp. **65**(216), 1439–1461 (1996)
[10] Chalabi, A., Seghir, D.: Convergence of relaxation schemes for initial boundary value problems for conservation laws Preprint 1999
[11] Collet, J.F., Rascle, M.: Convergence of the relaxation approximation to a scalar nonlinear hyperbolic equation arising in chromatography. Z. Angew. Math. Phys. **47**, 400–409 (1996)
[12] Cockburn, B., Coquel, F., LeFloch, P.G.: Convergence of the finite volume method for multidimensional conservation laws. SIAM J. Numer. Anal. **32**(3), 687–705 (1995)
[13] Crandall, M., Majda, A.: Monotone difference approximations for scalar conservation laws. Math. of Comp . **34**, 1–21 (1980)
[14] Desprès et Frédéric Lagoutière, B: Un schéma non linéaire anti-dissipatif pour l'équation d'advection linéaire C. R. Acad. Sci. Paris Sér. I Math.. **328**(10), 939–944 (1999)

[15] DiPerna, R. J.: Convergence of approximate solutions to conservation laws. Arch. Rational Mech. Anal. **82**(1), 27–70 (1983)

[16] Dubois, F., LeFloch, P.: Boundary conditions for nonlinear hyperbolic systems of conservation laws. J. Differential Equations. **71**(1), 93–122 (1988)

[17] Gisclon, M., Serre, D.: Conditions aux limites pour un système strictement hyperbolique fournies par le schéma de Godunov". RAIRO Modélisation Mathématique et Analyse Numérique. **31**(3), 359–380 (1997)

[18] Goodman, J.B.: Initial-boundary value problems for hyperbolic systems of conservation laws. Thesis, Stanford University, 1981

[19] Gustafsson, B., Kreiss, H.-O., Sundström, A.: Stability theory of difference approximations for mixed initial boundary value problems. II. Math. Comp. **26**, 649–686 (1972)

[20] Hanouzet, B., Natalini, R.: Weakly coupled systems of quasilinear hyperbolic equations, Differential Integral Equations **9**(6), 1279–1292 (1996)

[21] Harten, A.: On a class of high resolution total-variation-stable finite difference schemes, SIAM J.Num.Anal. **21**, 1–23 (1984)

[22] Harten, A., Lax, P.D., van Leer, B.: On upstream differencing and Godunov–type schemes for hyperbolic conservation laws. SIAM Rev. **25**, 35–61 (1983)

[23] James, F.: Convergence results for some conservation laws with a reflux boundary condition and a relaxation term arising in chemical engineering. SIAM J. Math. Anal., 1998. in press.

[24] Jin, S., Xin, Z.: The relaxation schemes for systems of conservation laws in arbitrary space dimensions. Comm. Pure Appl. Math.. **48**, 235–277 (1995)

[25] Kan, P.T., Santos, M.M., Xin, Z.: Initial-boundary value problem for conservation laws. Comm. Math. Phys. **186**(3), 701–730 (1997)

[26] Kreiss, H.-O.: Initial-boundary value problems for hyperbolic systems. Comm. Pure Appl. Math. **23**, 277–298 (1970)

[27] Kružkov, S.N.: First order quasilinear equations in several independent variables. Math. USSR Sb. **10**, 217–243 (1970)

[28] Le Roux, A.-Y.: étude du problème mixte pour une équation quasi-linéaire du premier ordre. C. R. Acad. Sci. Paris. Sér. A-B **285**(5), 351–354 (1977)

[29] Le Roux, A.-Y.: Approximation of initial and boundary value problems for quasilinear first order equations. Computational mathematics (Warsaw, 1980), 21–31, (1984)

[30] Liu, T.-P.: Hyperbolic conservation laws with relaxation. Comm. Math. Phys. **108**, 153–175 (1987)

[31] Liu, T.-P.: Initial-boundary value problems for gaz dynamics. Arch. Rational Mech. Anal. **64**(2), 137–168 (1977)

[32] Liu, H., Yong, W.-A.: Time-asymptotic stability of boundary-layers for a hyperbolic relaxation system. Comm. Partial Differential Equations **26**(7-8), 1323–1343 (2001)

[33] Málek et al, J.: Weak and measure-valued solutions to evolutionary PDEs. Chapman & Hall, London, 1996

[34] Milišić, V.: Stability and convergence of discrete kinetic approximations to an initial-boundary value problem for conservation laws. Proceeding of the Amer. Math. Soc. to appear

[35] Natalini, R.: Convergence to equilibrium for the relaxation approximations of conservation laws. Comm. Pure Appl. Math. **49**(8), 795–823 (1996)

[36] Natalini, R.: A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws. J. Differential Equations. **148**(2), 292–317 (1998)

[37] Natalini, R., Terracina, A.: Convergence of a relaxation approximation to a boundary value problem for conservation laws. Comm. Partial Differential Equations **26**(7-8), 1235–1252 (2001)

[38] Nishibata, S.: The initial-boundary value problems for hyperbolic conservation laws with relaxation. J. Differential Equations. **130**(1), 100–126 (1996)

[39] Nouri, A., Omrane, A., Vila, J. P.: Boundary conditions for scalar conservation laws from a kinetic point of view. J. Statist. Phys., **94**(5-6), 779–804 (1999)

[40] Otto, F.: Initial-boundary value for a scalar conservation law. C. R. Acad. Sci. Paris Sér. I Math. **322**(8), 729–734 (1996)

[41] Perthame, B.: Boltzmann type schemes for gas dynamics and the entropy property. SIAM J. Numer. Anal., **27**(6), 1405–1421 (1990)

[42] Sanders, R.H., Prendergast, K.H.: On the origin of the 3-kiloparsec arm. Ap. J. **188**, 489–500 (1974)

[43] Serre, D.: Relaxation semi-linéaire et cinétique des systèmes de lois de conservation. Ann. Inst. H. Poincaré Anal. Non Linéaire. **17**(2), 169–192 (2000)

[44] Szepessy, A.: Convergence of a streamline diffusion finite element method for scalar conservation laws with boundary conditions. RAIRO Modél. Math. Anal. Numér. **25**(6), 749–782 (1991)

[45] Vovelle, J.: Convergence of finite volume monotone schemes for scalar conservation laws on bounded domains. Numer. Math. **90**(3), 563–596 (2002)

[46] Wang, W.C., Xin, Z.: Asymptotic limit of initial-boundary value problems for conservation laws with relaxational extensions. Comm. Pure Appl. Math. **51**(5), 505–535 (1998)

[47] Whitham, J.: Linear and nonlinear waves. Wiley, New York, 1974

[48] Woodward, P., Colella, P.: The numerical simulation of two-dimensional fluid flow with strong shocks. J. Comput. Phys. **54**(1), 115–173 (1984)

[49] Yong, W.A.: Boundary conditions for hyperbolic systems with stiff source terms. Indiana Univ. Math. J. **48**(1), 115–137 (1999)