Analysis of non-overlapping models with a weighted infinite delay

Thierno Mamadou Baldé

Univ Brest, CNRS UMR 6205, Laboratoire de Mathématiques de Bretagne Atlantique 6, Avenue Victor Le Gorgeu, 29200 Brest, France

Vuk Milisic

Univ Brest, CNRS UMR 6205, Laboratoire de Mathématiques de Bretagne Atlantique 6, Avenue Victor Le Gorgeu, 29200 Brest, France

The framework of this article is cell motility modeling. Approximating cells as rigid spheres we take into account for both non-penetration and adhesions forces. Adhesions are modeled as a memory-like microscopic elastic forces. This leads to a delayed and constrained vector valued system of equations. We prove that the solution of these equations converges when ε, the linkages turnover parameter, tends to zero to the a constrained model with friction. We discretize the problem and penalize the constraints to get an unconstrained minimization problem. The well-posedness of the constrained problem is obtained by letting the penalty parameter to tend to zero. Energy estimates *à la* De Giorgi are derived accounting for delay. Thanks to these estimates and the convexity of the constraints, we obtain compactness uniformly with respect to the discretisation step and ε, this is the mathematically involved part of the article. Considering that the characteristic bonds lifetime goes to zero, we recover a friction model comparable to [Venel *et al*, ESAIM, 2011] but under more realistic assumptions on the external load, this part being also one of the challenging aspects of the work.

Keywords: Adhesions, contact models, Volterra equations, optimal conditions, friction.

Mathematics Subject Classification: xxx, xxx

1. Introduction

Cells migration is driven by various extracellular guidance cues which are of chemical or mechanical type. The first kind of response is due to gradient of diffusible cues that are either attractive or repulsive, we call this mechanism *chemotaxis*. The chemotaxis may include bacteria migrating for nutrients [Jen06], lymphocytes responding to chemokines gradients in order to locate sites of immune response [SBTG90]. In [YDMW02], the authors prove that molecules of Family Growth Factor of type 4 and 8 respectively control the attractive and repulsive chemotaxis during the chicken gastrulation. In recent years *durotaxis* (mechanical substrate compliance) has been investigated in many papers. In [EMB22], the elastic properties of the migratory substrate bias single and collective cells migration. The authors proved as well that cells exert higher traction and increase the areas when exposed to stiffer surfaces or stiff gradient and may alter their contractility to withstand the mechanical properties of the migratory substrate. Furthermore the authors of [EMB22] prove that human cancer cells have stronger phenotypes when exposed to stiffer substrate, and collective epithelial cells undergo durotaxis even if the cells taken individually do not necessarily do so. These mechanisms, chemotaxis and durotaxis are are both investigated in [PB22]. There the authors underline the similarity but also the remarkable diversity of cells' response to their local environment.

In order to account for this locality, we model contacts between neighboring cells. When considering the literature related to this field, sweeping processes are the starting point. In his seminal paper [Mor77], Moreau considers a point $q(t)$ in a moving closed and convex set $C(t)$ of a Hilbert space H without external perturbation. The particle stays at rest as long as it happens to lie in the interior of C; and once caught up by the boundary $\partial C(t)$, it can only move in the inward normal direction : it always belongs to $C(t)$. Many other authors have been attempting to either weaken the hypotheses or add some external perturbation into the Moreau's system since. For instance in [CDHV93], in finite dimension, the authors considered the set valued function C as the complement of a convex set. Moreover, the authors introduced a bounded, closed and convex valued multifunction. In [CMM95], the perturbation is supposed to be upper semi-continuous with *linear compact growth*, and C is Hausdorff continuous and satisfies the so-called *interior ball condition*. To weaken the convexity of $C(t)$, Colombo et al. introduce prox-regular sets. A prox-regular set (defined below in a more formal way) can be of any shape (non-convex for instance) but it is possible to project points on it if these are close enough. The authors deal first with an unperturbed problem before adding external perturbations. More recently, Juliette Venel uses similar arguments to deal with non-penetration models in the case of human crowd motion and emergency exits [Ven08]. Pedestrians are idealized as rigid disks whose radii centers are respectively $r_i > 0$ and $q_i \in \mathbb{R}^2$ and the individuals centers are collected in a single vector called global configuration. Venel models crowd's dynamics where individuals do not overlap. She perturbs the model by adding an individualistic (or idealized) velocity (the velocity that individuals aim in the absence of others) represented by Lipschitz bounded function. The actual velocity is then the closest velocity from the idealized one.

Here we model adhesions using a microscopic description of bounds as a continuous deterministic death and birth process. This approach was used in the pioneering work of Oelz and Schmeiser [OS10]. The model is based on the microscopic description of the dynamics and interactions of individual filaments, called the Filament-Based Lamellipodium Model. The adhesion forces inside this model rely on a microscopic description of proteic linkages. The authors in [OS10] derived a formal limit (when the rate of linkages turnover ε is small enough). They end up with a gradient flow model with classical friction terms for adhesion of actin filaments to the substrate and cross-links. Using **minimizing movements** *à la* De Giorgi, they prove that the semi-discretisation in time of the problem converges and provides existence and uniqueness of the limit problem. Since then various attempts were made to make this formal computation rigorous [MO11], [MO16], [MO18],[Mil20]. To simplify the problem, a single adhesion point was considered. Its position is the first unknown of the problem and a population of bonds related to this point is the second one. The equation for the position is a Volterra equation accounting for forces balance between the elastic forces of the linkages and an external load. The population density solves an age-structured problem with a non-local birth term modelling saturation of bonds. This equation depends as well on ε. In [MO16], the authors considered the fully-coupled case (the death-rate of linkages depends on the unknown position). They proved that if the balance between the on-rate of the linkages and the external force is violated then the velocity of the particles blows up as the density vanishes. This blow-up mimics detachment of the binding site from the substrate. In a further step, space-dependence was taken into account as well (see [MO18], [Mil20]). In [Mil20], a delayed harmonic map is considered on the sphere. A complete asymptotic study of a scalar fourth order penalized and delayed problem was achieved recently [MS24], the authors considered limits with respect to ϵ and for large times.

In the present work, we model time dependent positions of several cells. These minimize an energy functional under non-linear overlapping constraints. The energy contains two parts : a delay term representing the adhesive energy and a coercive and strictly convex function representing the energy of the external load. The adhesive terms in the total energy rely on the same memory models presented above. Their presence does not allow straightforward proofs of existence neither provides compactness. This is why we discretize the problem with respect to time and age. This approach leads to delayed minimizing movements in the spirit of [Mil20]. We extend energy estimates provided by classical *minimizing movements* [OS10] to the case with memory. The crucial property enabling this step is the monotonicty of the binding kernels. These estimates and convexity assumptions on the source term (the position dependent *external load*) are used in order to prove compactness. Precisely we prove that the time derivative of the solution is bounded in $L^2(0,T)$ for any $T > 0$. We prove that the discrete minimization scheme is equivalent to a variational inclusion and show that the discrete approximation of the solution converges toward the solution of the continuous problem. We show as well that when ε, the instantaneous turn-over parameter of our model tends to zero then the limit function solves the model investigated in [Ven08] weighted by friction coefficients. Nevertheless, as we only assume coercivity and convexity of the external load, we cannot apply the same techniques as in [Ven08] : while the Lipshitz assumption made on the external load allows for the use of Uzawa's method in [Ven08], this assumption is not made here and we propose a new alternative approach. Indeed in [Ven08] the Lipschitz hypothesis is contradicted even for the simplest quadratic potentials. Instead, here, at each time step, we penalize the discrete constraint and let the penalty parameter to tend to zero. This extends the well-posedness of our discrete constrained problem and applies as well to [Ven08]. Moreover in [Ven08], the Lipschitz feature of the external load guarantees the boundedness of the discrete time derivative of the solution. Here, since we weakened this hypothesis, the arguments of [Ven08] do not apply in the asymptotics with respect to ε (the delay operator is not uniformly bounded with respect to ε). In order to overcome this difficulty, we test the Euler-Lagrange equations against a regular enough test function and transpose the delay operator on it [Mil20].

The paper is organized as follows: in Section 2, we set the framework of the problem. We first remind the notion of non-overlapping introduced in [Ven08], then we define the contact adhesion model and lastly we set some assumptions on the data. Section 3 is devoted to the results of this paper. In this section we prove first the well-posedness of the discrete solution, we then establish a compactness criterion which we use to prove the convergence of our model toward a weighted differential inclusion. All the results are extended on the torus as well. We end section 3 by some numerical simulations.

2. Definition of the model

2.1. *Preliminaries*

Consider N_p particles which we idealize as rigid disks whose centers coordinate in the (x, y)-axis and radii are $q_i := (q_i^x, q_i^y)$ and $r_i > 0$, $i = 1, \cdots, N_p$ respectively. We identify the ith particle (q_i, r_i). The global configuration of all particles is given by

$$\boldsymbol{q} := \left(q_1, q_2, \cdots, q_{N_p}\right) \in \mathbb{R}^{2N_p}. \tag{2.1}$$

For $i < j$, we define $D_{ij}(\boldsymbol{q})$ the signed distance between (q_i, r_i) and (q_j, r_j) by

$$D_{ij}(\boldsymbol{q}) := |q_j - q_i| - (r_i + r_j), \tag{2.2}$$

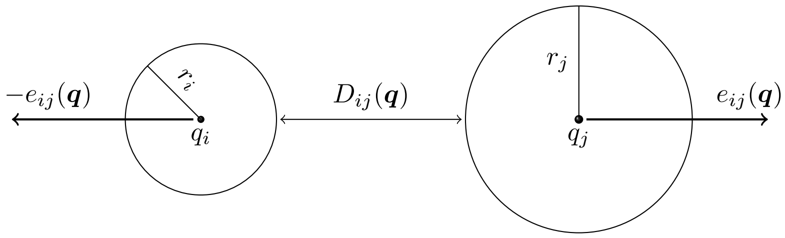


Fig. 1. The signed distance

see Figure 1. Here $|\cdot|$ denotes the Euclidean norm.

Therefore the gradient vector of D_{ij} naturally involves the oriented vector $e_{ij}(\boldsymbol{q})$ in Figure 1 and reads

$$\boldsymbol{G}_{ij}(\boldsymbol{q}) := \nabla D_{ij}(\boldsymbol{q}) = \left(0, \cdots 0, \underset{i}{-e_{i,j}}(\boldsymbol{q}), 0 \cdots 0, \underset{j}{e_{i,j}}(\boldsymbol{q}), 0, \cdots, 0\right), \quad e_{ij}(\boldsymbol{q}) := \frac{q_j - q_i}{|q_j - q_i|}, \quad \forall i < j.$$

The particles should not overlap, so that we define $\boldsymbol{Q}_0$ the set of global configurations for which D_{ij} is nonegative for any distinct particles. Precisely

$$\boldsymbol{Q}_0 := \left\{\boldsymbol{q} \in \mathbb{R}^{2N_p},\, D_{ij}(\boldsymbol{q}) \geq 0,\, \forall i < j\right\}. \tag{2.3}$$

$\boldsymbol{Q}_0$ is called the set of feasible configurations.

2.2. *Definition of the adhesion contact model*

Let $T > 0$ be any time value and ε be a nonnegative parameter. In this article the positions of N_p particles in $\mathbb{R}^2$ at time t are represented by $\boldsymbol{z}_\varepsilon(t) \in \mathbb{R}^{2N_p}$ and solve the minimization problem:

$$\begin{cases} \boldsymbol{z}_\varepsilon(t) = \underset{\boldsymbol{q} \in \boldsymbol{Q}_0}{\arg\min}\, E_t^\varepsilon(\boldsymbol{q}), \quad t \in (0, T], \\ \boldsymbol{z}_\varepsilon(t) = \boldsymbol{z}_p(t), \quad \forall t \leq 0, \end{cases} \tag{2.4}$$

where the energy functional reads

$$E_t^\varepsilon(\boldsymbol{q}) := \frac{1}{2\varepsilon} \sum_{i=1}^{N_p} \int_{\mathbb{R}_+} |q_i - z_{\varepsilon,i}(t - \varepsilon a)|^2 \rho_i(a) da + F(\boldsymbol{q}),$$

$\boldsymbol{z}_p$ represents the positions for negative times and $F : \mathbb{R}^{2N_p} \to \mathbb{R}$ is the energy associated to the external load. The parameter ε represents the maximal lifetime of the linkages (an adimensionalized parameter representing a ratio between a characteristic time divided by a characteristic age of the bonds) and its inverse is assumed to be proportional to the linkages' stiffness.
Furthermore we assume that the linkages density is independent of time and ε and solves an age structured equation. Precisely for any particle, ρ_i solves the following equation

$$\begin{cases} \partial_a \rho_i(a) + (\zeta_i \rho_i)(a) = 0, \quad a > 0, \\ \rho_i(0) = \beta_i \left(1 - \displaystyle\int_0^\infty \rho_i(a) da\right), \end{cases} \tag{2.5}$$

where the linkages' off-rate $\zeta_i : \mathbb{R}_+ \to \mathbb{R}_+$ and the on-rates $\beta_i \in \mathbb{R}_+$ are given constants.
We mention that the non-local term between the parentheses in (2.5) is a saturation term: if the integral is close enough to 0, more births occur while if it is large enough then $\rho_i(0)$ is small. We define the vector density of linkages $\boldsymbol{\rho} \in (\mathbb{R}_+)^{N_p}$, as well as the vector on-rates $\boldsymbol{\beta}$ and off-rates $\boldsymbol{\zeta}$.

2.3. *Main objective*

We aim in this paper at proving that the global configuration $\boldsymbol{z}_\varepsilon$ satisfies

$$\begin{cases} \boldsymbol{\mathcal{L}}_\varepsilon[\boldsymbol{z}_\varepsilon] + \nabla F(\boldsymbol{z}_\varepsilon) \in -N\left(\boldsymbol{K}(\boldsymbol{z}_\varepsilon), \boldsymbol{z}_\varepsilon\right), & \text{a.e. } t \in (0,T], \\ \boldsymbol{z}_\varepsilon(t) = \boldsymbol{z}_p(t), \quad \forall t \leq 0, \end{cases} \tag{2.6}$$

where the delay operator reads

$$\mathcal{L}_{\varepsilon,i}[\boldsymbol{z}_\varepsilon](t) := \frac{1}{\varepsilon}\int_0^\infty \left(z_{\varepsilon,i}(t) - z_{\varepsilon,i}(t-\varepsilon a)\right)\rho_i(a)da, \quad \forall i. \tag{2.7}$$

Moreover we prove that $\boldsymbol{z}_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \boldsymbol{z}_0$ in $C\left([0,T];\mathbb{R}^{2N_p}\right)$ where the limit function $\boldsymbol{z}_0$ solves

$$\begin{cases} \boldsymbol{\mu}_1\partial_t\boldsymbol{z}_0 + \nabla F(\boldsymbol{z}_0) \in -N\left(\boldsymbol{K}(\boldsymbol{z}_0), \boldsymbol{z}_0\right), & \text{a.e. } t \in (0,T], \\ \boldsymbol{z}_0(0) = \boldsymbol{z}_p(0). \end{cases} \tag{2.8}$$

and

$$\boldsymbol{\mu}_1\partial_t\boldsymbol{z}_0 = (\mu_{1,i}\partial_t z_{0,i})_{i=1,\cdots,N_p} \text{ and } \mu_{1,i} := \int_0^\infty \tilde{a}\rho_i(\tilde{a})d\tilde{a} \in \mathbb{R}, \quad \forall i.$$

We mention that $\boldsymbol{K}(\boldsymbol{z}_\varepsilon)$ (respectively $\boldsymbol{K}(\boldsymbol{z}_0)$) is the interior convex approximation of $\boldsymbol{Q}_0$ at $\boldsymbol{z}_\varepsilon$ (respectively at $\boldsymbol{z}_0$) and $N(\boldsymbol{K}(\boldsymbol{z}_\varepsilon), \boldsymbol{z}_\varepsilon)$ (respectively $N(\boldsymbol{K}(\boldsymbol{z}_0), \boldsymbol{z}_0)$) is the proximal-normal cone of $\boldsymbol{K}(\boldsymbol{z}_\varepsilon)$ (respectively $\boldsymbol{K}(\boldsymbol{z}_0)$) at $\boldsymbol{z}_\varepsilon$ (respectively at $\boldsymbol{z}_0$).
We remind that for any closed and nonempty set S of a Hilbert space H and $x \in S$, the proximal-normal cone of S at x (represented in Figure 2) is defined as

$$N(S,x) := \left\{v \in H;\ \exists \alpha > 0 \text{ s.t. } x \in P_S(x+\alpha v)\right\}. \tag{2.9}$$

To reach this main objective we proceed as follows: consider the discrete version of our problem, and prove that it converges to (2.6) by letting the discretization step to go to 0 for fixed ε which in turn converges when ε goes to 0.

2.4. *Notations and assumptions on the data*

2.4.1. *Notations*

For any $T > 0$, we note the following spaces: $\boldsymbol{\mathcal{C}} := \mathcal{C}([0,T];\mathbb{R}^{2N_p})$, $\boldsymbol{H}^1 := H^1([0,T];\mathbb{R}^{2N_p})$, $\boldsymbol{L}^2 := L^2([0,T];\mathbb{R}^{2N_p})$, $\boldsymbol{L}^\infty := L^\infty([0,T];\mathbb{R}^{2N_p})$.

2.4.2. *Assumptions*

(i) *The off-rate* is assumed to be Lipschitz i.e. there exists a constant $L_\zeta > 0$ such that

$$|\boldsymbol{\zeta}(a) - \boldsymbol{\zeta}(b)| \leq L_{\boldsymbol{\zeta}}\,|a-b|\,, \quad \forall a,b \in \mathbb{R}_+.$$

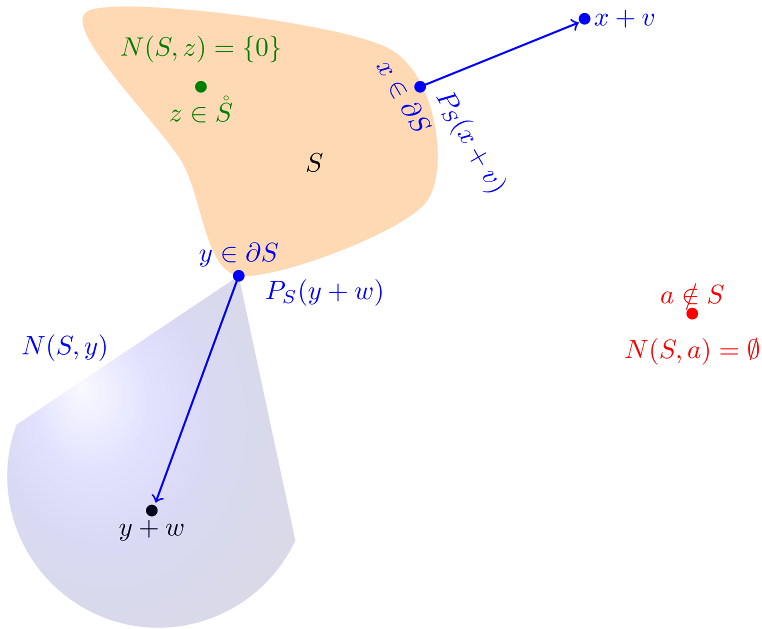


Fig. 2. The proximal-normal cone of S at $z \in \mathring{S}$, $x, y \in \partial S$ and $a \notin S$.

Moreover for any particle there exist $\underline{\zeta_i}$ and $\overline{\zeta_i}$ such that $0 < \underline{\zeta_i} < \zeta_i(a) < \overline{\zeta_i}$. We define $\underline{\zeta} := \min_i \underline{\zeta_i}$ (respectively $\overline{\zeta} := \max_i \overline{\zeta_i}$) as well.

(ii) *The source term* F is coercive (*cf.* Definition A.5), strictly convex and continuous.

(iii) *The past configurations* satisfy $\boldsymbol{z}_p \in Lip\left(\mathbb{R}_-; \boldsymbol{Q}_0\right) : \boldsymbol{z}_p(t) \in \boldsymbol{Q}_0, \forall t \leq 0$ and there exists $C_{\boldsymbol{z}_p} > 0$ such that

$$\left|\boldsymbol{z}_p(t_2) - \boldsymbol{z}_p(t_1)\right| \leq C_{\boldsymbol{z}_p}\left|t_2 - t_1\right|, \quad \forall t_1, t_2 \leq 0.$$

Note as well that in this particular case, the closed form of the linkages density is at hand. Precisely

$$\rho_i(a) = \frac{\beta_i}{1 + \beta_i \int_0^\infty e^{-\int_0^\sigma \zeta_i(\tilde{a})d\tilde{a}} d\sigma} e^{-\int_0^a \zeta_i(\tilde{a})d\tilde{a}}, \quad i = 1, \cdots, N_p. \tag{2.10}$$

And by assumptions 2.4.2 (i), the moments $\mu_{k,i} := \int_0^\infty a^k \rho_i(a)da, k \in \mathbb{N}$ are well defined. Particularly for any particle, there exists $\underline{\mu_{k,i}}, \overline{\mu_{k,i}}$ such that

$$0 < \underline{\mu_{k,i}} \leq \mu_{k,i} \leq \overline{\mu_{k,i}}.$$

2.5. *Time and age discretization and numerical approximations*

The age interval $\mathbb{R}_+$ is divided with constant discretization step Δa such that

$$\mathbb{R}_+ := \bigcup_{l=0}^{\infty} \big[l\Delta a, (l+1)\Delta a\big),$$

as well as the time interval with a discretization grid satisfying $\Delta t = \varepsilon \Delta a$ and $N := \left\lfloor \dfrac{T}{\Delta t} \right\rfloor$ and thus

$$[0,T) = \bigcup_{n=0}^{N-1} \big[n\Delta t, (n+1)\Delta t\big).$$

We set $t^n := n\Delta t$ and $a_l := l\Delta a$ for $n, l \in \{0, 1 \cdots, N\} \times \mathbb{N}$.

We discretize (2.5) using an implicit Euler scheme. This provides $R_{l,i}$ as a function of $R_{l-1,i}$ and reads:

$$R_{l,i} = R_{l-1,i}/\big(1 + \Delta a \zeta_{l,i}\big), \quad (l,i) \in \mathbb{N}^* \times \{1, 2, \cdots, N_p\} \tag{2.11}$$

while on the boundary

$$R_{0,i} = \frac{R_{b,i}}{1 + \frac{\Delta t}{\varepsilon}\zeta_{0,i}}, \quad \forall i \in \{1, 2, \cdots, N_p\} \tag{2.12}$$

For any particle i, the non-local condition relates $R_{b,i}$ to the mean of the density $\mu_{0,\Delta,i}$ as

$$R_{b,i} = \beta_i\big(1 - \Delta a \sum_{l=0}^{\infty} R_{l,i}\big) =: \beta_i(1 - \mu_{0,\Delta,i}). \tag{2.13}$$

By induction over l in (2.11) we have

$$R_{l,i} = \left(\prod_{r=1}^{l} \frac{1}{1 + \Delta a \zeta_{r,i}}\right) R_{0,i}, \quad \forall i \in \{1, 2, \cdots, N_p\},$$

so that we have the following system of two equations with two unknowns ($R_{b,i}$ and $R_{0,i}$) can be set :

$$\begin{cases} R_{b,i} - (1 + \Delta a \zeta_{0,i})\, R_{0,i} = 0 \\ R_{b,i} + \Delta a \beta_i \left(1 + \displaystyle\sum_{l=1}^{\infty} \prod_{r=1}^{l} \frac{1}{1 + \Delta a \zeta_{r,i}}\right) R_{0,i} = \beta_i, \end{cases}$$

which can be solved explicitly giving :

$$\begin{cases} R_{0,i} = \beta_i \left(1 + \Delta a \left(\beta_i + \zeta_{0,i} + \beta_i \displaystyle\sum_{l=1}^{\infty} \prod_{r=1}^{l} \frac{1}{1 + \Delta a \zeta_{r,i}}\right)\right)^{-1}, \\ R_{b,i} = \dfrac{\beta_i(1 + \Delta a \zeta_{0,i})}{1 + \Delta a \Big(\beta_i + \zeta_{0,i} + \beta_i \sum_{l=1}^{\infty} \prod_{r=1}^{l} \dfrac{1}{1 + \Delta a \zeta_{r,i}}\Big)}. \end{cases} \tag{2.14}$$

The discrete version of the minimization process (2.4) is performed

$$\begin{cases} \boldsymbol{Z}^n_\varepsilon = \underset{\boldsymbol{q} \in \boldsymbol{Q}_0}{\arg\min} \left\{ E_{n,\varepsilon}(\boldsymbol{q}) := \dfrac{\Delta a}{2\varepsilon} \displaystyle\sum_{i=1}^{N_p} \sum_{l=1}^{\infty} |q_i - Z^{n-l}_{\varepsilon,i}|^2 R_{l,i} + F(\boldsymbol{q}) \right\}, \quad n = 1, 2, \cdots, N \\ \boldsymbol{Z}^n_\varepsilon = \boldsymbol{Z}^n_p, \quad n \leq 0, \end{cases} \tag{2.15}$$

where the discrete average of positions for negative times is :

$$\boldsymbol{Z}^n_p = \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} \boldsymbol{z}_p(s)\, ds, \quad \forall n \in \mathbb{Z}_-.$$

We define as well

- the piecewise constant approximation functions

$$\boldsymbol{z}_{\varepsilon,\Delta}(t) := \sum_{n=1}^{N} \boldsymbol{Z}_{\varepsilon}^{n} \mathbb{1}_{(t^{n-1},t^{n}]}(t),\ \boldsymbol{z}_{p,\Delta}(t) := \sum_{n=-\infty}^{n=0} \boldsymbol{Z}_{p}^{-n} \mathbb{1}_{(t^{n-1},t^{n}]}(t), \tag{2.16}$$

- the piecewise linear interpolation

$$\tilde{\boldsymbol{z}}_{\varepsilon,\Delta}(t) := \sum_{n=1}^{N} \left\{ Z_{\varepsilon}^{n-1} + \frac{t-t^{n-1}}{\Delta t}(\boldsymbol{Z}_{\varepsilon}^{n} - \boldsymbol{Z}_{\varepsilon}^{n-1}) \right\} \mathbb{1}_{(t^{n-1},t^{n}]}(t), \tag{2.17}$$

- the piecewise linear constant of the linkages density

$$\boldsymbol{\rho}_{\Delta}(a) := \sum_{l=0}^{\infty} \boldsymbol{R}_{l} \mathbb{1}_{(l\Delta a,(l+1)\Delta a)}(a). \tag{2.18}$$

3. Results

We first prove that the piecewise constant approximation of the linkages density converges towards $\boldsymbol{\rho}$ when the age stepsize Δa is small enough.

Proposition 1. *Under the CFL conditions, for any particle, the solution $R_{l,i}$ of* (2.11) *is nonnegative.*

Proof. We perform the proof by induction over $l \in \mathbb{N}$. Indeed

- $l = 0$ since the birth-rate and death-rate are nonnegative, we have that $R_{b,i} \geq 0$ and $R_{0,i}$ for any particle (see (2.14))

- Assume that the claim hold until $l-1$.
- Let us prove that the claim is valid for l. We use the induction hypothesis ($R_{l,i} \geq 0$) and the fact that $\zeta_{l,i}$ is nonnegative in the definition (2.11). □

Lemma 1. *Under the CFL condition $\Delta t = \varepsilon \Delta a$, if linkages' density is defined as in* (2.11)*,*

$$R_{l,i} \geq 0 \Leftrightarrow \mu_{0,\Delta,i} \leq 1, \quad \forall i \in \{1,\dots,N_p\}.$$

Proof. The claim follows from the definition of the first order moment and the fact that the on-rate and the off-rate are nonnegative. Indeed,
$\Rightarrow$) assume that $R_{l,i} \geq 0, \quad \forall (l,i) \in \mathbb{N} \times \{1,2,\cdots,N_p\}$. By (2.12) and (2.13), we have that

$$R_{0,i} = \frac{R_{b,i}}{1+\Delta a \zeta_{0,i}} \geq 0 \implies R_{b,i} =: \beta_i(1-\mu_{0,\Delta,i}) \geq 0, \quad \forall i.$$

We've used the fact that $\zeta_{0,i} \geq 0$ in the latter denominator. The latter inequality gives needed result.
$\Leftarrow$) Assume that $\mu_{0,\Delta,i} \leq 1$. Since $\beta_i \geq 0$ for all i, by (2.13) we have that

$$R_{b,i} = \beta_i(1-\mu_{0,\Delta,i}) \geq 0, \quad \forall i,$$

so that $R_{b,i} \geq 0$ for all particles. This in turn by (2.12) and the fact that the death rate $\zeta_{0,i}$ is nonnegative gives that the initial linkages density $R_{0,i} \geq 0$ for all i. This, by induction over $l \in \mathbb{N}$

into equation (2.11) gives the nonnegative feature of the discrete linkages density. Furthermore note in this case that $\mu_{0,\Delta,i} \geq 0$ for all the particles. □

Define

$$\overline{\boldsymbol{\rho}}_{\Delta}(a) := \sum_{l=0}^{\infty} \overline{\boldsymbol{R}}_l \mathbb{1}_{(l\Delta a,(l+1)\Delta a)}(a) \text{ where } \overline{\boldsymbol{R}}_l = \frac{1}{\Delta a}\int_{l\Delta a}^{(l+1)\Delta a} \boldsymbol{\rho}(a)da$$

where $\boldsymbol{\rho}$ solves (2.5) as well as $\overline{\boldsymbol{\mu}}_{0,\Delta} = \dfrac{1}{\Delta a}\int_{l\Delta a}^{(l+1)\Delta a} \boldsymbol{\mu}_0(a)da$. We have

Lemma 2. *Under the same hypotheses as above if $\boldsymbol{\rho}$ solves* (2.5)*, we have that*

$$|\boldsymbol{\rho}_{\Delta} - \overline{\boldsymbol{\rho}}_{\Delta}|_{L^1_a} \leq O(\Delta a) \text{ and } |\overline{\boldsymbol{\rho}}_{\Delta} - \boldsymbol{\rho}|_{L^1_a} \leq O(\Delta a),$$

where $L^1_a := L^1\left(\mathbb{R}_+, \mathbb{R}^{N_p}\right)$ and $\boldsymbol{\rho}_{\Delta}$ is defined in (2.18).

Proof. Indeed due to the consistency of the scheme (2.11), we have that

$$\begin{aligned}\delta\overline{R}_{l,i} + \Delta a\zeta_{l,i}\overline{R}_{l,i} &= \frac{1}{\Delta a}\int_{l\Delta a}^{(l+1)\Delta a}(1+\zeta_{l,i}\Delta a)e^{-\int_0^{\Delta a}\zeta_i(s)ds}\rho_i(a)da - \frac{1}{\Delta a}\int_{l\Delta a}^{(l+1)\Delta a}\rho_i(a)da \\ &= \frac{1}{\Delta a}\int_{l\Delta}^{(l+1)\Delta a}\left(\Delta a(\zeta_{l,i} - \zeta_i(a)) + O(\Delta a^2)\right)\rho_i(a)da \leq L_{\boldsymbol{\zeta}}||\zeta_i||_{W^{1,\infty}_a}\Delta a^2\overline{R}_{l,i}.\end{aligned}$$

We've used the fact that

$$|\zeta_{l,i} - \zeta_i(a)| \leq \frac{1}{\Delta a}\int_{l\Delta a}^{(l+1)\Delta a}|\zeta_i(\sigma) - \zeta_i(a)|\,d\sigma, \quad \forall a \in (l\Delta a, (l+1)\Delta a), \forall i = 1, \cdots, N_p,$$

so that for any particle

$$\begin{aligned}|\zeta_{l,i} - \zeta_i(a)| &\leq \frac{1}{\Delta a}\int_{l\Delta}^{(l+1)\Delta a}|a-\sigma|\left|\frac{\zeta_i(\sigma)-\zeta_i(a)}{\sigma - a}\right|d\sigma \\ &\leq L_{\boldsymbol{\zeta}}\int_{l\Delta a}^{(l+1)\Delta a}||\partial_a\zeta_i||_{L^\infty_a}\,d\sigma \leq \Delta a\,||\partial_a\zeta_i||_{L^\infty_a}.\end{aligned}$$

On the other hand, setting $E_i := \Delta a\sum_{l=0}^{\infty}(R_{l+1,i} - \overline{R}_{l+1,i})$ for any particle, we have that

$$\begin{aligned}|E_i| = \Delta a\sum_{l=0}^{\infty}\left|\frac{R_{l,i}}{1+\Delta a\zeta_{l+1,i}} - \overline{R}_{l+1,i}\right| &\leq \frac{\Delta a}{1+\Delta a\underline{\zeta}_i}\left(E_i + \sum_{l=0}^{\infty}\left|(1+\Delta a\zeta_{l,i})\overline{R}_{l+1,i} + \overline{R}_{l,i}\right|\right) \\ &\leq \frac{\Delta a E_i}{1+\Delta a\underline{\zeta}_i} + \frac{C}{1+\Delta a\underline{\zeta}_i}\Delta a^2, \quad \forall i,\end{aligned}$$

which gives $|E_i| \leq C\Delta a$, $\forall i \in \{1, 2, \cdots, N_p\}$ implying that $|\boldsymbol{E}| \lesssim C\Delta a$. It follows that

$$\int_0^{\infty}|\boldsymbol{\rho}_{\Delta} - \overline{\boldsymbol{\rho}}_{\Delta}|(a)da \leq \int_0^{\infty}\sum_{l=0}^{\infty}|\boldsymbol{R}_l - \overline{\boldsymbol{R}}_l|\mathbb{1}_{(l\Delta,(l+1)\Delta a)}(a)da \leq C\Delta a,$$

so that $|\boldsymbol{\rho}_{\Delta} - \boldsymbol{\rho}_{\Delta}|_{L^1_a} \leq O(\Delta a)$, which is the first claim. Next

$$\begin{aligned}\int_0^{\infty}|\overline{\boldsymbol{\rho}}_{\boldsymbol{\Delta}}(a) - \boldsymbol{\rho}(a)|\,da &= \int_0^{\infty}\left|\boldsymbol{\rho}(a) - \frac{1}{\Delta a}\sum_{l=0}^{\infty}\Big(\int_{l\Delta a}^{(l+1)\Delta a}\boldsymbol{\rho}(\sigma)d\sigma\Big)\mathbb{1}_{(l\Delta,(l+1)\Delta a)}(a)da\right|da \\ &\leq \frac{1}{\Delta a}\sum_{l=0}^{\infty}\int_0^{\infty}\left|\boldsymbol{\rho}(a) - \int_{l\Delta a}^{(l+1)\Delta a}\boldsymbol{\rho}(\sigma)d\sigma\right|\mathbb{1}_{(l\Delta a,(l+1)\Delta l)}(a)da.\end{aligned}$$

Define the space $U := \left\{ f \in L^1_a \text{ s.t. } \limsup_{\sigma \to 0} \int_0^\infty |\frac{f(a+\sigma) - f(a)}{\sigma}| da < \infty \right\}$ endowed with the norm

$$||f||_U := ||f||_{L^1_a} + \limsup_{\sigma \to 0} \int_0^\infty \left| \frac{f(a+\sigma) - f(a)}{\sigma} \right| da,$$

we have by the Lemma Appendix B.2 p.36 [Mil20] that

$$\int_0^\infty |\overline{\boldsymbol{\rho}}_{\boldsymbol{\Delta}}(a) - \boldsymbol{\rho}(a)| \, da \leq \Delta a \, |\boldsymbol{\rho}|_U \, .$$

Thus, taking Δa small enough, gives the second claim. □

3.1. *Existence and uniqueness of solution of the constrained problem*

Since $\boldsymbol{Q}_0$ is nonconvex (see Figure 4 below), we consider its interior convex approximation $\boldsymbol{K}(\boldsymbol{Z}^{n-1}_\varepsilon)$ defined as follows

$$\boldsymbol{K}(\boldsymbol{Z}^{n-1}_\varepsilon) := \left\{ \boldsymbol{q} \in \mathbb{R}^{2N_p} : \varphi^{n,\varepsilon}_{ij}(\boldsymbol{q}) \leq 0, \; \forall \, i < j \right\}, \tag{3.1}$$

where for any n and ε fixed, the constraints functions $\varphi^{n,\varepsilon}_{ij} : \mathbb{R}^{2N_p} \longrightarrow \mathbb{R}$ are affine and read

$$\varphi^{n,\varepsilon}_{ij}(\boldsymbol{q}) := -D_{ij}(\boldsymbol{Z}^{n-1}_\varepsilon) - \boldsymbol{G}_{ij}(\boldsymbol{Z}^{n-1}_\varepsilon) \cdot (\boldsymbol{q} - \boldsymbol{Z}^{n-1}_\varepsilon), \quad i < j. \tag{3.2}$$

The minimization problem over this convex set reads : find $\boldsymbol{Z}^n_\varepsilon \in \mathbb{R}^{2N_p}$ s.t.

$$\begin{cases} \boldsymbol{Z}^n_\varepsilon = \underset{\boldsymbol{q} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_\varepsilon)}{\arg\min} \; E_{n,\varepsilon}(\boldsymbol{q}), \quad n \geq 1, \\ \boldsymbol{Z}^n_\varepsilon = \boldsymbol{Z}^n_p, \quad n \leq 0. \end{cases} \tag{3.3}$$

Due to Lemma 5 below we have that (2.15) is equivalent to (3.3), so that instead of (2.15), we may deal with (3.3) in the following investigations.

Theorem 1. *Lets fix the integer $n \geq 1$ and assume that $\boldsymbol{Z}^{n-1} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1})$. Moreover suppose that assumptions 2.4.2 (i)-(iii) hold and consider the penalised problem : find $\boldsymbol{Z}^n_{\varepsilon,\delta}$ such that*

$$\begin{cases} \boldsymbol{Z}^n_{\varepsilon,\delta} = \underset{\boldsymbol{q} \in \mathbb{R}^{2N_p}}{\arg\min} \left\{ E^\delta_{n,\varepsilon}(\boldsymbol{q}) := E_{n,\varepsilon}(\boldsymbol{q}) + \dfrac{1}{2\delta} \displaystyle\sum_{i<j} \max\left(\varphi^{n,\varepsilon}_{ij}(\boldsymbol{q}), 0\right)^2 \right\}, \\ \boldsymbol{Z}^n_{\varepsilon,\delta} = \boldsymbol{Z}^n_p, \quad n \leq 0. \end{cases} \tag{3.4}$$

Then there exists a unique $\boldsymbol{Z}^n_{\varepsilon,\delta} \in \mathbb{R}^{2N_p}$ solving the above problem. Moreover when letting the penalty parameter δ to go to 0, $\boldsymbol{Z}^n_{\varepsilon,\delta}$ converges to $\boldsymbol{Z}^n_\varepsilon$ solving (3.3)*. Again, one has that $\boldsymbol{Z}^n_\varepsilon \in \boldsymbol{K}(Z^n_\varepsilon)$. The result is then true for any $n \in \mathbb{N}^*$*

Proof. Thanks to asumption 2.4.2.(iii), one has that $\boldsymbol{Z}^0_\varepsilon \equiv \boldsymbol{z}_p(0)$ is such that $\boldsymbol{Z}^0_\varepsilon \in \boldsymbol{K}(\boldsymbol{Z}^0_\varepsilon)$ which is thus non-empty. We check hereafter the hypotheses of Theorem A.8. Indeed

(1) for $\varepsilon > 0$ and $n \in \mathbb{N}^*$ fixed, $\boldsymbol{q} \mapsto E_{n,\varepsilon}(\boldsymbol{q})$ is continuous, coercive and strictly convex. Indeed, this is by definition since the sum of continuous (respectively coercive, strictly convex) function is continuous (respectively coercive, strictly convex). Let us mention that this ensures the existence and uniqueness of $\boldsymbol{Z}^n_{\varepsilon,\delta}$ solution of (3.4).

(2) Let's define $\boldsymbol{K}(\boldsymbol{p}) := \{\boldsymbol{q} \in \mathbb{R}^{2N_p} \ : \ \varphi_{ij}(\boldsymbol{p},\boldsymbol{q}) \leq 0, \ i<j\}$, where $\varphi_{ij}(\boldsymbol{p},\boldsymbol{q}) := -D_{ij}(\boldsymbol{p}) - \boldsymbol{G}_{ij}(\boldsymbol{p}) \cdot (\boldsymbol{q}-\boldsymbol{p})$. Assume that $\boldsymbol{p} \in \mathbb{R}^{2N_p}$ is s.t. $D_{ij}(\boldsymbol{p}) \geq 0$ for all $i<j$. Then we claim that $\boldsymbol{K}(\boldsymbol{p})$ is a closed convex, non-empty set. Indeed, $\boldsymbol{p} \in \boldsymbol{K}(\boldsymbol{p})$ which implies that it is non-empty. Since $\boldsymbol{q} \mapsto D_{ij}(\boldsymbol{q})$ is convex, it is easy to check that $\boldsymbol{K}(\boldsymbol{p})$ is convex as finite intersection of convex sets. It is closed as finite intersection of closed sets : as

$$\boldsymbol{K}(\boldsymbol{p}) = \bigcap_{i<j} (\varphi_{ij}(\boldsymbol{p},\cdot))^{-1}((-\infty,0]),$$

so that since the maps $\boldsymbol{q} \mapsto \varphi_{ij}(\boldsymbol{p},\boldsymbol{q})$ are continuous and $(-\infty,0]$ is a closed interval, $\boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$ is closed as intersection of reciprocal images of closed subsets by continuous functions. Thus, $\boldsymbol{K}(Z^{n-1}_{\varepsilon})$ is a closed, convex and non empty set since $\boldsymbol{Z}^{n-1}_{\varepsilon} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$.

(3) The map $\psi^{n,\varepsilon} : \mathbb{R}^{2N_p} \longrightarrow \mathbb{R}$ defined by

$$\psi^{n,\varepsilon}(\boldsymbol{q}) := \frac{1}{2}\sum_{i<j} \max\left(\varphi^{n,\varepsilon}_{ij}(\boldsymbol{q}),0\right)^2,$$

satisfies (A.1), namely it is continuous, convex and satisfies

$$\psi^{n,\varepsilon}(\boldsymbol{q}) \geq 0 \text{ for every } \boldsymbol{q} \in \mathbb{R}^{2N_p} \text{ and } \psi^{n,\varepsilon}(\boldsymbol{q}) = 0 \iff \boldsymbol{q} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon}).$$

We prove first the continuity. Indeed for any $n \in \mathbb{N}$ and $\varepsilon > 0$ fixed, the maps $f^{n,\varepsilon}_{ij}(\boldsymbol{q}) := \max(\cdot,0)^2 \circ \varphi^{n,\varepsilon}_{ij}(\boldsymbol{q}), \ i<j$ are continuous as composition of continuous functions, so that $\psi^{n,\varepsilon}(\boldsymbol{q}) := \sum_{i<j} f^{n,\varepsilon}_{ij}(\boldsymbol{q})$ is continuous. For the convexity we use properties of composition and sum of convex functions. Indeed the functions $f^{n,\varepsilon}_{ij}$ are convex as composition of convex functions, so that $\psi^{n,\varepsilon}$ is convex as sum of convex functions. Furthermore, by definition $\psi^{n,\varepsilon}(\boldsymbol{q}) \geq 0, \forall \boldsymbol{q} \in \mathbb{R}^{2N_p}$ and $\psi^{n,\varepsilon}(\boldsymbol{q}) = 0 \iff \boldsymbol{q} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$. Indeed

$$\sum_{i<j} f^{n,\varepsilon}_{ij}(\boldsymbol{q}) = 0 \implies \max\left(\varphi^{n,\varepsilon}_{ij}(\boldsymbol{q}),0\right) = 0, \ \forall i<j \implies \varphi^{n,\varepsilon}_{ij}(\boldsymbol{q}) \leq 0, \quad \forall i<j.$$

Conversely let $\boldsymbol{q} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$, we have

$$\varphi^{n,\varepsilon}_{ij}(\boldsymbol{q}) \leq 0, \ \forall i<j \implies \max(\varphi^{n,\varepsilon}_{ij}(\boldsymbol{q}),0)^2 = 0, \ \forall i<j \implies \sum_{i<j} f^{n,\varepsilon}_{ij}(\boldsymbol{q}) = 0.$$

This shows the claim.

Now having fulfilled all hypotheses of Theorem A.8, we have that the solution $\boldsymbol{Z}^n_{\varepsilon}$ of (3.3) exists as limit of $\boldsymbol{Z}^n_{\varepsilon,\delta}$, the unique solution of (3.4) when δ goes to 0. Since $\boldsymbol{Z}^n_{\varepsilon}$ satisfies the constraint, $\boldsymbol{Z}^n_{\varepsilon} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$ the proof extends to every $n \in \mathbb{N}^*$ by induction. □

3.2. *The constrained problem in term of primal-dual problem*

We aim at proving there exists (in general not a unique) a dual variable called the Lagrange variable such that the *primal* problem (3.3) (whose variable $\boldsymbol{Z}^n_{\varepsilon}$ is called the primal variable) is equivalent to a involving both primal and dual variables : the *primal-dual* problem.

Definition 1. (Feasible direction) Let $\boldsymbol{q} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$ be a feasible configuration and $\boldsymbol{w} \in \mathbb{R}^{2N_p}$, we say that $\boldsymbol{w}$ is a feasible direction if and only if there exists $\eta > 0$ such that for any $0 < s \leq \eta$ we have $\boldsymbol{q} + s\boldsymbol{w} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$.
In other words, $\boldsymbol{q}$ is a feasible direction if from $\boldsymbol{q}$ one can move at least of η by still staying in

$\boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$. In figure 3 we have the possible directions for $\boldsymbol{q}$ strictly interior in the domain on one hand and $\boldsymbol{q}$ on the boundary of the domain on the other hand.

Let $\boldsymbol{q}$, $\tilde{\boldsymbol{q}} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$ such that $\boldsymbol{q} \neq \tilde{\boldsymbol{q}}$. Since $\boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$ is convex, we have $[\boldsymbol{q}, \tilde{\boldsymbol{q}}] \subset \boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$ and $\boldsymbol{w} = \tilde{\boldsymbol{q}} - \boldsymbol{q}$ is a feasible direction.

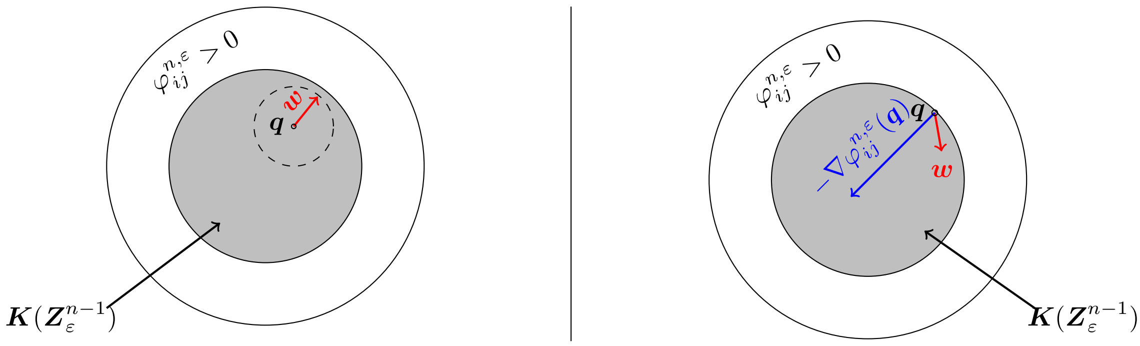


Fig. 3. feasible directions for $\boldsymbol{q}$ strictly interior to $\boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$ (left) vs. $\boldsymbol{q}$ on the boundary (right).

Definition 2. *All05* Let $\boldsymbol{q} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$, for any fixed $\varepsilon > 0$ we define the cone of feasible directions at $\boldsymbol{q}$ by

$$\boldsymbol{C}(\boldsymbol{q}) = \left\{ \boldsymbol{w} \in \mathbb{R}^{2N_p},\, \exists \boldsymbol{q}^r \in \left(\boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})\right)^{\mathbb{N}}, \exists\, \delta^r \in (\mathbb{R}^*_+)^{\mathbb{N}}, \boldsymbol{q}^r \to \boldsymbol{q},\, \delta^r \to 0 \text{ and } \lim_{r\to\infty} \frac{\boldsymbol{q}^r - \boldsymbol{q}}{\delta^r} = \boldsymbol{w} \right\}.$$

Remark 1. $\boldsymbol{C}(\boldsymbol{q})$ is a cone in the sense that $\boldsymbol{0} \in \boldsymbol{C}(\boldsymbol{q})$ (take $\boldsymbol{q}^r = \boldsymbol{q}$ for any r) and if $\boldsymbol{w} \in \boldsymbol{C}(\boldsymbol{q})$ we have that $\lambda \boldsymbol{w} \in \boldsymbol{C}(\boldsymbol{q})$ for any $\lambda > 0$. Moreover we have the followings

- If $\boldsymbol{q}$ is strictly interior to the domain $\boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$, we have that $C(\boldsymbol{q}) = \mathbb{R}^{2N_p}$. It suffices to take $\boldsymbol{q}^r = \boldsymbol{q} + \dfrac{1}{r}\boldsymbol{w}$ for all $\boldsymbol{w} \in \mathbb{R}^{2N_p}$ and r large enough (see figure the left hand side of 2).
- Since $\boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$ is convex $\boldsymbol{C}(\boldsymbol{q}) = \left\{\boldsymbol{w} - \boldsymbol{q} \text{ for all } \boldsymbol{w} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})\right\}$. It suffices to take $\boldsymbol{q}^r = \boldsymbol{q} + \dfrac{1}{r}(\boldsymbol{w} - \boldsymbol{q})$ for all r.

For any $\boldsymbol{q} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$, the cone $\boldsymbol{C}(\boldsymbol{q})$ in Definition 2 can be seen as the set of all vectors which are tangent at $\boldsymbol{q}$ to a curve lying in $\boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$ and passing through $\boldsymbol{q}$. More precisely $\boldsymbol{C}(\boldsymbol{q})$ is the set of all possible directions of variation from $\boldsymbol{q}$ which guarantee that one stays in $\boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$. But the main issue here is the fact that we cannot always handle a closed form of $\boldsymbol{C}(\boldsymbol{q})$. Nevertheless in some specific cases; called the *qualification conditions* one may obtain an explicit form of $\boldsymbol{C}(\boldsymbol{q})$.
For any $\boldsymbol{q} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$, we have that:

- if $\varphi^{n,\varepsilon}_{ij}(\boldsymbol{q}) < 0$, for any direction $\boldsymbol{w} \in \mathbb{R}^{2N_p}$ and $\eta > 0$ small enough, we have that $\varphi^{n,\varepsilon}_{ij}(\boldsymbol{q} + \eta\boldsymbol{w}) \leq 0$ (see Figure 2 on the left hand side). We say that the constraint ij is *nonactive*.
- If $\varphi^{n,\varepsilon}_{ij}(\boldsymbol{q}) = 0$ we want the direction $\boldsymbol{w}$ to satisfy the condition $\varphi^{n,\varepsilon}_{ij}(\boldsymbol{q} + \eta\boldsymbol{w}) \leq 0$ for $i < j$, in order to ensure that all the constraints are satisfied for $\boldsymbol{q} + \eta\boldsymbol{w}$ (see Figure 2 on the right hand side). Such conditions are called *qualification conditions*.

But since the functions $\varphi_{ij}^{n,\varepsilon}$ are affine, for any $\boldsymbol{w} \in \mathbb{R}^{2N_p}$ and $\eta > 0$ we have

$$\varphi_{ij}^{n,\varepsilon}(\boldsymbol{q}) = 0 \implies \varphi_{ij}^{n,\varepsilon}(\boldsymbol{q} + \eta \boldsymbol{w}) = -\eta \boldsymbol{G}_{ij}(\boldsymbol{Z}_{\varepsilon}^{n-1}) \cdot \boldsymbol{w}, \quad \forall i < j.$$

So that if there exists a direction $\overline{\boldsymbol{w}} \in \mathbb{R}^{2N_p}$ such that $\varphi_{ij}^{n,\varepsilon}(\boldsymbol{q} + \eta \overline{\boldsymbol{w}}) \leq 0$, we necessarily have $\boldsymbol{G}_{ij}(\boldsymbol{Z}_{\varepsilon}^{n-1}) \cdot \overline{\boldsymbol{w}} \geq 0$. Such a direction exists : it suffices to take $\overline{\boldsymbol{w}} = \boldsymbol{0}$. We say that the constraints (3.1) are qualified at $\boldsymbol{q}$.

Remark 2. Note that $\boldsymbol{q}$ above is chosen arbitrarily. Moreover $\boldsymbol{Z}_{\varepsilon}^{n}$ belongs to $\boldsymbol{K}(\boldsymbol{Z}_{\varepsilon}^{n-1})$ for any time step so that, the constraints (3.1) are qualified at $\boldsymbol{Z}_{\varepsilon}^{n}$.

Definition 3. [All05] Let $\boldsymbol{q} \in \boldsymbol{K}(\mathbf{Z}_{\varepsilon}^{n-1})$, we define the set of active constraints by

$$Ind(\boldsymbol{q}) := \left\{ 1 \leq i < j \leq N_p : \varphi_{ij}^{n,\varepsilon}(\boldsymbol{q}) = 0 \right\}.$$

$Ind(\boldsymbol{q})$ is also called the set of saturated constraints.

Remark 3. Let $\boldsymbol{q} \in \boldsymbol{K}(\boldsymbol{Z}_{\varepsilon}^{n-1})$. We have that

$$\boldsymbol{C}(\boldsymbol{q}) = \left\{ \boldsymbol{w} \in \mathbb{R}^{2N_p} : \boldsymbol{G}_{ij}(\boldsymbol{Z}_{\varepsilon}^{n-1}) \cdot \boldsymbol{w} \geq 0, \ \forall i,j \in Ind(\boldsymbol{Z}_{\varepsilon}^{n}) \right\}. \tag{3.5}$$

Definition 4. [Cia89] Let V and M be two subsets consider $L : V \times M \longrightarrow \mathbb{R}$.
The couple of points $(u, \lambda) \in V \times M$ is called saddle point of L if u is the minimum of $L(\cdot, \lambda) : v \in V \longmapsto L(v, \lambda) \in \mathbb{R}$ and λ is the maximum of $L(u, \cdot) : \mu \in M \longmapsto L(u, \mu) \in \mathbb{R}$. In other words (u, λ) is a saddle point of L if it satisfies

$$\sup_{\mu \in M} L(u, \mu) = L(u, \lambda) = \inf_{v \in V} L(v, \lambda).$$

From now on $V := \mathbb{R}^{2N_p}$ and $M := (\mathbb{R}_+)^{N_c}$ where $N_c := N_p(N_p - 1)/2$ is the maximal number of contacts. We introduce the Euler-Lagrange equations associated with (3.3) and investigate the existence of optimal points. To this end for $\boldsymbol{\mu} = (\mu_{ij})_{i<j}$, we define the Lagrangian $L : \mathbb{R}^{2N_p} \times \mathbb{R}_+^{N_c} \longrightarrow \mathbb{R}$ by

$$L(\boldsymbol{q}, \boldsymbol{\mu}) = \frac{\Delta a}{2\varepsilon} \sum_{i=1}^{N_p} \sum_{l=1}^{\infty} \left| q_i - Z_{\varepsilon,i}^{n-l} \right|^2 R_{l,i} + F(\boldsymbol{q}) + \sum_{i<j} \mu_{ij} \varphi_{ij}^{n,\varepsilon}(\boldsymbol{q}). \tag{3.6}$$

Since for all n, the mappings E_n and $\varphi_{ij}^{n,\varepsilon}$, $i < j$ are convex, continuous in $\mathbb{R}^{2N_p}$ and differentiable in $\boldsymbol{K}(\boldsymbol{Z}_{\varepsilon}^{n-1})$ and the constraints are qualified at $\boldsymbol{Z}_{\varepsilon}^{n}$, the KKT theorem (cf. Theorem A.9) guarantees that (3.3) is equivalent to the existence of $\boldsymbol{\lambda}_{\varepsilon}^{n} = (\lambda_{ij}^{n,\varepsilon})_{i<j} \in (\mathbb{R}_+)^{N_c}$ such that $(\boldsymbol{Z}_{\varepsilon}^{n}, \boldsymbol{\lambda}_{\varepsilon}^{n})$ is a saddle point of the Lagrangian (3.6) in $\mathbb{R}^{2N_p} \times \mathbb{R}_+^{N_c}$. This can be rephrased as $\boldsymbol{Z}_{\varepsilon}^{n}$ is a solution of (3.3) if and only if there exists $\boldsymbol{\lambda}_{\varepsilon}^{n} = \boldsymbol{\lambda}_{\varepsilon}^{n}(\boldsymbol{Z}_{\varepsilon}^{n})$ such that

$$\boldsymbol{\varphi}^{n,\varepsilon}(\boldsymbol{Z}_{\varepsilon}^{n}) \leq \boldsymbol{0}, \ \boldsymbol{\lambda}_{\varepsilon}^{n}(\boldsymbol{Z}_{\varepsilon}^{n}) \geq \boldsymbol{0}, \ \boldsymbol{\lambda}_{\varepsilon}^{n}(\boldsymbol{Z}_{\varepsilon}^{n}) \cdot \boldsymbol{\varphi}(\boldsymbol{Z}_{\varepsilon}^{n}) = 0; \ \boldsymbol{E}_{n}^{'}(\boldsymbol{Z}_{\varepsilon}^{n}) + \sum_{i<j} \lambda_{ij}^{n,\varepsilon}(\boldsymbol{Z}_{\varepsilon}^{n}) (\varphi_{ij}^{n,\varepsilon})^{'}(\boldsymbol{Z}_{\varepsilon}^{n}) = \boldsymbol{0}, \tag{3.7}$$

where $\boldsymbol{\varphi}_{\varepsilon}^{n}(\boldsymbol{q}) := \left(\varphi_{ij}^{n,\varepsilon} \right)_{i<j} : \mathbb{R}^{2N_p} \longrightarrow \mathbb{R}^{N_c}$ is vectorized form of the constraints functions.

3.3. *Energy estimates and compactness criterion*

Proposition 2. *Under assumptions 2.4.2, if $(\boldsymbol{R}_l)_{l\in\mathbb{N}}$ and $(\boldsymbol{Z}^n_\varepsilon)_{n=1,2\cdots,N}$ are defined as above, there exists a constant K_0 independent either of ε or Δa such that*

$$\frac{\Delta a}{2\varepsilon}\sum_{i=1}^{N_p}\sum_{l=1}^{\infty}\left|Z^n_{\varepsilon,i}-Z^{n-l}_{\varepsilon,i}\right|^2 R_{l,i}+\Delta t\sum_{m=1}^{n}D^m_\varepsilon+F(\boldsymbol{Z}^n_\varepsilon)\leq K_0+F(\boldsymbol{Z}^0_p), \tag{3.8}$$

where the dissipation term reads

$$D^n_\varepsilon:=\frac{\Delta a}{2}\sum_{i=1}^{N_p}\sum_{l=1}^{\infty}|U^{n-1}_{l,\varepsilon,i}|^2 R_{l+1,i}\zeta_{l+1,i},\ \text{and}\ U^n_{l,\varepsilon,i}:=\frac{1}{\varepsilon}(Z^n_{\varepsilon,i}-Z^{n-l}_{\varepsilon,i}),\quad\forall i=1,\cdots,N_p,\ l\in\mathbb{N}^*.$$

Proof. By definition of the minimization process

$$E_{n,\epsilon}(\boldsymbol{Z}^n_\varepsilon)\leq E_{n,\varepsilon}(\boldsymbol{Z}^{n-1}_\varepsilon)=\frac{\Delta a}{2\varepsilon}\sum_{i=1}^{N_p}\sum_{l=2}^{\infty}|Z^{n-1}_{\varepsilon,i}-Z^{n-l}_{\varepsilon,i}|^2R_{l,i}+F(\boldsymbol{Z}^{n-1}_\varepsilon),$$

so that by a change of index,

$$I_{n,\varepsilon}+F(\boldsymbol{Z}^n_\varepsilon)\leq\frac{\Delta a}{2\varepsilon}\sum_{i=1}^{N_p}\sum_{l=1}^{\infty}|Z^{n-1}_{\varepsilon,i}-Z^{n-1-l}_{\varepsilon,i}|^2R_{l+1,i}+F(\boldsymbol{Z}^{n-1}_\varepsilon),$$

where we've set

$$I_{n,\varepsilon}:=\frac{\Delta a}{2\varepsilon}\sum_{i=1}^{N_p}\sum_{l=1}^{\infty}|Z^n_{\varepsilon,i}-Z^{n-l}_{\varepsilon,i}|^2R_{l,i}.$$

Since $R_{l,i}$ solves (2.5), we have that

$$I_{n,\varepsilon}+F(\boldsymbol{Z}^n_\varepsilon)+\frac{\Delta a}{2\varepsilon}\frac{\Delta t}{\varepsilon}\sum_{i=1}^{N_p}\sum_{l=1}^{\infty}|Z^{n-1}_{\varepsilon,i}-Z^{n-1-l}_{\varepsilon,i}|^2R_{l+1,i}\zeta_{l+1,i}\leq I_{n-1,\varepsilon}+F(\boldsymbol{Z}^{n-1}_\varepsilon),$$

so that by induction over n

$$I_{n,\varepsilon}+F(\boldsymbol{Z}^n_\varepsilon)+\frac{\Delta a}{2\varepsilon}\frac{\Delta t}{\varepsilon}\sum_{m=1}^{n}\sum_{i=1}^{N_p}\sum_{l=1}^{\infty}|Z^{n-1}_{\varepsilon,i}-Z^{n-1-l}_{\varepsilon,i}|^2R_{l+1,i}\zeta_{l+1,i}\leq I_{0,p}+F(\boldsymbol{Z}^0_p).$$

Now we need to find an upper bound for $I_{0,p}$. Indeed for any $i\in\{1,2,\cdots,N_p\}$ fixed,

$$\left|Z^0_{\varepsilon,i}-Z^{-l}_{\varepsilon,i}\right|\leq\varepsilon\Delta aC_{z_{p,i}}l,$$

so that

$$I_{0,p}:=\frac{\Delta a}{2\varepsilon}\sum_{i=1}^{N_p}\sum_{l=1}^{\infty}\left|Z^0_{\varepsilon,i}-Z^{-l}_{\varepsilon,i}\right|^2R_{l,i}\leq\frac{\varepsilon}{2}\sum_{i=1}^{N_p}C^2_{z_{p,i}}\mu_{2,i}.$$

It then follows that

$$I_{n,\varepsilon}+\Delta t\sum_{m=1}^{n}D^m_\varepsilon+F(\boldsymbol{Z}^n_\varepsilon)\leq\underbrace{\frac{\varepsilon}{2}\sum_{i=1}^{N_p}C^2_{z_{p,i}}\mu_{2,i}}_{:=K_0}+F(\boldsymbol{Z}^0_p),$$

which is the claim. □

Lemma 3. *Under the same hypotheses as in Proposition 2, the sequence* $(\boldsymbol{Z}^n_\varepsilon)_{n\in\mathbb{N}}$ *is bounded.*

Proof. Assume that there exists a subsequence $(\boldsymbol{Z}^{n_k}_\varepsilon)_{k\in\mathbb{N}}$ such that $|\boldsymbol{Z}^{n_k}_\varepsilon| \underset{k\to\infty}{\longrightarrow} \infty$. Since F is coercive, we have for all $M>0$, there exists $k_0 \in \mathbb{N}$ such that $\forall k > k_0$, $F(\boldsymbol{Z}^{n_k}_\varepsilon) > M$, which contradicts the fact that $F(\boldsymbol{Z}^n_\varepsilon) \leq K_0 + F(\boldsymbol{Z}^0_\varepsilon)$. This prove that any sub-sequence $(\boldsymbol{Z}^{n_k}_\varepsilon)_k$ is bounded. Thus $\boldsymbol{Z}^n_\varepsilon$ is bounded. □

Theorem 2. (*Compactness*) *Under assumptions 2.4.2 (i)–(iii), there exists a constant* $C>0$, *depending only on* $\overline{\mu}_2, \underline{\mu_0}, \overline{\mu_0}, \overline{\zeta}$ *such that*

$$\Delta t \sum_{n=1}^{N} \sum_{i=1}^{N_p} \left| \frac{Z^n_{\varepsilon,i} - Z^{n-1}_{\varepsilon,i}}{\Delta t} \right|^2 \leq C. \tag{3.9}$$

Before perform the proof, we set the following notations $\delta \boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon := \boldsymbol{Z}^n_\varepsilon - \boldsymbol{Z}^{n-1}_\varepsilon$, $\quad \delta \boldsymbol{\mathcal{L}}^{n-\frac{1}{2}}_\varepsilon := \boldsymbol{\mathcal{L}}^n_\varepsilon - \boldsymbol{\mathcal{L}}^{n-1}_\varepsilon$, where the discrete delay operator is $\boldsymbol{\mathcal{L}}^n_\varepsilon = (\mathcal{L}^n_\varepsilon)_i$ and $\mathcal{L}^n_{\varepsilon,i} = \dfrac{\Delta a}{\varepsilon} \sum_{l=1}^{\infty} (Z^n_{\varepsilon,i} - Z^{n-l}_{\varepsilon,i}) R_{l,i}, \quad \forall i \in \{1,\ldots,N_p\}$.

Proof. First we easily check that the global elongation variable solves

$$\varepsilon \frac{\mathbf{U}^n_{\varepsilon,l} - \mathbf{U}^{n-1}_{\varepsilon,l}}{\Delta t} + \frac{\mathbf{U}^{n-1}_{\varepsilon,l} - \mathbf{U}^{n-1}_{\varepsilon,l-1}}{\Delta a} = \frac{\mathbf{Z}^n_\varepsilon - \mathbf{Z}^{n-1}_\varepsilon}{\Delta t}.$$

So by multiplying this equation (taken component-wisely) by $R_{l,i}$ and summing over index $l \in \mathbb{N}^*$, we have

$$\frac{\varepsilon}{\Delta t} \delta \mathcal{L}^{n-\frac{1}{2}}_{\varepsilon,i} + \sum_{l=1}^{\infty} \big(U^{n-1}_{\varepsilon,l,i} - U^{n-1}_{\varepsilon,l-1,i}\big) R_{l,i} = \frac{1}{\Delta t} \underbrace{\left(\Delta a \sum_{l=1}^{\infty} R_{l,i}\right)}_{=:\theta_{\Delta,i}} \delta Z^{n-\frac{1}{2}}_{\varepsilon,i}, \quad i = 1,\cdots,N_p. \tag{3.10}$$

Moreover, since $R_{l,i}$ solves (2.11), we have that

$$\begin{aligned} \sum_{l=1}^{\infty} \big(U^{n-1}_{\varepsilon,l,i} - U^{n-1}_{\varepsilon,l-1,i}\big) R_{l,i} &= \sum_{l=1}^{\infty} U^{n-1}_{\varepsilon,l,i} R_{l,i} - \sum_{l=1}^{\infty} U^{n-1}_{\varepsilon,l-1,i} R_{l,i} = \sum_{l=1}^{\infty} U^{n-1}_{\varepsilon,l,i} R_{l,i} - \sum_{l=0}^{\infty} U^{n-1}_{\varepsilon,l,i} R_{l+1,i} \\ &= \Delta a \sum_{l=1}^{\infty} U^{n-1}_{\varepsilon,l,i} \zeta_{l+1,i} R_{l+1,i}, \quad i = 1,\cdots,N_p, \end{aligned}$$

which plugged into (3.10) gives

$$\frac{\varepsilon}{\Delta t} \delta \mathcal{L}^{n-\frac{1}{2}}_{\varepsilon,i} + \Delta a \sum_{l=1}^{\infty} U^{n-1}_{\varepsilon,l,i} \zeta_{l+1,i} R_{l+1,i} = \theta_{\Delta,i} \frac{\delta Z^{n-\frac{1}{2}}_{\varepsilon,i}}{\Delta t}, \quad i = 1,\cdots,N_p.$$

On the other hand, setting

$$H^n_{\varepsilon,i} := \sum_{k<j} \lambda^{n,\varepsilon}_{kj} (\varphi^{n,\varepsilon}_{kj})'_i(\boldsymbol{Z}^n_\varepsilon)$$

the ith component of the non-penetration velocity, we have by the optimality conditions (3.7) that

$$\theta_{\Delta,i}\frac{\delta Z^{n-\frac{1}{2}}_{\varepsilon,i}}{\Delta t}+\frac{\varepsilon}{\Delta t}(H^n_{\varepsilon,i}-H^{n-1}_{\varepsilon,i}) = \Delta a \sum_{l=1}^{\infty} U^{n-1}_{\varepsilon,l,i}\zeta_{l+1,i}R_{l+1,i}-\frac{\varepsilon}{\Delta t}\left[F_i^{'}(\boldsymbol{Z}^n_\varepsilon)-F_i^{'}(\boldsymbol{Z}^{n-1}_\varepsilon)\right], \quad \forall i. \tag{3.11}$$

Since the mappings $\left(\boldsymbol{\varphi}^{n,\varepsilon}_{kj}\right)_{k<j}$ are convex and differentiable, using Proposition 10.1.4 [All05] we have

$$(\varphi^{n,\varepsilon}_{kj})^{'}(\boldsymbol{Z}^{n-1}_\varepsilon)\cdot\delta\boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon \leq \varphi^{n,\varepsilon}_{kj}(\boldsymbol{Z}^n_\varepsilon)-\varphi^{n,\varepsilon}_{kj}(\boldsymbol{Z}^{n-1}_\varepsilon) \leq (\varphi^{n,\varepsilon}_{kj})^{'}(\boldsymbol{Z}^n_\varepsilon)\cdot\delta\boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon.$$

Moreover since for any time step, $\sum_{k<j}\lambda^{n,\varepsilon}_{kj}\varphi^{n,\varepsilon}_{kj}(\boldsymbol{Z}^n_\varepsilon)=0$ with $\varphi^{n,\varepsilon}_{kj}(\boldsymbol{q})\leq 0$ and $\lambda^{n,\varepsilon}_{kj}\geq 0$, for any $k<j$,

$$0\leq -\sum_{k<j}\left\{\lambda^{n,\varepsilon}_{kj}\varphi^{n,\varepsilon}_{kj}(\boldsymbol{Z}^{n-1}_\varepsilon)+\lambda^{n-1,\varepsilon}_{kj}\varphi^{n-1,\varepsilon}_{kj}(\boldsymbol{Z}^n_\varepsilon)\right\} \leq (\boldsymbol{H}^n_\varepsilon-\boldsymbol{H}^{n-1}_\varepsilon)\cdot\delta\boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon.$$

We multiply (3.11) by $\delta\boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon$ in order to obtain

$$\underline{\theta}\frac{\left|\delta\boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon\right|^2}{\Delta t}\leq\left(\boldsymbol{S}^n_\varepsilon-\frac{\varepsilon}{\Delta t}(\boldsymbol{F}^{'}(\boldsymbol{Z}^n_\varepsilon)-\boldsymbol{F}^{'}(\boldsymbol{Z}^{n-1}_\varepsilon))\right)\cdot\delta\boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon, \tag{3.12}$$

where $\underline{\theta}:=\min_i\theta_i$ and $S^n_{\varepsilon,i}:=\Delta a\sum_{l=1}^{\infty}\boldsymbol{U}^{n-1}_{\varepsilon,l,i}\zeta_{l+1,i}R_{l+1,i}$, for all i. As F is strictly convex we have $\left(\boldsymbol{F}^{'}(\boldsymbol{Z}^n_\varepsilon)-\boldsymbol{F}^{'}(\boldsymbol{Z}^{n-1}_\varepsilon)\right)\cdot(\boldsymbol{Z}^n_\varepsilon-\boldsymbol{Z}^{n-1}_\varepsilon)>0$, so that

$$\underline{\theta}\frac{\left|\delta\boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon\right|^2}{\Delta t}\leq\boldsymbol{S}^n_\varepsilon\cdot\delta\boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon\leq\frac{\Delta t}{\gamma}\left|\boldsymbol{S}^n_\varepsilon\right|^2+\frac{\gamma}{\Delta t}\left|\delta\boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon\right|^2, \quad \forall\gamma>0,$$

where we've used the Young's inequality. It follows that

$$(\underline{\theta}-\gamma)\frac{\left|\delta\boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon\right|^2}{\Delta t}\leq\frac{\Delta t}{\gamma}\left|\boldsymbol{S}^n_\varepsilon\right|^2, \quad \forall\gamma>0.$$

Moreover

$$|\boldsymbol{S}^n_\varepsilon|^2=\sum_{i=1}^{N_p}\Delta a^2\left|\sum_{l=1}^{\infty}U^{n-1}_{l,\varepsilon,i}R_{l+1,i}\zeta_{l+1,i}\right|^2\leq\underbrace{2\Delta a\overline{\zeta}\,\overline{R}}_{:=K_1}\left(\frac{\Delta a}{2}\sum_{i=1}^{N_p}\sum_{l=1}^{\infty}|U^{n-1}_{l,\varepsilon,i}|^2R_{l+1,i}\zeta_{l+1,i}\right)\leq K_1D^n_\varepsilon,$$

where the first inequality is due to Jensen. It follows that

$$(\underline{\theta}-\gamma)\frac{\left|\delta\boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon\right|^2}{\Delta t}\leq\frac{K_1}{\gamma}\Delta tD^n_\varepsilon, \quad \forall n=1,2\cdots,N.$$

So that the sum over n in the latter inequality gives

$$(\underline{\theta}-\gamma)\sum_{n=1}^{N}\frac{\left|\delta\boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon\right|^2}{\Delta t}\leq\frac{K_1}{\gamma}\left(\Delta t\sum_{n=1}^{N}D^n_\varepsilon\right), \quad \forall\gamma>0,$$

which by the energy estimate (3.8) gives

$$(\underline{\theta}-\gamma)\sum_{n=1}^{N}\frac{\left|\delta\boldsymbol{Z}^{n-\frac{1}{2}}_\varepsilon\right|^2}{\Delta t}\leq\frac{K_1}{\gamma}K_0+\frac{K_1}{\gamma}\left(F(\boldsymbol{Z}^0_p)-F(\boldsymbol{Z}^N_\varepsilon)\right), \quad \forall\gamma>0.$$

By Lemma 3, there exist two constants K_2 and K_3 independent of ε and Δt

$$K_2 := \frac{K_1}{\gamma} K_0 \ \text{ and } K_3 \geq \frac{K_1}{\gamma}\left(F(\boldsymbol{Z}_p^0) - F(\boldsymbol{Z}_\varepsilon^N)\right),$$

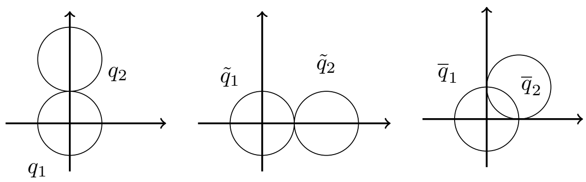
so that

$$(\underline{\theta} - \gamma) \sum_{n=1}^{N} \frac{\left|\delta \boldsymbol{Z}_\varepsilon^{n-\frac{1}{2}}\right|^2}{\Delta t} \leq K_2 + K_3, \quad \forall \gamma > 0.$$

Hence there exists a constant $C := \frac{K_2+K_3}{\underline{\theta}-\gamma}$ such that (3.9) holds. This gives a bound on the discrete time derivative of $\tilde{\boldsymbol{z}}_{\varepsilon,\Delta}$ in $L^2((0,T))$ and ends the proof. □

3.4. *Convergences toward variational inclusions*

This part is devoted to the convergence of the discrete model's solution toward the solution of the continuous variational inclusion when Δa goes to 0 and $\varepsilon > 0$ is fixed. Then we let ε to go to 0 and prove that the resulting limit $\boldsymbol{z}_0$ solves a weighted differential inclusion. To this end, we prove that the constrained minimization problem is equivalent to a variational inclusion (by the use of projections onto closed, nonempty and convex sets) in order to deal with the convergence of the discrete problem to the continuous one, when Δa is small enough.
We mention that the set of admissible configurations is not convex (see Figure 4) so that the projection onto $\boldsymbol{Q}_0$ is not well defined. Nevertheless as shown in [Ven08], there exists $\eta > 0$ such that $P_{\boldsymbol{Q}_0}\boldsymbol{q}$ is well defined for $\boldsymbol{q} \in \mathbb{R}^{2N_p}$ satisfying $dist(\boldsymbol{Q}_0, \boldsymbol{q}) < \eta$. We say that $\boldsymbol{Q}_0$ is *η-prox-regular* or uniformly *prox-regular*, see Appendix A or [Ven08] for more details.



$\boldsymbol{q} = (q_1, q_2)$ $\quad$ $\tilde{\boldsymbol{q}} = (\tilde{q}_1, \tilde{q}_2)$ $\quad$ $\overline{\boldsymbol{q}} = \frac{1}{2}(\boldsymbol{q} + \tilde{\boldsymbol{q}})$

Fig. 4. Lack of convexity of $\boldsymbol{Q}_0$.

3.4.1. *Expression of the contact model as a variational inclusion*

We use the fact that $\boldsymbol{K}(\boldsymbol{Z}_\varepsilon^{n-1})$ is convex to write the constrained minimization problem as a projection on a convex set.

Proposition 3. *Suppose that assumption 2.4.2 (iii) hold. For any $\varepsilon > 0$, the solution of* (2.15) *also satisfies :*

$$\boldsymbol{Z}_\varepsilon^n = P_{\boldsymbol{K}(\boldsymbol{Z}_\varepsilon^{n-1})}\left(\boldsymbol{Z}_\varepsilon^n - \Delta t \boldsymbol{\mathcal{L}}_\varepsilon^n - \Delta t \boldsymbol{F}^{'}(\boldsymbol{Z}_\varepsilon^n)\right), \quad n = 0, \cdots, N-1. \tag{3.13}$$

Proof. Since $\boldsymbol{K}(\boldsymbol{Z}_\varepsilon^{n-1})$ is nonempty closed and convex and the map $\boldsymbol{q} \mapsto E_{n,\varepsilon}(\boldsymbol{q})$ is differentiable at $\boldsymbol{Z}_\varepsilon^n$, by Euler inequality (see [All05]) we have that

$$\langle (\boldsymbol{E}_{n,\varepsilon})'(\boldsymbol{Z}_\varepsilon^n), \boldsymbol{q} - \boldsymbol{Z}_\varepsilon^n \rangle \geq 0, \quad \forall \boldsymbol{q} \in \boldsymbol{K}(\boldsymbol{Z}_\varepsilon^{n-1}).$$

This, since $\Delta t > 0$, is equivalent to

$$\langle \big(\boldsymbol{Z}_\varepsilon^n - \Delta t (\boldsymbol{E}_{n,\varepsilon})'(\boldsymbol{Z}_\varepsilon^n)\big) - \boldsymbol{Z}_\varepsilon^n, \boldsymbol{q} - \boldsymbol{Z}_\varepsilon^n \rangle \leq 0, \quad \forall \boldsymbol{q} \in K(\boldsymbol{Z}_\varepsilon^{n-1}).$$

The latter inequality is nothing but the characterization of the projection onto $\boldsymbol{K}(\boldsymbol{Z}_\varepsilon^{n-1})$ [Bre11] i.e.

$$\boldsymbol{Z}_\varepsilon^n = P_{\boldsymbol{K}(\boldsymbol{Z}_\varepsilon^{n-1})}\left(\boldsymbol{Z}_\varepsilon^n - \Delta t (E_{n,\varepsilon})'(\boldsymbol{Z}_\varepsilon^n)\right),$$

which gives the claim. □

By definition of the proximal-normal cone (see (2.9)) for convex sets, (3.13) is equivalent to

$$\boldsymbol{\mathcal{L}}_\varepsilon^n + \boldsymbol{F}'(\boldsymbol{Z}_\varepsilon^n) \in -N\left(\boldsymbol{K}(\boldsymbol{Z}_\varepsilon^{n-1}), \boldsymbol{Z}_\varepsilon^n\right). \tag{3.14}$$

Proposition 4. *Assume that assumption 2.4.2 (iii) holds, the discrete inclusion* (3.14) *has a unique solution* $\boldsymbol{Z}_\varepsilon^n$.

Proof. The existence and uniqueness of solutions of (2.15) is given in Theorem 1, by Proposition 3, this solution also satisfies (3.13) which ends the proof. □

3.4.2. *Convergence for a fixed $\varepsilon > 0$ when Δa goes to 0*

Let $\varepsilon > 0$, we need to check that the above inclusion is satisfied for the stepsize linear function $\boldsymbol{z}_{\varepsilon,\Delta}$ and then take the limit when Δa goes to 0. Consider the time stepsize constant functions

$$\psi_\Delta|_{(t^{n-1},t^n]} := t^{n-1},\ \theta_\Delta|_{(t^{n-1},t^n]} := t^n, \text{ and } \psi_\Delta(0) = 0,\ \theta_\Delta(0) = 0.$$

Lemma 4. *Under the same condition as in Proposition 4, given the sequence* $(\boldsymbol{Z}_\epsilon^n)_{n\in\{0,N\}}$, *the piecewise linear interpolation* $\tilde{\boldsymbol{z}}_{\varepsilon,\Delta}$ *defined in* (2.17) *satisfies the following inclusion*

$$\tilde{\boldsymbol{\mathcal{L}}}_{\varepsilon,\Delta}(t) + \boldsymbol{F}'(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta}(t)) \in -N\Big(\boldsymbol{K}\left(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta}(\psi_\Delta(t))\right), \tilde{\boldsymbol{z}}_{\varepsilon,\Delta}(\theta_\Delta(t))\Big) \text{ a.e. } t \in [0,T], \tag{3.15}$$

where $\tilde{\boldsymbol{\mathcal{L}}}_{\varepsilon,\Delta}$ *is the linear interpolation of* $\boldsymbol{\mathcal{L}}_\varepsilon^n$.

Proof. Indeed we have that

$$\boldsymbol{\mathcal{L}}_\varepsilon^n + \boldsymbol{F}'(\boldsymbol{Z}_\varepsilon^n) \in -N\left(\boldsymbol{K}(\boldsymbol{Z}_\varepsilon^{n-1}), \boldsymbol{Z}_\varepsilon^n\right), \forall\, n < N.$$

On the other hand, evaluating the latter inequality at two time steps t^n and t^{n-1} and using the definition of $\boldsymbol{z}_{\varepsilon,\Delta}$ and $\boldsymbol{\mathcal{L}}_{\varepsilon,\Delta}$, we have that

$$\tilde{\boldsymbol{\mathcal{L}}}_{\varepsilon,\Delta}(t) + \boldsymbol{A}_{\varepsilon,\Delta}(t) \in -\frac{t-t^{n-1}}{\Delta t} N\left(\boldsymbol{K}(\boldsymbol{Z}_\varepsilon^{n-1}), \boldsymbol{Z}_\varepsilon^n\right) - \big(1 - \frac{t-t^{n-1}}{\Delta t}\big) N\left(\boldsymbol{K}(\boldsymbol{Z}_\varepsilon^{n-2}), \boldsymbol{Z}_\varepsilon^{n-1}\right),\ t \in (t^{n-1}, t^n)$$

where $\boldsymbol{A}_{\varepsilon,\Delta}(t) := \dfrac{t-t^{n-1}}{\Delta t}\boldsymbol{F}'(\boldsymbol{Z}_\varepsilon^n) + (t^n - t)/\Delta t)\boldsymbol{F}'(\boldsymbol{Z}_\varepsilon^{n-1})$. □

Let $\varepsilon > 0$ be fixed we prove that the piecewise constant function (2.16) uniformly converges toward the solution of our continuous problem as the subdivision step Δa goes to 0. Moreover the limit function satisfies a variational inclusion.

Lemma 5. [Ven08] *Let $\boldsymbol{q} \in \boldsymbol{Q}_0$, we have equality between the cones*

$$N(\boldsymbol{Q}_0, \boldsymbol{q}) = N(\boldsymbol{K}(\boldsymbol{q}), \boldsymbol{q}). \tag{3.16}$$

So that we shall consider $N\left(\boldsymbol{Q}_0, \boldsymbol{Z}^n_\varepsilon\right)$ instead of $N\big(\boldsymbol{K}(\boldsymbol{Z}^{n-1}_\varepsilon), \boldsymbol{Z}^n_\varepsilon\big)$ in what follows.

Theorem 3. *Let $\varepsilon > 0$ be fixed and $T > 0$. If the assumptions 2.4.2 (i)-(iii) hold, then the piecewise linear interpolation $\tilde{\boldsymbol{z}}_{\varepsilon,\Delta}$ uniformly converges in $\mathcal{C}([0,T];\boldsymbol{Q}_0)$ when $\Delta a \to 0$. Moreover the limit function denoted by $\boldsymbol{z}_\varepsilon$ satisfies*

$$\begin{cases} \boldsymbol{\mathcal{L}}_\varepsilon[\boldsymbol{z}_\varepsilon](t) + \boldsymbol{F}^{'}(\boldsymbol{z}_\varepsilon(t)) \in -N(\boldsymbol{Q}_0, \boldsymbol{z}_\varepsilon(t)),\, t > 0, \\ \boldsymbol{z}_\varepsilon(t) = \boldsymbol{z}_p(t),\ t \leq 0, \end{cases} \tag{3.17}$$

where $\boldsymbol{\mathcal{L}}_\varepsilon(t) = \big(\mathcal{L}_{\varepsilon,1}(t), \cdots, \mathcal{L}_{\varepsilon,N_p}(t)\big)$ and for any particle $\mathcal{L}_{\varepsilon,i}$ is defined in (2.7).

Proof. In this proof, we aim at using the theorem due to Ascoli. To this purpose, we use compactness arguments as in [Ven08]. We have the followings

- By definition the piecewise linear interpolation $\tilde{\boldsymbol{z}}_{\varepsilon,\Delta}$ is equicontinuous on $[0,T]$.
- Moreover by Lemma 3, $\boldsymbol{Z}^n_\varepsilon$ is bounded uniformly with respect to the discretization step Δa for any time $t^n = n\Delta t$. This implies that $\tilde{\boldsymbol{z}}_{\varepsilon,\Delta}$ admits a L^∞-bound uniformly with respect to Δa.

Let $(\Delta_m)_{m\in\mathbb{N}}$ be a sequence of discretization steps decreasing to 0. Thanks to Arzelà-Ascoli's theorem, there exists a subsequence still denoted by $(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m})_{m\in\mathbb{N}}$ which uniformly converges to $\boldsymbol{z}_\varepsilon \in \boldsymbol{\mathcal{C}}$. We prove first that the limit function belongs to $\boldsymbol{Q_0}$ for all $t \in [0,T]$. Indeed since

$$\tilde{\boldsymbol{z}}_{\varepsilon,\Delta}|_{(t^{n-1},t^n)} = \left(\frac{t - t^{n-1}}{\Delta t}\right)\boldsymbol{Z}^n_\varepsilon + \left(1 - \frac{t-t^{n-1}}{\Delta t}\right)\boldsymbol{Z}^{n-1}_\varepsilon,$$

and $\boldsymbol{Z}^n_\varepsilon, \boldsymbol{Z}^{n-1}_\varepsilon \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_\varepsilon)$ which is convex, we have that $\tilde{\boldsymbol{z}}_{\varepsilon,\Delta} \in \boldsymbol{K}(\boldsymbol{Z}^{n-1}_\varepsilon) \subset \boldsymbol{Q}_0$ for all $n = 1, 2, \cdots, N$. On the other hand, since $\boldsymbol{Q}_0$ is closed for the $\mathcal{C}$-topology we have that

$$\boldsymbol{z}_\varepsilon(t) =: \lim_{m\to\infty} \tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(t) \in \boldsymbol{Q}_0, \quad \forall\, t \in [0,T].$$

Combining this with the fact that $\boldsymbol{z}_\varepsilon \in \boldsymbol{\mathcal{C}}$, we claim that $\boldsymbol{z}_\varepsilon \in \mathcal{C}([0,T],\boldsymbol{Q}_0)$.
We prove now that $\boldsymbol{\pi}_\varepsilon := \boldsymbol{\mathcal{L}}_\varepsilon[\boldsymbol{z}_\varepsilon] + \boldsymbol{F}^{'}(\boldsymbol{z}_\varepsilon) \in -N\left(\boldsymbol{Q}_0, \boldsymbol{z}_\varepsilon\right)$. In fact, thanks to (3.16), it suffices to prove that $\boldsymbol{\mathcal{L}}_\varepsilon[\boldsymbol{z}_\varepsilon] + \boldsymbol{F}^{'}(\boldsymbol{z}_\varepsilon) \in -N\left(\boldsymbol{K}(\boldsymbol{z}_\varepsilon), \boldsymbol{z}_\varepsilon\right), \quad \forall t \in [0,T]$.

- **Convergence:** First, we prove that the linear interpolation of the delay operator converges to the continuous limit with respect to the norm $||\cdot||_{\boldsymbol{\mathcal{C}}}$.
  Indeed for any $i = 1, 2, \cdots, N_p$, we have that

$$\begin{aligned}\tilde{\mathcal{L}}_{\varepsilon,\Delta,i} &= \frac{\mu_{\Delta,i}}{\varepsilon}\sum_{n=1}^{N}\left\{\left(Z^n_{\varepsilon,i} + \frac{t-t^{n-1}}{\Delta t}(Z^n_{\varepsilon,i} - Z^{n-1}_{\varepsilon,i})\right)\right\}\mathbb{1}_{J_n}(t) \\ &\quad - \frac{\Delta a}{\varepsilon}\sum_{n=1}^{N}\left\{\sum_{l=0}^{\infty}\left(Z^{n-l-1}_{\varepsilon,i} + \frac{t-t^{n-1}}{\Delta t}(Z^{n-l}_{\varepsilon,i} - Z^{n-l-1}_{\varepsilon,i})\right)R_{l,i}\right\}\mathbb{1}_{J_n}(t) =: I^1_{\Delta,i} - I^2_{\Delta,i},\end{aligned}$$

where we've set $J_n := \big((n-1)\Delta t, n\Delta t\big)$. To deal with the convergence of $I^1_{\Delta,i}$, we use the fact that $|\boldsymbol{\rho}_\Delta - \boldsymbol{\rho}|_{L^1_a} \underset{\Delta\to 0}{\longrightarrow} 0$ which for any particle gives

$$I^1_{\Delta,i} = \frac{1}{\varepsilon}\tilde{z}_{\varepsilon,\Delta,i}(t)\int_{\mathbb{R}_+}\rho_{\Delta,i}(a)da \xrightarrow[\Delta\longrightarrow 0]{} \frac{1}{\varepsilon}z_{\varepsilon,i}(t)\int_0^\infty \rho_i(a)da, \text{ in } \boldsymbol{\mathcal{C}},$$

On the other hand, we split the second term as follows

$$\begin{aligned}
I^2_{\Delta,i} &= \frac{1}{\varepsilon}\sum_{n=1}^{N}\left\{\Delta a\sum_{l=0}^{\infty} Z^{n-l-1}_{\varepsilon,i}R_{l,i} + \frac{t-t^{n-1}}{\Delta t}\Delta a\sum_{l=0}^{\infty}(Z^{n-l}_{\varepsilon,i} - Z^{n-l-1}_{\varepsilon,i})R_{l,i}\right\}\mathbb{1}_{J_n}(t)\\
&= \frac{1}{\varepsilon}\sum_{n=1}^{N}\left(\frac{t-t^{n-1}}{\Delta t}\int_{\mathbb{R}_+}\left(z_{\Delta,i}(n\Delta t-\varepsilon a) - z_{\Delta,i}(n\Delta t - \varepsilon\Delta a - \varepsilon a)\right)\rho_{\Delta,i}(a)da\right)\mathbb{1}_{J_n}(t)\\
&\quad + \frac{1}{\varepsilon}\sum_{n=1}^{N}\left(\int_{\mathbb{R}_+} z_{\varepsilon,\Delta,i}(n\Delta t - \varepsilon\Delta a - \varepsilon a)\rho_{\Delta,i}(a)da\right)\mathbb{1}_{J_n}(t) =: \frac{1}{\varepsilon}I^{2,1}_{\Delta,i} + \frac{1}{\varepsilon}I^{2,2}_{\Delta,i}.
\end{aligned}$$

Let us now estimate $|\boldsymbol{I}^2_\Delta - \tilde{\boldsymbol{I}}_\Delta|$ where for any particle

$$\tilde{I}_{\Delta,i} := \frac{1}{\varepsilon}\int_{\mathbb{R}_+}\tilde{z}_{\varepsilon,i}(t-\varepsilon\Delta a - \varepsilon a)\rho_{\Delta,i}(a)da$$

We prove that $\boldsymbol{I}^2_\Delta, \tilde{\boldsymbol{I}}_\Delta \in \boldsymbol{L}^2$. Indeed

$$\begin{aligned}
\int_0^T |I^{2,2}_{\Delta,i}(t)|^2dt &\le \sum_{n=1}^{N}\int_{J_n}\left|\int_{\mathbb{R}_+} z_{\varepsilon,\Delta,i}(n\Delta t - \varepsilon\Delta a - \varepsilon a)\rho_{\Delta,i}(a)da\right|^2 dt\\
&\le \sum_{n=1}^{N}\int_{J_n}\int_{\mathbb{R}_+}\rho_{\Delta,i}(\sigma)d\sigma\int_{\mathbb{R}_+}\left|z_{\varepsilon,\Delta,i}(n\Delta t - \varepsilon\Delta a - \varepsilon a)\right|^2\rho_{\Delta,i}(a)dadt, \quad \forall i,
\end{aligned}$$

where we've used the Jensen's inequality in the latter inequality. Furthermore, since

$$\int_{\mathbb{R}_+}\rho_{\Delta,i}(a)da = \mu_{0,\Delta,i} < \infty, \quad \forall i,$$

we have that

$$\int_0^T |I^{2,2}_{\Delta,i}(t)|^2dt \le \mu_{0,\Delta,i}\Delta t\sum_{n=1}^{N}\Delta a\sum_{l=0}^{\infty}\left|Z^{n-l-1}_{\varepsilon,i}\right|^2 R_{l,i},$$

which can be bounded uniformly with respect to ε since

$$\Delta t\sum_{n=1}^{N}\Delta a\sum_{l=0}^{\infty}\left|Z^{n-l-1}_{\varepsilon,i}\right|^2 R_{l,i} \le T\left(|z_{\varepsilon,\Delta,i}|^2_{L^\infty_t} + C^2_{z_{p,i}} + |z^{-1}_{p,i}|^2\right)\int_{\mathbb{R}_+}(1+a)^2\rho_{\Delta,i}(a)da, \quad \forall i = 1,\cdots,N_p.$$

In the latter inequality, we've split the sum over the ages into $l \in \{0,1,\cdots,n-1\}$ and $l \in \{n, n+1, \cdots\}$. In the first part we've inserted the past data then use the bound provided by (3.9) and in the second part we use the Lipschitz condition of the past data. The same arguments guarantee that $\boldsymbol{I}^{1,2}_\Delta$ and $\tilde{\boldsymbol{I}}_\Delta$ belongs to $\boldsymbol{L}^2$.
Furthermor since the past data are Lipschitz and we have the bound (3.9), it follows

$$\int_0^T\left|\boldsymbol{I}^2_\Delta(t) - \tilde{\boldsymbol{I}}_\Delta(t)\right|dt \lesssim \Delta t\sum_{n=1}^{N}\Delta a\sum_{l=0}^{\infty}\left|Z^{n-l-1}_{\varepsilon,i} - Z^{n-l-2}_{\varepsilon,i}\right|^2 R_{l,i} \le O(\Delta a).$$

Thus $||\tilde{\boldsymbol{\mathcal{L}}}_{\varepsilon,\Delta_m} - \boldsymbol{\mathcal{L}}_\varepsilon||_{\boldsymbol{\mathcal{C}}} \longrightarrow 0$ as m grows to infinity.
Furthermore, using the fact that F is continuously differentiable and $\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m} \to \boldsymbol{z}_\varepsilon$, we have that

$$\tilde{\boldsymbol{\pi}}_{\varepsilon,\Delta_m} := \tilde{\boldsymbol{\mathcal{L}}}_{\varepsilon,\Delta_m} + \boldsymbol{F}^{'}(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}) \xrightarrow[m\to\infty]{} \boldsymbol{\pi}_\varepsilon =: \boldsymbol{\mathcal{L}}_\varepsilon[\boldsymbol{z}_\varepsilon] + \boldsymbol{F}^{'}(\boldsymbol{z}_\varepsilon), \quad \forall t \in [0,T] \text{ and } \forall \varepsilon > 0,$$

which gives the convergence.

- **Inclusion:** here we use the same arguments as in [Ven08].
We need to prove that

$$\boldsymbol{\pi}_\varepsilon(t) \in -N\left(\boldsymbol{K}(\boldsymbol{z}_\varepsilon(t)), \boldsymbol{z}_\varepsilon(t)\right), \quad \text{ a.e. } t \in [0,T].$$

By Lemma A.6, (3.15) is equivalent to

$$\langle \tilde{\boldsymbol{\pi}}_{\varepsilon,\Delta_m}, \boldsymbol{\xi}\rangle \leq \big|\tilde{\boldsymbol{\pi}}_{\varepsilon,\Delta_m}(t)\big| d_{\boldsymbol{K}(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\psi_{\Delta_m}(t)))}\big(\boldsymbol{\xi} + \tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\theta_{\Delta_m}(t))\big), \quad \forall \boldsymbol{\xi} \in \mathbb{R}^{2N_p}.$$

Replacing $\boldsymbol{\xi}$ by $-\boldsymbol{\xi}$ in the above inequality, we have that

$$\langle \tilde{\boldsymbol{\pi}}_{\varepsilon,\Delta_m}, \boldsymbol{\xi}\rangle \leq \big|\tilde{\boldsymbol{\pi}}_{\varepsilon,\Delta_m}(t)\big| d_{\boldsymbol{K}(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\psi_{\Delta}(t)))}\big(-\boldsymbol{\xi} + \tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\theta_{\Delta_m}(t))\big), \quad \forall \boldsymbol{\xi} \in \mathbb{R}^{2N_p}.$$

Let us now prove that $|\tilde{\boldsymbol{\pi}}_{\varepsilon,\Delta_m}|$ is bounded uniformly with respect Δa. Indeed, on one hand since $\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}$ and F is continuously differentiable, there exists a constant K_F independent of ε and Δa such that $\big|\boldsymbol{F}^{'}(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m})\big| \leq K_F$. On the other hand, using the energy estimates and the Jensen's inequality, we have

$$|\boldsymbol{\mathcal{L}}^n_\varepsilon|^2 \leq \frac{2C_0}{\varepsilon}\sum_{i=1}^{N_p}\frac{\Delta a}{2\varepsilon}\sum_{l=1}^{\infty}|Z^n_{\varepsilon,i} - Z^{n-l}_{\varepsilon,i}|^2 R_{l,i} \leq \frac{2C_0}{\varepsilon}\left|K_0 + F(\boldsymbol{Z}^0_p) - F(\boldsymbol{Z}^n_\varepsilon)\right|, \tag{3.18}$$

so that $|\tilde{\boldsymbol{\mathcal{L}}}_{\varepsilon,\Delta_m}| \leq \dfrac{K}{\sqrt{\varepsilon}}$ with $K > 0$ is independent of Δa and ε, moreover

$$|\tilde{\boldsymbol{\pi}}_{\varepsilon,\Delta_m}| \leq \left|\tilde{\boldsymbol{\mathcal{L}}}_{\varepsilon,\Delta_m}\right| + \left|\boldsymbol{F}^{'}(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m})\right| \leq \frac{K}{\sqrt{\varepsilon}} + K_F. \tag{3.19}$$

The sum of the two latter inequalities implies that

$$\big|\langle \tilde{\boldsymbol{\pi}}_{\varepsilon,\Delta_m}, \boldsymbol{\xi}\rangle\big| \leq \left(\frac{K}{\sqrt{\varepsilon}} + K_F\right) d_{\boldsymbol{K}(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\psi_{\Delta_m}(t)))}\big| -\boldsymbol{\xi} + \tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\theta_{\Delta_m}(t))\big|, \quad \forall \varepsilon > 0. \tag{3.20}$$

Using the fact that the distance to a nonempty, closed and convex set is 1-Lipschitz and setting

$$\tilde{I}_{\varepsilon,\Delta_m}(t) := \big|d_{\boldsymbol{K}(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\psi_{\Delta_m}(t)))}\big(-\boldsymbol{\xi} + \tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\theta_{\Delta_m}(t))\big) - d_{\boldsymbol{K}(\boldsymbol{z}_\varepsilon(t))}\big(-\boldsymbol{\xi} + \boldsymbol{z}_\varepsilon(t)\big)\big|,$$

we have that

$$\begin{aligned}
\tilde{I}_{\varepsilon,\Delta_m} \leq & \big|d_{\boldsymbol{K}(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\psi_{\Delta_m}(t)))}\big(-\boldsymbol{\xi} + \tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\theta_{\Delta_m}(t))\big) - d_{\boldsymbol{K}(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\psi_{\Delta_m}(t)))}\big(-\boldsymbol{\xi} + \boldsymbol{z}_\varepsilon(t)\big)\big| \\
& + \big|d_{\boldsymbol{K}(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\psi_{\Delta_m}(t)))}\big(\langle -\boldsymbol{\xi} + \boldsymbol{z}_\varepsilon(t)\rangle\big) - d_{\boldsymbol{K}(\boldsymbol{z}_\varepsilon(t))}\big(-\boldsymbol{\xi} + \boldsymbol{z}_\varepsilon(t)\big)\big| \\
\leq & \big|\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\theta_\Delta(t)) - \boldsymbol{z}_\varepsilon(t)\big| + \underbrace{\big|d_{\boldsymbol{K}(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\psi_{\Delta_m}(t)))}\big(\langle -\boldsymbol{\xi} + \boldsymbol{z}_\varepsilon(t)\rangle\big) - d_{\boldsymbol{K}(\boldsymbol{z}_\varepsilon(t))}\big(-\boldsymbol{\xi} + \boldsymbol{z}_\varepsilon(t)\big)\big|}_{\tilde{J}_{\varepsilon,\Delta_m}(t)}.
\end{aligned}$$

Moreover by Proposition A.8, there exists $\nu > 0$ such that for all $\boldsymbol{\xi} \in \mathbb{R}^{2N_p}$ satisfying $|\boldsymbol{\xi}| \leq \nu$, $\tilde{J}_{\varepsilon,\Delta_m}(t) \underset{m\to\infty}{\longrightarrow} 0$.
Thus for any $\boldsymbol{\xi} \in \mathbb{R}^{2N_p}$, there exists $\nu > 0$ satisfying $|\boldsymbol{\xi}| \leq \nu$ and

$$0 \leq \tilde{I}_{\varepsilon,\Delta_m} \leq \left|\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\theta_{\Delta_m}(t)) - \boldsymbol{z}_{\varepsilon}(t)\right| \underset{m\to\infty}{\longrightarrow} 0,$$

i.e.

$$d_{\boldsymbol{K}(\tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\psi_{\Delta_m}(t)))}\big(-\boldsymbol{\xi} + \tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(\theta_{\Delta_m}(t))\big) \underset{m\to\infty}{\longrightarrow} d_{\boldsymbol{K}(\boldsymbol{z}_{\varepsilon}(t))}\big(-\boldsymbol{\xi} + \boldsymbol{z}_{\varepsilon}(t)\big).$$

Since $\varepsilon > 0$ is fixed, equation (3.20) finally gives

$$\forall \boldsymbol{\xi} \in \mathbb{R}^{2N_p}, |\boldsymbol{\xi}| \leq \nu, \quad |\langle \boldsymbol{\pi}_{\varepsilon}(t), \boldsymbol{\xi}\rangle| \leq \left(\frac{K}{\sqrt{\varepsilon}} + K_F\right) d_{\boldsymbol{K}(\boldsymbol{z}_{\varepsilon}(t))}\big| -\boldsymbol{\xi} + \boldsymbol{z}_{\varepsilon}(t))\big|,$$

which using back Lemma A.6 is equivalent to

$$\boldsymbol{\pi}_{\varepsilon}(t) \in -N(\boldsymbol{K}(\boldsymbol{z}_{\varepsilon}(t)), \boldsymbol{z}_{\varepsilon}(t)), \quad \forall \varepsilon > 0,$$

ending the proof once we prove that $\tilde{J}_{\varepsilon,\Delta_m}$; but this is a consequence of Proposition A.8. □

3.4.3. *Uniqueness of solutions of the continuous problem*

Theorem 4. *Let $\varepsilon > 0$ and $T > 0$ be fixed. Under assumptions 2.4.2 (i)-(iii), the variational inclusion* (3.17) *has a unique solution $\boldsymbol{z}_{\varepsilon}$ in $\boldsymbol{\mathcal{C}}$.*

Proof. The existence of the limit $\boldsymbol{z}_{\varepsilon}$ is due to compactness. Indeed $\boldsymbol{z}_{\varepsilon}(t) = \lim_{m\to\infty} \tilde{\boldsymbol{z}}_{\varepsilon,\Delta_m}(t)$ in $\boldsymbol{\mathcal{C}}$.
For the uniqueness, we use the fact that $\boldsymbol{z}_{\varepsilon} \in \boldsymbol{Q}_0$. Indeed since $\boldsymbol{z}_{\varepsilon} \in \boldsymbol{Q}_0$ and solves (3.17), the same arguments as above give

$$\begin{cases} (E_t^{\varepsilon})^{'}(\boldsymbol{z}_{\varepsilon}) \in -N\left(\boldsymbol{K}(\boldsymbol{z}_{\varepsilon}), \boldsymbol{z}_{\varepsilon}\right), \quad t > 0 \\ \boldsymbol{z}_{\varepsilon}(t) = \boldsymbol{z}_p(t), \quad t \leq 0, \end{cases} \iff \begin{cases} \boldsymbol{z}_{\varepsilon}(t) = \underset{\boldsymbol{q}\,\in\,\boldsymbol{K}(\boldsymbol{z}_{\varepsilon})}{\arg\min}\, E_t^{\varepsilon}(\boldsymbol{q}), \quad \forall t > 0 \\ \boldsymbol{z}_{\varepsilon}(t) = \boldsymbol{z}_p(t), \quad t \leq 0. \end{cases}$$

For same seasons as in (3.7), the latter equation in turn is equivalent to the existence of saddle point $(\boldsymbol{\lambda}_{\varepsilon}, \boldsymbol{z}_{\varepsilon})$ such that

$$\boldsymbol{\mathcal{L}}_{\varepsilon}[\boldsymbol{z}_{\varepsilon}] + \boldsymbol{F}^{'}(\boldsymbol{z}_{\varepsilon}) + \sum_{i<j} \lambda_{ij}^{\varepsilon} (\boldsymbol{\varphi}_{ij}^{\varepsilon})^{'}(\boldsymbol{z}_{\varepsilon}) = \boldsymbol{0}, \tag{3.21}$$

where the functions $\varphi_{ij}^{\varepsilon}$ define the interior convex approximation set $\boldsymbol{K}(\boldsymbol{z}_{\varepsilon})$.
Consider two solutions $\boldsymbol{z}_{\varepsilon}^1, \boldsymbol{z}_{\varepsilon}^2$ of (3.21) sharing the same positions for negative times $\boldsymbol{z}_p$ and the same linkages density $\boldsymbol{\rho}$. We have

$$\langle \hat{\boldsymbol{\mathcal{L}}}_{\varepsilon}, \hat{\boldsymbol{z}}_{\varepsilon}\rangle + \langle \boldsymbol{F}^{'}(\boldsymbol{z}_{\varepsilon}^2) - \boldsymbol{F}^{'}(\boldsymbol{z}_{\varepsilon}^1), \hat{\boldsymbol{z}}_{\varepsilon}\rangle + \left\langle \sum_{i<j} \left[\lambda_{ij}^{\varepsilon,2} (\boldsymbol{\varphi}_{ij}^{\varepsilon,2})^{'}(\boldsymbol{z}_{\varepsilon}^2) - \lambda_{ij}^{\varepsilon,1} (\boldsymbol{\varphi}_{ij}^{\varepsilon,1})^{'}(\boldsymbol{z}_{\varepsilon}^1)\right], \hat{\boldsymbol{z}}_{\varepsilon} \right\rangle = 0,$$

where $\hat{\boldsymbol{z}}_{\varepsilon} := \boldsymbol{z}_{\varepsilon}^2 - \boldsymbol{z}_{\varepsilon}^1$ and $\hat{\boldsymbol{\mathcal{L}}}_{\varepsilon} := \boldsymbol{\mathcal{L}}_{\varepsilon}[\boldsymbol{z}_{\varepsilon}^2] - \boldsymbol{\mathcal{L}}_{\varepsilon}[\boldsymbol{z}_{\varepsilon}^1]$. Notice once again that since F is convex, we have that

$$\langle \boldsymbol{F}^{'}(\boldsymbol{z}_{\varepsilon}^2) - \boldsymbol{F}^{'}(\boldsymbol{z}_{\varepsilon}^1), \boldsymbol{z}_{\varepsilon}^2 - \boldsymbol{z}_{\varepsilon}^1\rangle \geq 0.$$

So that

$$\langle \hat{\boldsymbol{\mathcal{L}}}_\varepsilon, \hat{\boldsymbol{z}}_\varepsilon \rangle + \left\langle \sum_{i<j} \left[\lambda_{ij}^{\varepsilon,2} (\boldsymbol{\varphi}_{ij}^{\varepsilon,2})'(\boldsymbol{z}_\varepsilon^2) - \lambda_{ij}^{\varepsilon,1} (\boldsymbol{\varphi}_{ij}^{\varepsilon,1})'(\boldsymbol{z}_\varepsilon^1) \right], \hat{\boldsymbol{z}}_\varepsilon \right\rangle \leq 0. \tag{3.22}$$

Let's consider the second term on the right hand side. Since $\varphi_{ij}^{\varepsilon,1}$ and $\varphi_{ij}^{\varepsilon,2}$ are convex, by the same arguments as in the proof of Theorem 2, we have that

$$\langle (\boldsymbol{\varphi}_{ij}^{\varepsilon,k})'(\boldsymbol{z}_\varepsilon^1), \hat{\boldsymbol{z}}_\varepsilon \rangle \leq \boldsymbol{\varphi}_{ij}^{\varepsilon,k}(\boldsymbol{z}_\varepsilon^2) - \boldsymbol{\varphi}_{ij}^{\varepsilon,k}(\boldsymbol{z}_\varepsilon^1) \leq \langle (\boldsymbol{\varphi}_{ij}^{\varepsilon,k})'(\boldsymbol{z}_\varepsilon^1), \hat{\boldsymbol{z}}_\varepsilon \rangle, \quad k \in \{1,2\} \text{ and } i<j,$$

so that, since the Lagrange multipliers $\lambda_{ij}^{\varepsilon,k}(t) \geq 0$ for all $i < j$ and $t \in [0,T])$ and $\sum_{i<j} \lambda_{ij}^{\varepsilon,k} \varphi_{ij}^{\varepsilon,k}(\boldsymbol{z}_\varepsilon^k) = 0,\ k \in \{1,2\}$ we have that

$$0 \leq \sum_{i<j} \left[\langle \lambda_{ij}^{\varepsilon,2} (\boldsymbol{\varphi}_{ij}^{\varepsilon,2})'(\boldsymbol{z}_\varepsilon^2) - \lambda_{ij}^{\varepsilon,1} (\boldsymbol{\varphi}_{ij}^{\varepsilon,1})'(\boldsymbol{z}_\varepsilon^1), \hat{\boldsymbol{z}}_\varepsilon \rangle \right].$$

By (3.22), this means that

$$\langle \hat{\boldsymbol{\mathcal{L}}}_\varepsilon, \hat{\boldsymbol{z}}_\varepsilon \rangle \leq 0. \tag{3.23}$$

Then using [a], we have that

$$\frac{1}{2\varepsilon} \sum_{i=1}^{N_p} \int_0^\infty \left| \hat{z}_{\varepsilon,i}(t) \right|^2 (t) \rho_i(a) da - \frac{1}{2\varepsilon} \sum_{i=1}^{N_p} \int_0^{t/\varepsilon} |\hat{z}_{\varepsilon,i}(t - \varepsilon a)|^2 \rho_i(a) da \leq \langle \hat{\boldsymbol{\mathcal{L}}}_\varepsilon, \hat{\boldsymbol{z}}_\varepsilon \rangle,$$

so that by definition of $\boldsymbol{\rho}$,

$$\frac{\mu_{0,m}}{2\varepsilon} |\hat{\boldsymbol{z}}_\varepsilon(t)|^2 - \frac{1}{2\varepsilon} \sum_{i=1}^{N_p} \int_0^{t/\varepsilon} |\hat{z}_{\varepsilon,i}(t - \varepsilon a)|^2 \rho_i(a) da \leq \langle \hat{\boldsymbol{\mathcal{L}}}_\varepsilon, \hat{\boldsymbol{z}}_\varepsilon \rangle, \quad \forall \varepsilon > 0 \text{ fixed.} \tag{3.24}$$

Combining (3.23) and (3.24), we have

$$|\hat{\boldsymbol{z}}_\varepsilon(t)|^2 \leq \frac{\overline{\rho}}{\mu_{0,m}} \int_0^t |\hat{\boldsymbol{z}}_\varepsilon(s)|^2 ds, \quad \forall t \in [0,T],$$

which thanks to the Gronwall's lemma gives $|\hat{\boldsymbol{z}}_\varepsilon| \equiv 0$, i.e. $\boldsymbol{z}_\varepsilon^1(t) = \boldsymbol{z}_\varepsilon^2(t)$, a.e. $t \in [0,T]$. □

3.4.4. *Convergence when ε is small enough*

In this section we are interested in the asymptotic when the linkages remodelling rate becomes large enough. We prove the convergence of $\boldsymbol{z}_\varepsilon$ in $\boldsymbol{\mathcal{C}}$. Nevertheless we are not in conditions of using the same arguments of [Ven08], because the delay operator is not uniformly bounded with respect to ε (see (3.18)).

Theorem 5. *Let $T > 0$ be fixed. Under assumptions 2.4.2 (i)-(iii), when ε tends to 0 we have that*

$$\int_0^T \langle \mathcal{L}_\varepsilon[\boldsymbol{z}_\varepsilon], \boldsymbol{\psi}(t) \rangle dt \longrightarrow \langle \boldsymbol{z}_\varepsilon(T), \boldsymbol{\psi}(T) \rangle - \langle \boldsymbol{z}_\varepsilon(0), \boldsymbol{\psi}(0) \rangle - \int_0^T \langle \boldsymbol{\mu}_1 \boldsymbol{z}_0, \partial_t \boldsymbol{\psi}(t) \rangle dt, \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}^1. \tag{3.25}$$

[a] $\langle a - b, a \rangle \geq \frac{1}{2}\left(|a|^2 - |b|^2\right)$ for any $a, b \in \mathbb{R}^{2N_p}$

Proof. Let $\boldsymbol{z}_\varepsilon$ be the unique solution of (3.17). By the energy estimates there exists a constant C independent of ε such that

$$\sum_{i=1}^{N_p}\int_0^T\int_0^\infty \rho_i|u_{\varepsilon,i}|^2\zeta_i(a)dadt \leq C < \infty.$$

On the other, since the death rate ζ has a lower bound, we have that

$$\int_0^T |\boldsymbol{\mathcal{L}}_\varepsilon|^2\, dt = \sum_{i=1}^{N_p}\int_0^T \left|\int_0^\infty \frac{\rho_i u_{\varepsilon,i}(a,t)\zeta_i(a)}{\zeta_i(a)}da\right|^2 dt \leq \frac{1}{\underline{\zeta}^2}\sum_{i=1}^{N_p}\int_0^T\left|\int_0^\infty \rho_i u_{\varepsilon,i}(a,t)\zeta_i(a)da\right|^2 dt,$$

so that by the Jensen inequality

$$\frac{1}{\underline{\zeta}^2}\sum_{i=1}^{N_p}\int_0^T\left|\int_0^\infty \rho_i u_{\varepsilon,i}(a,t)\zeta_i(a)da\right|^2 dt \leq \frac{\overline{\zeta}}{\underline{\zeta}^2}\mu_{0,M}\sum_{i=1}^{N_p}\int_0^T\int_0^\infty \rho_i|u_{\varepsilon,i}|^2\zeta_i(a)dadt.$$

This shows that the delay operator belongs to $\boldsymbol{L}^2$ uniformly with respect in ε. There there exists $\boldsymbol{\mathcal{L}}_0 \in \boldsymbol{L}^2$ such that $\boldsymbol{\mathcal{L}}_\varepsilon$ weakly converges to $\boldsymbol{\mathcal{L}}_0$ in $\boldsymbol{L}^2$ when ε tends to 0, implying that

$$\int_0^T \langle \boldsymbol{\mathcal{L}}_\varepsilon, \boldsymbol{\psi}(t)\rangle dt \xrightarrow[\varepsilon\longrightarrow 0]{} \int_0^T\langle \boldsymbol{\mathcal{L}}_0, \boldsymbol{\psi}(t)\rangle dt, \quad \forall \boldsymbol{\psi}\in \boldsymbol{L}^2.$$

As it stands, we have

i) $\partial_t \tilde{\boldsymbol{z}}_{\varepsilon,\Delta} \in \boldsymbol{L}^2$,
ii) $\tilde{\boldsymbol{z}}_{\varepsilon,\Delta} \in \boldsymbol{\mathcal{C}}$ and
iii) $||\tilde{\boldsymbol{z}}_{\varepsilon,\Delta} - \boldsymbol{z}_\varepsilon||_{\boldsymbol{\mathcal{C}}} \xrightarrow[\Delta\longrightarrow 0]{} 0$.

Setting $I[\boldsymbol{\rho},\boldsymbol{z}_\varepsilon,\boldsymbol{\psi}] := \int_0^T\langle \boldsymbol{\mathcal{L}}_\varepsilon[\boldsymbol{z}_\varepsilon],\boldsymbol{\psi}\rangle dt$, we split the integral as follows

$$\begin{aligned}I[\boldsymbol{\rho},\boldsymbol{z}_\varepsilon,\boldsymbol{\psi}] &= \frac{1}{\varepsilon}\sum_{i=1}^{N_p}\int_0^\infty\int_0^T\langle z_{\varepsilon,i}(t),\psi_i(t)-\psi_i(t+\varepsilon a)\rangle\rho_i(a)dadt \\ &+ \frac{1}{\varepsilon}\sum_{i=1}^{N_p}\int_0^T\int_0^\infty\left\{\langle z_{\varepsilon,i}(t),\psi_i(t+\varepsilon a)\rangle - \langle z_{\varepsilon,i}(t-\varepsilon a),\psi_i(t)\rangle\right\}\rho_i(a)dadt =: I_1^\varepsilon + I_2^\varepsilon. \quad (3.26)\end{aligned}$$

By the dominated Lebesgue's theorem, we have that

$$I_1^\varepsilon \xrightarrow[\varepsilon\to 0]{} -\int_0^T\langle \boldsymbol{\mu}_1\boldsymbol{z}_0,\partial_t\boldsymbol{\psi}\rangle dt, \quad \forall \boldsymbol{\psi}\in\boldsymbol{H}^1.$$

Splitting I_2^ε into $I_{2,1}^\varepsilon$ and $I_{2,2}^\varepsilon$ and using the same arguments as in [Mil20] we have that

$$I_{2,1}^\varepsilon := \frac{1}{\varepsilon}\sum_{i=1}^{N_p}\int_0^T\int_0^\infty\langle z_{\varepsilon,i}(t),\psi_i(t+\varepsilon a)\rangle\rho_i(a)dadt \xrightarrow[\varepsilon\to 0]{} \langle\boldsymbol{\mu}_1\boldsymbol{z}_0(T),\boldsymbol{\psi}(T)\rangle,$$

and

$$I_{2,2}^\varepsilon := \frac{1}{\varepsilon}\sum_{i=1}^{N_p}\int_0^T\int_0^\infty\langle z_{\varepsilon,i}(t-\varepsilon a),\psi_i(t)\rangle\rho_i(a)dadt \xrightarrow[\varepsilon\to 0]{} \langle\boldsymbol{\mu}_1\boldsymbol{z}_0(0),\boldsymbol{\psi}(0)\rangle.$$

We gather the above convergences to obtain

$$I[\boldsymbol{\rho}, \boldsymbol{z}_\varepsilon, \boldsymbol{\psi}] \longrightarrow \langle \boldsymbol{\mu}_1 \boldsymbol{z}_0(T), \boldsymbol{\psi}(T)\rangle - \langle \boldsymbol{\mu}_1 \boldsymbol{z}_0(0), \boldsymbol{\psi}(0)\rangle - \int_0^T \langle \boldsymbol{\mu}_1 \boldsymbol{z}_0, \partial_t \boldsymbol{\psi}\rangle dt, \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}^1.$$

On the other hand since $\partial_t \boldsymbol{z}_0 \in \boldsymbol{L}^2$ and $\boldsymbol{z}_0 \in \boldsymbol{L}^\infty$ we have that $\boldsymbol{z}_0 \in \boldsymbol{\mathcal{C}}$, so that the integration by parts is well-defined in $\boldsymbol{H}^1$ and we have

$$\langle \boldsymbol{\mu}_1 \boldsymbol{z}_0(T), \boldsymbol{\psi}(T)\rangle - \langle \boldsymbol{\mu}_1 \boldsymbol{z}_0(0), \boldsymbol{\psi}(0)\rangle - \int_0^T \langle \boldsymbol{\mu}_1 \boldsymbol{z}_0, \partial_t \boldsymbol{\psi}\rangle dt = \int_0^T \langle \mu_1 \partial_t \boldsymbol{z}_0, \boldsymbol{\psi}\rangle dt, \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}^1.$$

This gives that

$$\int_0^T \langle \boldsymbol{\mathcal{L}}_0 - \boldsymbol{\mu}_1 \partial_t \boldsymbol{z}_0, \boldsymbol{\psi}\rangle dt = 0, \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}^1 \iff \boldsymbol{\mathcal{L}}_0 = \boldsymbol{\mu}_1 \partial_t \boldsymbol{z}_0 \text{ a.e. } t \in [0,T],$$

ending by the same way the proof. □

Theorem 6. *Let $\boldsymbol{z}_\varepsilon$ be the unique solution of* (3.21)*. Under hypotheses 2.4.2 (i)–(iii) there exists $\boldsymbol{\lambda}_0 = \left(\lambda_{ij}^0\right)_{i<j} \in L^2\left([0,T]; (\mathbb{R}_+)^{N_c}\right)$ depending only on time such that*

$$\sum_{i<j} \int_0^T \lambda_{ij}^\varepsilon(t) \langle \boldsymbol{G}_{ij}(\boldsymbol{z}_\varepsilon), \boldsymbol{\psi}(t)\rangle dt \xrightarrow[\varepsilon \to 0]{} \sum_{i<j} \int_0^T \lambda_{ij}^0(t) \langle \boldsymbol{G}_{ij}(\boldsymbol{z}_0), \boldsymbol{\psi}(t)\rangle dt, \quad \forall \boldsymbol{\psi} \in \boldsymbol{L}^2.$$

Proof. Let $\boldsymbol{U} := \boldsymbol{\mathcal{L}}_\varepsilon - \boldsymbol{F}^{'}(\boldsymbol{z}_\varepsilon)$ and

$$\boldsymbol{\Lambda}^\varepsilon_{\boldsymbol{z}_\varepsilon, \boldsymbol{U}} := \left\{ \boldsymbol{\lambda}_\varepsilon \in \mathbb{R}^{N_c},\ \sum_{i<j} \lambda_{ij}^\varepsilon \boldsymbol{G}_{ij}(\boldsymbol{z}_\varepsilon) =: \boldsymbol{U},\ \lambda_{ij}^\varepsilon \geq 0 \text{ and } \lambda_{ij}^\varepsilon = 0 \text{ if } D_{ij}(\boldsymbol{z}_\varepsilon) > 0 \right\}.$$

If $\boldsymbol{\Lambda}^\varepsilon_{\boldsymbol{z}_\varepsilon, \boldsymbol{U}} \neq \emptyset$, the same arguments as in ^{Ven08} guarantee that

$$\forall \boldsymbol{\lambda}_\varepsilon \in \boldsymbol{\Lambda}^\varepsilon_{\boldsymbol{z}_\varepsilon, \boldsymbol{U}} \text{ and } \forall i<j, \text{ we've } \ \lambda_{ij}^\varepsilon \leq |\boldsymbol{U}| b^{N_p}, \text{ where } b := \frac{2\sqrt{n_v}}{\min\left(\sin\left(\dfrac{\pi}{n_v+1}\right), \sin\left(\dfrac{\pi}{N}\right)\right)}, \tag{3.27}$$

and n_v is the maximal number of neighbours that a particle may have.
It follows that

$$\int_0^T |\lambda_{ij}^\varepsilon|^2 dt \leq 2b^{2N_p} \int_0^T \left(|\boldsymbol{\mathcal{L}}_\varepsilon|^2 + \left|\boldsymbol{F}^{'}(\boldsymbol{z}_\varepsilon)\right|^2 \right) dt \lesssim 2b^{2N_p} \left(\frac{\overline{\zeta}\mu_{0,M}}{\underline{\zeta}} + K^2 T \right), \quad \forall i<j,$$

where we've used the fact that $\boldsymbol{\mathcal{L}}_\varepsilon \in \boldsymbol{L}^2$ on one hand and $|\boldsymbol{F}^{'}(\boldsymbol{z}_\varepsilon)| < \infty$ (since $\boldsymbol{F}^{'}$ is continuous and $\boldsymbol{z}_\varepsilon$ is bounded) on the other.
Furthermore, since $\boldsymbol{Q}_0$ is closed and $\boldsymbol{z}_\varepsilon \in \boldsymbol{Q}_0$, we have that $\boldsymbol{z}_0 := \lim\limits_{\varepsilon \to 0} \boldsymbol{z}_\varepsilon \in \boldsymbol{Q}_0$. On the other hand, since by definition $\boldsymbol{G}_{ij}$ is defined and continuous in $\boldsymbol{Q}_0$, we have that

$$\boldsymbol{G}_{ij}(\boldsymbol{z}_\varepsilon) \xrightarrow[\varepsilon \longrightarrow 0]{} \boldsymbol{G}_{ij}(\boldsymbol{z}_0) \text{ in } \boldsymbol{\mathcal{C}}, \quad \forall i<j.$$

For any $i<j$, we have that

$$\begin{cases} \boldsymbol{\lambda}_\varepsilon \rightharpoonup \boldsymbol{\lambda}^0 \text{ in } L^2\left([0,T]; (\mathbb{R}_+)^{N_c}\right) \\ \boldsymbol{G}_{ij}(\boldsymbol{z}_\varepsilon) \longrightarrow \boldsymbol{G}_{ij}(\boldsymbol{z}_0) \text{ in } \boldsymbol{\mathcal{C}}, \end{cases}$$

so that

$$\lambda_{ij}^{\varepsilon}\boldsymbol{G}_{ij}(\boldsymbol{z}_{\varepsilon}) \rightharpoonup \lambda_{ij}^{0}\boldsymbol{G}_{ij}(\boldsymbol{z}_{0}) \text{ in } \boldsymbol{L}^{2},$$

implying that

$$\sum_{i<j}\int_{0}^{T}\lambda_{ij}^{\varepsilon}(t)\langle \boldsymbol{G}_{ij}(\boldsymbol{z}_{\varepsilon})(t),\boldsymbol{\psi}(t)\rangle dt \xrightarrow[\varepsilon\longrightarrow 0]{} \sum_{i<j}\int_{0}^{T}\lambda_{ij}^{0}(t)\langle \boldsymbol{G}_{ij}(\boldsymbol{z}_{0}(t)),\boldsymbol{\psi}(t)\rangle dt, \quad \forall \boldsymbol{\psi}\in \boldsymbol{L}^{2}.$$

This is the end of the proof. □

Theorem 7. *Under hypotheses 2.4.2 (i)-(iii), the unique solution of* (3.17) *converges toward* $\boldsymbol{z}_0 \in \boldsymbol{\mathcal{C}}$ *which in turn solves*

$$\begin{cases} \boldsymbol{\mu}_1\partial_t \boldsymbol{z}_0 + \boldsymbol{F}^{'}(\boldsymbol{z}_0) \in -N\left(\boldsymbol{K}(\boldsymbol{z}_0),\boldsymbol{z}_0\right) \ \text{ a.e. } t\in(0,T], \\ \boldsymbol{z}_0(0) = \boldsymbol{z}_p(0), \end{cases}$$

where

$$\boldsymbol{\mu}_1\partial_t\boldsymbol{z}_0 = (\mu_{1,i}\partial_t z_{0,i})_{i=1,\cdots,N_p} \text{ and } \mu_{1,i} := \int_0^{\infty} a\rho_i(a)da \in \mathbb{R}, \quad \forall i.$$

Moreover the limit function $\boldsymbol{z}_0$ *is unique.*

Proof. *The primal-dual problem:* as in the proof of Theorem 4, it suffices to prove that there exists $\boldsymbol{\lambda}_0 \in L^2\left([0,T];(\mathbb{R}_+)^{N_c}\right)$ depending only on time such that

$$\boldsymbol{\mu}_1\partial_t\boldsymbol{z}_0 + \boldsymbol{F}^{'}(\boldsymbol{z}_0) - \sum_{i<j}\lambda_{ij}^{0}\boldsymbol{G}_{ij}(\boldsymbol{z}_0) = \boldsymbol{0}. \tag{3.28}$$

The existence of $\boldsymbol{z}_0$ is due to compactness, since $\boldsymbol{z}_0 := \lim_{\varepsilon\to 0}\boldsymbol{z}_{\varepsilon}$ where $\boldsymbol{z}_{\varepsilon}$ is the unique solution of (3.17). Furthermore, from Theorems 5 and 6 on one hand and the fact that F is continuously differentiable on the other, we have that $\boldsymbol{z}_0$ solves

$$\int_0^t < \boldsymbol{\mu}_1\partial_t\boldsymbol{z}_0 + \boldsymbol{F}^{'}(\boldsymbol{z}_0) - \sum_{i<j}\lambda_{ij}^{0}\boldsymbol{G}_{ij}(\boldsymbol{z}_0), \boldsymbol{\psi} > ds = 0, \quad \forall\boldsymbol{\psi}\in\boldsymbol{H}^1 \text{ and } \forall t\in[0,T].$$

Uniqueness: Let $\boldsymbol{z}_0^1$ and $\boldsymbol{z}_0^2$ be two solutions of (3.28) sharing the same initial positions i.e. $\boldsymbol{z}_0^1(0) = \boldsymbol{z}_0^2(0)$. We have

$$\int_0^t < \boldsymbol{\mu}_1\partial_t\hat{\boldsymbol{z}}_0 + \widehat{\boldsymbol{F}^{'}(\boldsymbol{z}_0)} - \sum_{i<j}\lambda_{ij}^{0}\boldsymbol{G}_{ij}(\boldsymbol{z}_0^2) + \sum_{i<j}\lambda_{ij}^{0}\boldsymbol{G}_{ij}(\boldsymbol{z}_0^1), \boldsymbol{\psi} > ds = 0, \ \forall\boldsymbol{\psi}\in\boldsymbol{\mathcal{C}}\cap\boldsymbol{H}^1,$$

where $\hat{\boldsymbol{z}}_0 := \boldsymbol{z}_0^2 - \boldsymbol{z}_0^1$ and $\widehat{\boldsymbol{F}^{'}(\boldsymbol{z}_0)} := \boldsymbol{F}^{'}(\boldsymbol{z}_0^2) - \boldsymbol{F}^{'}(\boldsymbol{z}_0^1)$. Let us choose $\boldsymbol{\psi} = \hat{\boldsymbol{z}}_0$ in the latter equation. Since the source term and the constraints functions are convex, by the same arguments as in proof of Theorem 4, we have that

$$\mu_{1,m}\int_0^t\langle\partial_t\hat{\boldsymbol{z}}_0,\hat{\boldsymbol{z}}_0\rangle dt \le 0 \implies |\hat{\boldsymbol{z}}_0(t)|^2 \le 0, \quad \forall t\in[0,T],$$

which proves that $|\hat{\boldsymbol{z}}_0(t)| \equiv 0$, meaning that $\boldsymbol{z}_0^1 = \boldsymbol{z}_0^2$ for almost every $t\in[0,T]$. □

3.5. *The periodic contact model*

3.5.1. *Definition of the periodic signed distance*

Proposition 5. *For any $x = (x_1, x_2) \in \mathbb{R}^2$ set $\overline{x}_1 := x_1 - \left\lfloor \dfrac{x_1}{L} \right\rfloor L$ and $\overline{x}_2 := x_2 - \left\lfloor \dfrac{x_2}{H} \right\rfloor H$. We have the following statements:*

- $(\overline{x}_1, \overline{x}_2) \in [0, L] \times [0, H]$,
- *moreover*

$$\min_{h,k\in\mathbb{Z}} |x - hLe_1 - kHe_2| = \min_{h,k\in\{0,1\}} |\overline{x} - hLe_1 - kHe_2|,$$

where e_1 and e_2 respectively denotes the first and second vector of the canonical basis of $\mathbb{R}^2$.

Proof. For sake of simplicity, we first perform the proof in one dimension i.e. $\mathcal{D} = [0, L]$. The 2D-case being obtained by extension.
Let $x \in \mathbb{R}$, since $\mathbb{R}$ is Archimedean there exists $n := \left\lfloor \dfrac{x}{L} \right\rfloor$ such that

$$n \leq \frac{x}{L} < n + 1,$$

which implies that

$$nL \leq x < nL + L \implies 0 \leq \overline{x} < L, \tag{3.29}$$

which proves the first claim.
For the second claim, we notice that

$$\min_{k\in\mathbb{Z}} |x - kL| = \min_{k\in\mathbb{Z}} |\overline{x} + nL - kL| = \min_{k\in\mathbb{Z}} |\overline{x} - kL|.$$

On the other hand, since there exists $k \in \mathbb{Z}$ such that $|\overline{x} - kL| < L$ (take $k = 0$ for instance), the map $A : k \mapsto |\overline{x} - kL|$ realizes its minimum for indices k_0 satisfying $A(k_0) < L$. But thanks to the first claim,

$$|\overline{x} - kL| < L \implies (k - 1)L < \overline{x} < (k + 1)L$$

then by (3.29) we conclude that $-1 < k < 2$. Or equivalently

$$\min_{k\in\mathbb{Z}} |\overline{x} - kL| = \min_{k\in\{0,1\}} |\overline{x} - kL|.$$

We conclude that

$$\min_{k\in\mathbb{Z}} |x - kL| = \min_{k\in\{0,1\}} |\overline{x} - kL| =: \min\left(\overline{x}, L - \overline{x}\right) = \begin{cases} \overline{x} & \text{if } \overline{x} \leq L/2 \\ L - \overline{x} & \text{if } \overline{x} > L/2. \end{cases}$$

This ends the proof. □

3.5.2. *The framework for the continuous periodic model*

Consider the following minimization process

$$\tilde{\boldsymbol{z}}_\varepsilon(t) = \arg\min_{\boldsymbol{q}\in\tilde{\mathcal{U}}} \mathcal{E}_t(\boldsymbol{q}) := \frac{1}{2\varepsilon}\sum_{i=1}^{N_p}\int_0^\infty \left|q_i - \tilde{z}_{\varepsilon,i}(t-\varepsilon a)\right|^2 \rho_i(a)da + F(\tilde{\boldsymbol{z}}_\varepsilon), \tag{3.30}$$

where the set of constraints $\tilde{\mathcal{U}}$ reads

$$\tilde{\mathcal{U}} := \left\{\boldsymbol{q}\in\mathbb{R}^{2N_p} \text{ s.t. } \phi^\varepsilon_{ij}(\boldsymbol{q}) := -d_{ij}(\tilde{\boldsymbol{z}}_\varepsilon) - \nabla d_{ij}(\tilde{\boldsymbol{z}}_\varepsilon)\cdot(\boldsymbol{q}-\tilde{\boldsymbol{z}}_\varepsilon) \le 0,\, \forall\, i<j\right\},$$

and the periodic distance

$$d_{ij}(\boldsymbol{q}) := \min_{(h,k)\in\mathbb{Z}^2}\left|q_j - q_i - hLe_1 - kHe_2\right| - (r_i + r_j). \tag{3.31}$$

We denote $\overline{\boldsymbol{q}} := (\overline{q_1},\cdots,\overline{q_{N_p}})$ the projection of particles' position in the 2D-torus. For any particle we denote $\overline{q_i} := (\overline{q_i^x},\overline{q_i^y})$ where $\overline{q_i^x}$ (resp. $\overline{q_i^y}$) is the projection in $[0,L]$ (resp. in $[0,H]$) of q_i^x (reps. q_i^x) as in Proposition 5. When accounting for adhesions, the corresponding energy represents past positions in the 2D plane, whereas contact forces occur on the torus. This is because we take into account the length of adhesions greater than the periodicity dimensions L and H; see Figure 5. By

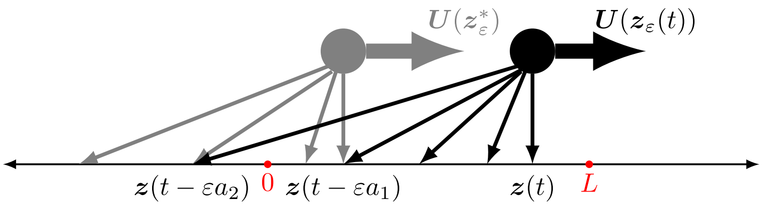


Fig. 5. Linkages associated to some past positions in the domain $[0,L]$ where $\boldsymbol{z}_\varepsilon^* := \boldsymbol{z}_\varepsilon(t-\varepsilon a_1)$.

Proposition 5, we have that

$$d_{ij}(\boldsymbol{q}) = \min_{(h,k)\in\{0,1\}^2}\left|\overline{q_j} - \overline{q_i} - hLe_1 - kHe_2\right| - (r_i + r_j).$$

Since this distance is well-defined i.e there exist are $\underline{h},\underline{k}\in\{0,1\}$ such that

$$d_{ij}(\boldsymbol{q}) = \left|\overline{q_j} - \overline{q_i} - \underline{h}Le_1 - \underline{k}He_2\right| - (r_i + r_j),$$

we define the gradient vector of d_{ij} in $\tilde{\boldsymbol{Q}}_0$ as

$$\tilde{\boldsymbol{G}}_{\boldsymbol{ij}} := \nabla d_{ij}(\boldsymbol{q}) = \Big(0,\cdots 0, \underbrace{-\tilde{e}_{i,j}(\boldsymbol{q})}_{i}, 0\cdots 0, \underbrace{\tilde{e}_{i,j}(\boldsymbol{q})}_{j}, 0,\cdots,0\Big), \quad i<j,$$

where $\tilde{e}_{ij}(\boldsymbol{q})$ is the action of the copy of particle i on particle j and is oriented towards j. It is unitary and reads

$$\tilde{e}_{ij}(\boldsymbol{q}) := \frac{q_j - q_i - (n_j^x - n_i^x + \underline{h})Le_1 - (n_j^y - n_i^y + \underline{k})He_2}{\left|q_j - q_i - (n_j^x - n_i^x + \underline{h})Le_1 - (n_j^y - n_i^y + \underline{k})He_2\right|}, \quad i<j,$$

where $n_k^x := \lfloor q_k^x/L\rfloor$ and $n_k^y := \lfloor q_k^y/H\rfloor$ for any particle k.

3.5.3. *The discrete periodic problem*

The same arguments as earlier in this paper lead to the discrete minimization problem,

$$\tilde{\boldsymbol{Z}}^n_\varepsilon = \underset{\boldsymbol{q}\,\in\,\tilde{\boldsymbol{K}}\left(\tilde{\boldsymbol{Z}}^{n-1}_\varepsilon\right)}{\arg\min} \left\{ \mathcal{E}_{n,\varepsilon}(\boldsymbol{q}) := \frac{\Delta a}{2\varepsilon}\sum_{i=1}^{N_p}\sum_{l=1}^{\infty}\left|q_i - \tilde{Z}^{n-l}_{\varepsilon,i}\right|^2 R_{l,i} + F(\boldsymbol{q})\right\}, \tag{3.32}$$

where the discrete constraints set reads

$$\tilde{\boldsymbol{K}}\left(\tilde{\boldsymbol{Z}}^{n-1}_\varepsilon\right) := \left\{\boldsymbol{q}\in\mathbb{R}^{2N_p} \text{ s.t. } \phi^{n,\varepsilon}_{ij}(\boldsymbol{q}) := -d_{ij}(\tilde{\boldsymbol{Z}}^{n-1}_\varepsilon) - \nabla d_{ij}(\tilde{\boldsymbol{Z}}^{n-1}_\varepsilon)\cdot\left(\boldsymbol{q} - \tilde{\boldsymbol{Z}}^{n-1}_\varepsilon\right) \le 0, \quad i<j\right\},$$

and

$$\nabla\phi^{n,\varepsilon}_{ij}(\boldsymbol{q}) = -\nabla d_{ij}(\tilde{\boldsymbol{Z}}^{n-1}_\varepsilon), \quad \forall \boldsymbol{q}\in\mathbb{R}^{2N_p}.$$

The same arguments as in Theorem 1 still hold and guarantee the existence and uniqueness of the solution $\tilde{\boldsymbol{Z}}^n_\varepsilon$ to (3.32). We define in the same way the sets of feasible configurations and of active constraints by $\tilde{\boldsymbol{Q}}_0$ and $\tilde{I}_q$ as in (2.3) and Definition 3 respectively. We only mention that, in the present case the periodic distance d_{ij} is considered instead of D_{ij}. The Lagrangian $\tilde{L}$ is defined from $\mathbb{R}^{2N_p}\times(\mathbb{R}_+)^{N_c}$ into $\mathbb{R}$ as well

$$\tilde{L}(\boldsymbol{q},\boldsymbol{\mu}) = \frac{\Delta a}{2\varepsilon}\sum_{i=1}^{N_p}\sum_{l=1}^{\infty}\left|q_i - \tilde{Z}^{n-l}_{\varepsilon,i}\right|^2 R_{l,i} + F(\boldsymbol{q}) + \sum_{i<j}\mu_{ij}\phi^{n,\varepsilon}_{ij}(\boldsymbol{q}). \tag{3.33}$$

All hypotheses of Theorem 9 hold and guarantee that (3.32) is equivalent to existence of saddle-point $(\tilde{\boldsymbol{Z}}^n_\varepsilon, \tilde{\boldsymbol{\lambda}}^n_\varepsilon)$ satisfying

$$\tilde{\boldsymbol{\lambda}}^n_\varepsilon \ge \boldsymbol{0},\ \boldsymbol{\phi}^{n,\varepsilon}(\tilde{\boldsymbol{Z}}^n_\varepsilon)\le\boldsymbol{0},\ \tilde{\boldsymbol{\lambda}}^n_\varepsilon\cdot\boldsymbol{\phi}^{n,\varepsilon}(\tilde{\boldsymbol{Z}}^{\boldsymbol{n}}_{\boldsymbol{\varepsilon}}) = 0 \text{ and } (\boldsymbol{\mathcal{E}}_{n,\varepsilon})'(\tilde{\boldsymbol{Z}}^n_\varepsilon) + \sum_{i<j}\tilde{\lambda}^{n,\varepsilon}_{ij}(\phi^n_{ij})'(\tilde{\boldsymbol{Z}}^n_\varepsilon) = \boldsymbol{0}, \tag{3.34}$$

where

$$\boldsymbol{\phi}^{n,\varepsilon}(\boldsymbol{q}) := \left(\phi^{n,\varepsilon}_{ij}(\boldsymbol{q})\right)_{i<j} : \mathbb{R}^{2N_p}\longrightarrow\mathbb{R}^{N_c}.$$

Note that the periodic distance locally coincides with the one defined in (2.2) in the sense that $d_{ij} = D_{ij}$ in $\mathcal{D}$. So that these two distances have the same properties. This yields the same results as those obtained above with the usual distance (energy estimates, compactness criterion, variational inclusion, etc) in the present case.

3.6. *Numerical approximation and simulations*

3.6.1. *Uzawa's algorithm*

Note that, due to the assumptions on F (see 2.4.2), the last equation in (3.7) is nonlinear with respect to $\boldsymbol{Z}^n_\varepsilon$ at each time step n. This induces the need of a nonlinear solver; such as the Newton solver in order to obtain the position from (3.7) at time $t^n = n\Delta t$. In order to overcome the numerical cost of such an implementation, we transform the external load to a source term depending on the solution at the previous time step. So that we have a linear problem with respect to the unknown position at each time step. More precisely consider the following problem

$$\boldsymbol{Z}^n_\varepsilon = \underset{\boldsymbol{q}\,\in\,\boldsymbol{K}(\boldsymbol{Z}^{n-1}_\varepsilon)}{\arg\min}\left\{\frac{\Delta a}{2\varepsilon}\sum_{i=1}^{N_p}\sum_{l=1}^{\infty}|q_i - Z^{n-l}_{\varepsilon,i}|^2 R_{l,i} + F(\boldsymbol{Z}^{n-1}_\varepsilon) + \boldsymbol{F}'(\boldsymbol{Z}^{n-1}_\varepsilon)\cdot(\boldsymbol{q} - \boldsymbol{Z}^{n-1}_\varepsilon)\right\}. \tag{3.35}$$

We are attempted to use the projected-gradient descent algorithm to numerically approximate the solution of (3.35). But the closed form of projection onto $\boldsymbol{K}(\boldsymbol{Z}^{n-1}_{\varepsilon})$ is not at hand here. To tackle this we pass through the dual problem and project the Lagrange multiplier onto $(\mathbb{R}_+)^{N_c}$ by simple truncation and then iterate the process in the spirit of Uzawa's algorithm. Precisely we build a sequence of primal variables $(\boldsymbol{Z}^{n,r}_{\varepsilon})_n \in \left(\mathbb{R}^{2N_p}\right)^{\mathbb{N}}$ and dual variables $(\boldsymbol{\lambda}^{n,r}_{\varepsilon})_n \in \left((\mathbb{R}_+)^{N_c}\right)^{\mathbb{N}}$ as follows:

$$\begin{cases} \boldsymbol{\lambda}^{n,0}_{\varepsilon} \in (\mathbb{R}_+)^{N_c} \\ L(\boldsymbol{Z}^{n,r}, \boldsymbol{\lambda}^{n,r}_{\varepsilon}) = \inf\limits_{\boldsymbol{q} \in \mathbb{R}^{2N_p}} L\left(\boldsymbol{q}, \boldsymbol{\lambda}^{n,r}_{\varepsilon}\right) \\ \boldsymbol{\lambda}^{n,r+1}_{\varepsilon} = \max\Big(\boldsymbol{\lambda}^{n,r}_{\varepsilon} + \eta \boldsymbol{\varphi}^{n,\varepsilon}(\boldsymbol{Z}^{n,r}_{\varepsilon}), \boldsymbol{0}\Big), \end{cases}$$

which ensures that solution $\boldsymbol{Z}^{n}_{\varepsilon}$ of (3.35) is well defined. We note that L in the above algorithm is the Lagrangian associated to the (3.35).

Proposition 6. *If $0 < \eta < 2\alpha/\varepsilon C^2$ with $\alpha := \mu_{\Delta}$ and $C := \sqrt{2N_c}$, Uzawa's algorithm converges. More precisely, for any initial Lagrange multiplier $\boldsymbol{\lambda}^{n,0}_{\varepsilon}$, the sequence $(\boldsymbol{Z}^{n,r}_{\varepsilon})_r$ converges to the solution of* (3.35) *when r tends to infinity.*

Proof. Let us check the hypotheses of [All05] . To do so

- $E_{n,\varepsilon}$ is twice-differentiable as sum of a quadratic function and an affine function and we have that
$$E^{''}_{n,\varepsilon}(\boldsymbol{q}) = diag\left(\alpha_1, \alpha_1, \cdots, \alpha_{N_p}\right), \quad \forall \boldsymbol{q} \in \mathbb{R}^{2N_p},$$
where $\alpha_i = \dfrac{\mu_{\Delta,i}}{\varepsilon}, \forall i$, so that $E_{n,\varepsilon}$ is uniformly convex with the constant $\alpha_m := \min_i \alpha_i$.
- $\boldsymbol{\varphi}^{n,\varepsilon}$ is convex, Lipschitz from $\mathbb{R}^{2N_p}$ into $(\mathbb{R}_+)^{N_c}$. Indeed the convexity is obvious. To prove the Lipschitz behavior, consider $\overline{\boldsymbol{q}}, \tilde{\boldsymbol{q}} \in \mathbb{R}^{2N_p}$. We have
$$\begin{aligned} \left|\varphi^{n,\varepsilon}_{ij}(\tilde{\boldsymbol{q}}) - \varphi^{n,\varepsilon}_{ij}(\overline{\boldsymbol{q}})\right| &= \left|\boldsymbol{G}_{ij}(\boldsymbol{Z}^{n-1}_{\varepsilon}) \cdot (\tilde{\boldsymbol{q}} - \boldsymbol{Z}^{n-1}_{\varepsilon}) - \boldsymbol{G}_{ij}(\boldsymbol{Z}^{n-1}_{\varepsilon}) \cdot (\overline{\boldsymbol{q}} - \boldsymbol{Z}^{n-1}_{\varepsilon})\right| \\ &= \left|\boldsymbol{G}_{ij}(\boldsymbol{Z}^{n-1}_{\varepsilon}) \cdot (\overline{\boldsymbol{q}} - \tilde{\boldsymbol{q}})\right| \\ &\leq \sqrt{2}\left|\tilde{\boldsymbol{q}} - \overline{\boldsymbol{q}}\right|, \quad \forall i < j. \end{aligned}$$
We've used the fact $\left|\boldsymbol{G}_{ij}(\boldsymbol{Z}^{n-1}_{\varepsilon})\right| = \sqrt{2}$ for all $i < j$ in the third row. Thus
$$|\boldsymbol{\varphi}^{n,\varepsilon}(\tilde{\boldsymbol{q}}) - \boldsymbol{\varphi}^{n,\varepsilon}(\overline{\boldsymbol{q}})| = \sqrt{\sum_{1 \leq i < j \leq N_p} \left|\varphi^{n,\varepsilon}_{ij}(\tilde{\boldsymbol{q}}) - \varphi^{n,\varepsilon}_{ij}(\overline{\boldsymbol{q}})\right|^2} \leq \sqrt{2N_c}\left|\tilde{\boldsymbol{q}} - \overline{\boldsymbol{q}}\right|.$$

Hence $\boldsymbol{\varphi}^{n,\varepsilon}$ is C-Lipschitz with $C = \sqrt{2N_c}$, which ends the proof. □

3.6.2. *Numerical simulations*

We consider here the quadratic case, namely, the external load reads $F(\boldsymbol{q}) = \frac{1}{2}\boldsymbol{q}^2$. We expect the cells to follow the gradient of descent and finally to cluster at the origin. Uzawa's algorithm is performed to get the position at each time step. We start with a given initial deterministic configuration $\boldsymbol{z}_p$. We estimate the MSD [b] (Mean Squared Displacement) which is a measure of the deviation of particles'

[b] $MSD(t) = \langle \boldsymbol{z}(t) - \boldsymbol{z}_{ref} \rangle = \dfrac{1}{N_p} \sum_{i=1}^{N_p} |z_i(t) - z_{ref,i}|^2$

positions with respect to a reference position $\boldsymbol{z}_{ref} = 0$. We compare MSDs performed with and without contact forces and compute the theoretical MSD in the same figure. To do so we consider first the deterministic case without contact forces whose explicit solution is at hand. Then we perturb it randomly by adding a Gaussian white noise in the system.

- Consider the following non contact model whose dynamic is described as

$$\begin{cases} \dot{\boldsymbol{z}} = -\nu \boldsymbol{z}, \quad t > 0 \\ \boldsymbol{z}(0) = \boldsymbol{z}_p(0), \end{cases} \tag{3.36}$$

where $\nu > 0$ can be seen as the inverse of the viscosity coefficient in our friction model. In figure 6 are represented the curves of the global deviation with respect to the origin with and without contacts (top) and the curve of the average activation of the Lagrange multipliers (bottom) see (3.37) below. In the top figure, the global deviation starts from $16m^2$ at $t = 0$ and decreases to

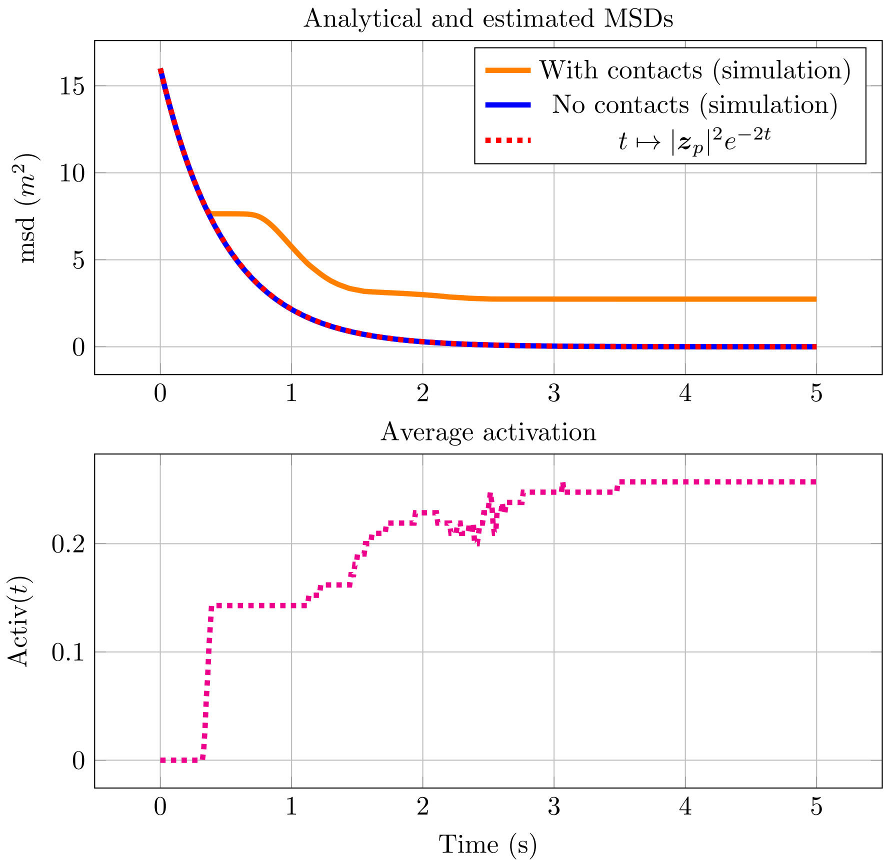


Fig. 6. Deterministic MSDs with respect to $\mathbf{0}$ (top) and the average activation of multipliers (bottom).

end up by following horizontal lines ($H = 0$ for the red and blue curves and $H \simeq 3$ for the orange one). This is what we expected in the absence of contact forces. Indeed in the absence of contacts (blue curve), each particle may be represented only by its center as a simple dot without any radius; allowing by the same way the overlapping between particles. Due to the source term, the particles are like attracted by the origin. The orange curve in Figure 6 represents the estimated MSD in the case of contacts between spheres. We remark that from $t = 0s$ to $t \simeq 0.35s$ the curve is the same as the one without contact forces. Indeed as long as the particles are not in contact the Lagrange multipliers are not activate so that particles' trajectories are driven only by external loads. Once the signed distance vanishes (from $t \simeq 0.35s$ to $t = 5s$), the contact forces are active (see (3.21)). The trajectories are no longer governed only by external load. The contacts induce a diminution of the velocity and the decay of the mean square displacement is no longer the same. This is illustrated by the change in the shape of the orange curve around $t = 0.35s$. The particles rearrange and move toward the origin increasing the contacts' number. As high congestion occurs, the particles move very slightly and end up by being motionless around the origin. This jamming leads to a complete steady state.

The bottom pink curve in Figure 6 represents the average activation of the Lagrange multipliers over time defined as follows

$$Activ(t) := \frac{2}{N_p(N_p-1)} \sum_{1 \le i < j \le N_p} \mathbb{1}_{\{\lambda^{\varepsilon}_{ij}(t) \neq 0\}}. \tag{3.37}$$

We may mention that the activation in (3.37) is by definition the cardinal of the set of active constraints $Ind(\boldsymbol{z}_\varepsilon)$ defined above (see Definition 3) over the maximal number of constraints. Precisely the average activation represents the ratio of the number of active constraints by the maximal number of constraints. Moreover, by definition of the Lagrange multipliers we have that

$$supp(\lambda^{\varepsilon}_{ij}) \subset \{t \ge 0 \text{ s.t. } D_{ij}(\boldsymbol{z}_\varepsilon(t)) = 0\}, \quad \forall i < j,$$

so that the multipliers are activated once the signed distance vanishes. Here (the bottom curve of Figure 6), the Lagrange multipliers are inactive for small times; in fact there is no contacts at the debut. The jump at $t \simeq 0.35s$ is due to the fact that some particles i and j are in contact; the Lagrange multipliers are positive . After that first jump, the average activation of the multipliers is constant equal to 0.15 for less than one second, because the particles in contact try to rearrange until they reach a steady state.

- Consider now the stochastic model where a random perturbation is added in the the previous model. We obtain the Ornstein-Uhlenbeck process

$$\begin{cases} \dot{\boldsymbol{z}} = -\nu \boldsymbol{z} + \sigma \boldsymbol{\eta}_t, & t > 0 \\ \boldsymbol{z}_0 = \boldsymbol{z}_p(0), \end{cases} \tag{3.38}$$

where $(\boldsymbol{\eta}_t)_{t \ge 0}$ denotes the $\mathbb{R}^{2N_p}$-Gaussian white noise. The explicit solution (3.38) as well as its second order moment are given in Appendix B. We compare the approximated MSD computed using the solutions of our algorithm and the theoretical value at each time step in Figure 7. We observe similar trends as in the deterministic case : the deviation exponentially decreases from $16m^2$ to end up by following horizontal lines ($H = 1/2$ for the red and blue curves and $H \simeq 4$ for the orange curve) for large times. Indeed by Appendix B, we have that

$$\lim_{t \to 0} \mathbb{E}|\boldsymbol{z}_t|^2 = |\boldsymbol{z}_p(0)|^2 \text{ and } \lim_{t \to \infty} \mathbb{E}|\boldsymbol{z}_t|^2 = \frac{1}{2},$$

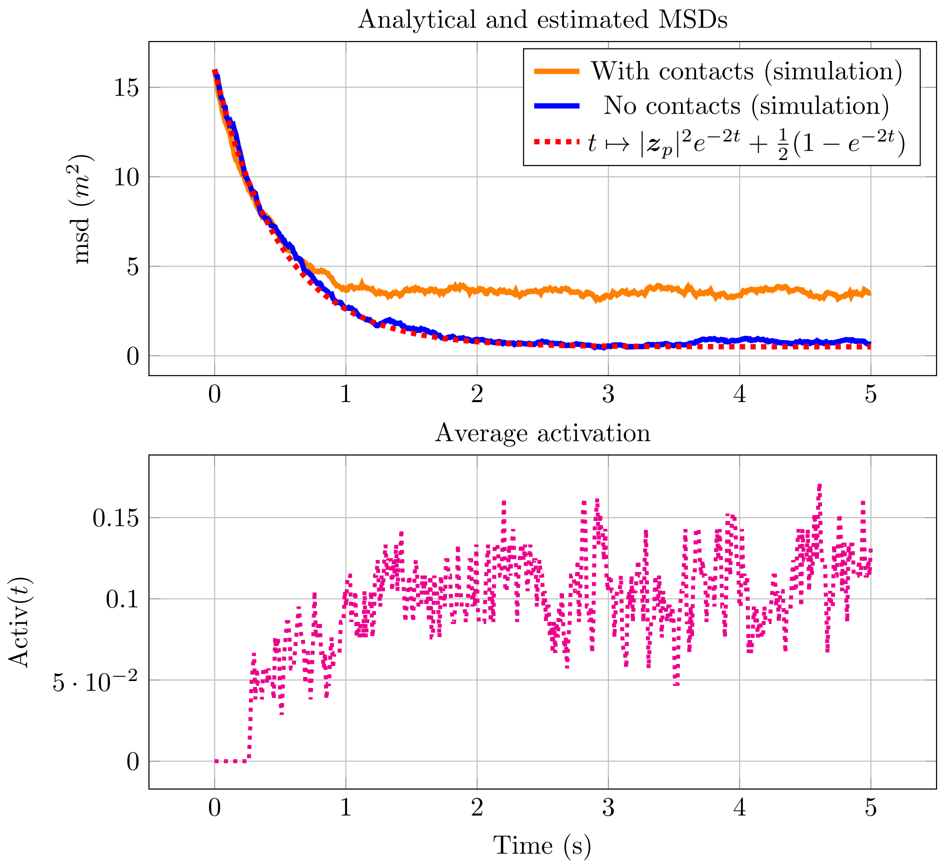


Fig. 7. Stochastic MSDs with respect to $\mathbf{0}$ (top), the average activation of the multipliers (bottom).

so that the particle never cluster at the origin as in the deterministic case, even without contacts. The red curve represents the second order moment of the trajectory (B.1) when particles do not interact.

4. Conclusions

In this paper we dealt with non-penetration models with adhesion forces. The position of cells at each time minimizes a convex energy functional with non-penetration constraints. The energy contains two terms : a delay part and the external load. By penalizing the constraints and letting the penalty parameter to go to zero, we prove that the solution of the constrained problem exists and is unique. We obtain energy estimates and we use convexity of the constraints and of the external load to obtain compactness. We then apply Arzela-Ascoli in order to obtain existence the continuous problem for a fixed $\epsilon > 0$. Finally, we prove that, if the characteristic of lifetime of the bonds tends to zero, our model converges to the model investigates in [Ven08] with a weighted by friction coefficients related to the microscopic adhesions. The same results are obtained on the torus ; the periodic distance is considered instead of the signed one defined by the Euclidian norm. Numerical simulations are made to validate the mathematical analysis.

A. Theoretical background of contact modelling

Definition 5. Let $n \in \mathbb{N}^*$ and $J : \mathbb{R}^n \to \mathbb{R}$ be a real valued function. J is called coercive if $J(v) \longrightarrow \infty$ as $|v| \to \infty$.

Theorem 8. [Cia89] *Let $J : \mathbb{R}^n \to \mathbb{R}$ be a continuous, coercive and strictly convex function, U a nonempty, convex and closed subset of $\mathbb{R}^n$ and $\psi : \mathbb{R}^n \to \mathbb{R}$ a continuous, convex function satisfying*

$$\psi(v) \geq 0 \text{ for every } v \text{ in } \mathbb{R}^n \text{ and } \psi(v) = 0 \iff v \in U, \tag{A.1}$$

Then for every $\delta > 0$, there exists a unique element u_δ satisfying

$$u_\delta \in \mathbb{R}^n \text{ and } J_\delta(u_\delta) = \inf_{v \in \mathbb{R}^n} J_\delta(v), \quad J_\delta(v) := J(v) + \frac{1}{\delta}\psi(v).$$

Moreover, $u_\delta \to u$ when δ goes to 0, where u is the unique solution of

$$\begin{cases} \text{find } u \text{ such that} \\ u \in U, \quad J(u) = \inf_{v \in U} J(v). \end{cases}$$

Theorem 9. [All05] *Let V be a Hilbert space and $M \in \mathbb{N}^*$ and K defined as follows*

$$K = \{v \in V : F_i(v) \leq 0, \ \forall 1 \leq i \leq M\}$$

Assume that J and $F_1, \cdots F_M$ are convex continuous in V and differentiable in K and define the associated Lagrangian

$$\mathcal{L}(v, q) = J(v) + q \cdot F(v), \quad \forall (v, q) \in V \times (\mathbb{R}_+)^M.$$

Let u be a point at which the constraints are qualified. Then u us a global minimum of J if and only if there exists $p \in (\mathbb{R}_+)^M$ such that (u, p) is a saddle-point of $\mathcal{L}$ on $V \times (\mathbb{R}_+)^M$ or equivalently, such that

$$F(u) \leq 0, \ p \geq 0, \ p \cdot u = 0, \ J^{'}(u) + \sum_{i=1}^{M} \lambda_i F^{'}(u_i) = 0,$$

where $F = (F_1, \cdots, F_M)$.

Definition 6. [Ven08] Let H be a Hilbert space and $S \subset H$ be a closed and nonempty subset and $x \in H$ we define:

- the set of nearest points of $x \in S$
$$P_S(x) := \{v \in S : d_S(x) = |x - v|\}, \quad d_S(x) := \inf_{u \in S} |x - u|,$$
- the proximal normal cone to S at x
$$N^P(S, x) := \{v \in H : \exists \alpha > 0, x \in P_S(x + \alpha v)\},$$
- the limiting normal cone to S at x by
$$N^L(S, x) := \left\{v \in H : \exists (x_n) \subset (S)^{\mathbb{N}}, \exists (v_n) \subset (N(S, x_n))^{\mathbb{N}} \text{ s.t } x_n \to x, v_n \rightharpoonup v\right\},$$

Note that if S is convex, the proximal normal cone coincides with the outward normal cone $N(S, x)$ to S at x into which we have $x = P_S(x + \alpha v)$ in the definition of $N^P(S, x)$.

Definition 7. [ET06] Let $S \subset H$ be a non empty closed subset of a Hilbert space H. For any fixed $\eta > 0$, we say that S is η-prox-regular (or uniformly prox-regular) if for any $v \in N^L(S,x)$ such that $|v| < 1$, $x \in P_S(x+\eta v)$.

Proposition 7. [ET06] *Let S be closed nonempty set of a Hilbert space H. S is η-prox-regular if and only if a nonzero proximal normal $v \in N^L(S,x)$ can be realized by an η-ball, that is for all $x \in S$ and $v \in N(S,x) \setminus \{0\}$,*

$$S \cap B\left(x + \frac{\eta}{|v|}v, \eta\right) = \emptyset.$$

In other words for any $x \in S$ and $v \in N(S,x)$,

$$\langle v, y - x\rangle \leq \frac{|v|}{2\eta}\left|y-x\right|^2, \quad \forall y \in S.$$

Furthermore S is convex if and only if it is ∞-prox-regular.

Theorem 10. [Ven08] *The set of admissible constraints $\boldsymbol{Q}_0$ is η-prox-regular where*

$$\eta = \frac{1}{N_p n_n}\left(\frac{\min\left(\sin\left(\dfrac{\pi}{n_n+1}\right), \sin\left(\dfrac{2\pi}{N_p}\right)\right)}{2\sqrt{n_n}}\right)^{N_p} \min_{i,j}(r_i + r_j), \tag{A.2}$$

where n_n is the number of maximal neighbors that a particle can have.

Lemma 6. *(page 90 in [Ven08]) Let S be a convex and closed subset of a Hilbert space H. Let $x \in S$ and $\omega \in H$ we have the following equivalences*

$$\omega \in N(S,x) \overset{def}{\Leftrightarrow} x = P_S(x+\omega) \tag{A.3}$$
$$\Leftrightarrow \forall y \in S, \quad \langle \omega, y-x\rangle \leq 0 \tag{A.4}$$
$$\Leftrightarrow \forall y \in H, \quad \langle \omega, y-x\rangle \leq |\omega| d_S(y) \tag{A.5}$$
$$\Leftrightarrow \forall \xi \in H, \quad \langle \omega, \xi\rangle \leq |\omega| d_S(\xi + x) \tag{A.6}$$
$$\Leftrightarrow \exists \eta > 0, \forall v \in H, |v| < \eta \quad \langle \omega, v\rangle \leq |\omega| d_S(v+x) \tag{A.7}$$
$$\Leftrightarrow \exists k > 0, \exists \eta > 0, \forall v \in H, |v| < \eta \quad \langle \omega, v\rangle \leq k d_S(v+x) \tag{A.8}$$

Proposition 8 (page 76 in [Ven08]). *Let $\boldsymbol{q} \in \boldsymbol{Q}_0$ and $(\boldsymbol{q}_{\Delta_m})$ be a sequence in $\boldsymbol{Q}_0$ satisfying $\boldsymbol{q}_{\Delta_m} \to \boldsymbol{q}$. For any $\boldsymbol{z} \in \mathbb{R}^{2N_p}$ we denote by $\boldsymbol{p} = P_{K(\boldsymbol{q})}(\boldsymbol{z})$ and $p_{\Delta_m} = P_{K(\boldsymbol{q}_{\Delta_m})}(\boldsymbol{z})$ there exists ν such that for any $\boldsymbol{z} \in \boldsymbol{B}(\boldsymbol{q}, \nu)$ we have $\boldsymbol{p}_{\Delta_m} \to \boldsymbol{p}$. Particularly $d_{K(\boldsymbol{q}_{\Delta_m})}(\boldsymbol{z}) \longrightarrow d_{K(\boldsymbol{q})}(\boldsymbol{z})$ as m goes to infinity.*

Definition 8. [Bre11] Let $(E, ||\cdot||_E)$ be a norm vector space and $A \subset E$ be a subset of E. The convex hull of A, we denote here by $conv(A)$ is the intersection of all convex sets of E containing A.

Theorem 11. [Bre11]. *Let $(E, ||\cdot||_E)$ be a vector space and $(x_n)_n$ a sequence that weakly converges to x in E. There exists a sequence of real numbers $(y_k)_{k=n\cdots,N(n)}$ (where $N : \mathbb{N} \to \mathbb{N}$) taking values in the convex hull of the set of values of the sequence $\left(x_n\right)_n$, satisfying*

$$y_n = \sum_{k=n}^{N(n)} \lambda_{n,k} x_k \text{ with } \sum_{k=n}^{N(n)} \lambda_{n,k} = 1, \quad \forall k \in \{n, \cdots, N(n)\}, \quad \lambda_{n,k} \in \mathbb{R}_+$$

and converges to x i.e.

$$||y_n - x||_E \to 0.$$

B. Mean Squared Displacement and Orsntein-Uhlenbeck process

Here we remind some properties of the Ornstein-Uhlenbeck (in the absence of contact forces), for which we compute the explicit formula of its MSD. To this end consider the following equation (3.38) (with $\nu = \sigma = 1$). By the variation of constants method we have

$$\boldsymbol{z}_t = \boldsymbol{z}_p(0)e^{-t} + \int_0^t e^{-(t-r)}\eta_r dr.$$

Due to the null expectation of the white noise, we have that

$$\mathbb{E}(\boldsymbol{z}_t) = \boldsymbol{z}_p(0)e^{-t}.$$

On the other hand for any $t, s \geq 0$, setting $t \wedge s := \min(t,s)$ we have

$$\begin{aligned} E[\boldsymbol{z}_t \cdot \boldsymbol{z}_s] &= |\boldsymbol{z}_p(0)|^2 e^{-(t+s)} + \mathbb{E}\left[(\int_0^t e^{-(t-r_1)}\eta_{r_1} dr_1) \cdot (\int_0^s e^{-(s-r_2)}\eta_{r_2} dr_2) \right] \\ &= |\boldsymbol{z}_p(0)|^2 e^{-(t+s)} + \int_0^{t\wedge s} e^{-(t+s-2r)} dr, \end{aligned}$$

where we've used the Ito's isometry at the second equality. Thus if $s = t$ we have

$$\mathbb{E}|\boldsymbol{z}_t|^2 = |\boldsymbol{z}_p(0)|^2 e^{-2t} + e^{-2t}\int_0^t e^{2r} d_r,$$

which gives the explicit form

$$\mathbb{E}|\boldsymbol{z}_t|^2 = |\boldsymbol{z}_p(0)|^2 e^{-2t} + \frac{1}{2}\left(1 - e^{-2t}\right). \tag{B.1}$$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments (All sources of funding of the study must be disclosed)

We would like to thank you for following the instructions above very closely in advance. It will definitely save us lot of time and expedite the process of your paper's publication.

Conflict of interest

References

All05. G. Allaire. *Analyse numérique et optimisation: une introduction à la modélisation mathématique et à la simulation numérique.* Editions Ecole Polytechnique, 2005.

Bre11. H. Brezis. *Sobolev Spaces and Partial Differential Equations.* Springer, 2011.

CDHV93. C. Castaing, T. X. D. Ha, and M. Valadier. Evolution equations governed by the sweeping process. *Set-Valued Anal.*, 1(2):109–139, 1993.

Cia89. P. G. Ciarlet. *Introduction to numerical algebra and optimization.* Cambridge University, 1989.

CMM95. C. Castaing and M. D. P. Monteiro-Marques. BV periodic solutions of an evolution problem associated with continuous moving convex sets. *Set-Valued Anal.*, 3(4):381–399, 1995.

EMB22. J. A. Espina, L. Marchant and E. H. Barriga. Durotaxis: the mechanical control of directed cell migration. *The FEBS Journal*, 289:2736–2754, 2022.

ET06. J. F. Edmond and L. Thibault. BV solutions of nonconvex sweeping process differential inclusion with perturbation. *J. Differ. Equations*, 226(1):135–179, 2006.

Jen06. H. S. Jennings. *Behavior of the lower organisms.* Columbia University Press, 1906.

Mil20. V. Milisic. From delayed and constrained minimizing movements to the harmonic map heat equation. *J. Funct. Anal.*, 279(2):50, 2020. Id/No 108520.

MO11. V. Milisic and D. Oelz. On the asymptotic regime of a model for friction mediated by transient elastic linkages. *J. Math. Pures Appl. (9)*, 96(5):484–501, 2011.

MO16. V. Milisic and D. Oelz. Tear-off versus global existence for a structured model of adhesion mediated by transient elastic linkages. *Commun. Math. Sci.*, 14(5):1353–1372, 2016.

MO18. V. Milisic and D. Oelz. Space dependent adhesion forces mediated by transient elastic linkages: new convergence and global existence results. *Journal of Differential Equations*, 256:6049–6082, 2018.

Mor77. J. J. Moreau. Evolution problem associated with a moving convex set in Hilbert space. *J. Differ. Equations*, 26:347–374, 1977.

MS24. V. Milisic and Ph. Souplet. From transient elastic linkages to friction: a complete study of a penalized fourth order equation with delay. *accepted for publication in Journ. Math. Pure et Appl. (2024)*, 2024.

OS10. D. Oelz and C. Schmeiser. Derivation of a model for symmetric lamellipodia with instantaneous cross-link turnover. *Arch. Ration. Mech. Anal.*, 198(3):963–980, 2010.

PB22. C. A. Parent and J. E. Bear. The principles of the directed cell migration. *The FEBS Journal*, 8:529–547, 2022.

SBTG90. T. J. Schall, K. Bacon, K. J. Toy, and D. V. Goeddel. Selective attraction of monocytes and T lymphocytes of memory phenotypes by cytokines rantes. *Nature Publishing Group*, 347:669–671, 1990.

Ven08. J. Venel. *Modélisation mathématique et numérique de mouvements de foule*. PhD thesis, Université Paris Sud Facultés des Sciences d'Orsay, 2008.

YDMW02. X. Yang, D. Dormann, A. E. Munsterberg, and C. J. Weijer. Cell movement patterns during gastrulation in the chick are controlled by positive and negative chemotaxis mediated by fgf4 and fgf8. *Developpement Cell*, 3:3, 2002.