A priori convergence estimates for a rough Poisson-Dirichlet problem with natural vertical boundary conditions

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Abstract

Stents are medical devices designed to modify blood flow in aneurysm sacs, in order to prevent their rupture. Some of them can be considered as a locally periodic rough boundary. In order to approximate blood flow in arteries and vessels of the cardio-vascular system containing stents, we use multi-scale techniques to construct boundary layers and wall laws. Simplifying the flow we turn to consider a 2-dimensional Poisson problem that conserves essential features related to the rough boundary. Then, we investigate convergence of boundary layer approximations and the corresponding wall laws in the case of Neumann type boundary conditions at the inlet and outlet parts of the domain. The difficulty comes from the fact that correctors, for the boundary layers near the rough surface, may introduce error terms on the other portions of the boundary. In order to correct these spurious oscillations, we introduce a vertical boundary layer. Trough a careful study of its behavior, we prove rigorously decay estimates. We then construct complete boundary layers that respect the macroscopic boundary conditions. We also derive error estimates in terms of the roughness size ϵ either for the full boundary layer approximation and for the corresponding averaged wall law.

Keywords: wall-laws, rough boundary, Laplace equation, multi-scale modelling, boundary layers, error estimates.

AMS subject classifications :76D05, 35B27, 76Mxx, 65Mxx

1 Introduction

A common therapeutic treatment to prevent rupture of aneurysms, in large arteria or in blood vessels in the brain, consists in placing a device inside the aneurysm sac. The device is designed to modify the blood flow in this region, so that the blood contained in the sac coagulates and the sac can be absorbed into the surrounding tissue. The traditional technique consists in obstructing the sac with a long coil. In a more recent procedure, a device called *stent*, that can be seen as a second artery wall, is placed so as to close the inlet of the sac. We are particularly interested in stents produced by a company called Cardiatis, which are designed as multi-layer wired structures. Clinical tests show surprising bio-compatibility features of these particular devices and one of our objectives is to understand how the design of these stents affect their effectiveness. As stent thicknesses are small compared to the characteristic dimensions of the flow inside an artery, studying their properties is a challenging multi-scale problem.

In this work we focus on the fluid part and on the effects of the stent rugosity on the fluid flow. We simplify the geometry to that of a 2-dimensional box Ω^ϵ, that represents a longitudinal

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cut through an artery: the rough base represents the shape of the wires of the stent (see fig. 2, left). We also simplify the flow model and consider a Poisson problem for the axial component of the velocity. Our objective is to analyze precisely multi-scale approximations of this simplified model, in terms of the rugosity.

In [4] we considered periodic inflow and outflow boundary conditions on the vertical sides $\Gamma_{\text{in}} \cup \Gamma_{\text{out}}$ of Ω^ϵ. Here, we study the case of more realistic Neumann conditions on these boundaries, which are consistent with the modelling of a flow of blood.

As a zeroth order approximation to u^ϵ, we consider the solution $\tilde{u}^0$ of the same PDE, posed on a smooth domain Ω^0 strictly contained in Ω^ϵ. We introduce boundary layer correctors β and τ that correct the incompatibilities between the domain and $\tilde{u}^0$. These correctors induce in turn perturbations on the vertical sides $\Gamma_{\text{in}} \cup \Gamma_{\text{out}}$ of Ω^ϵ. We therefore consider additional correctors ξ_{in} and ξ_{out}, that should account for these perturbations (see fig 1). We also introduce a first order approximation, defined in Ω^0, that satisfies a mixed boundary condition (called Saffman-Joseph wall law) on a fictitious interface Γ^0 located inside Ω^ϵ.

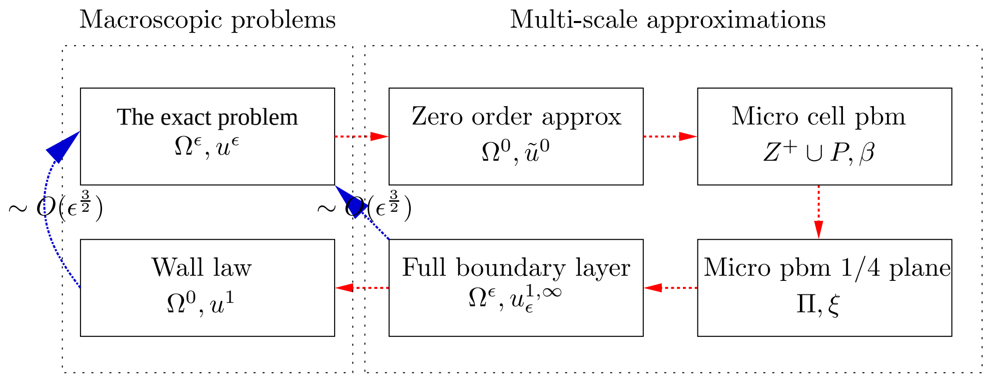

Figure 1: The exact solution, the multi-scale framework and wall laws

For the case of Navier-Stokes equations and the Poiseuille flow the problem was already considered in Jäger *et al.* [11, 10] but the authors imposed Dirichlet boundary conditions on $\Gamma_{\text{in}} \cup \Gamma_{\text{out}}$ for the vertical velocity and pressure. Their approach provided a localized vertical boundary layer in the ϵ-close neighborhood of $\Gamma_{\text{in}} \cup \Gamma_{\text{out}}$. A convergence proof for the boundary layer approximation and the wall law was given wrt to ϵ, the roughness size. These arguments are specific to the case of Poiseuille flow and differ from the general setting given in the homogenization framework [19]. In this work, we address the case of Neumann boundary conditions, where the above methods do not apply. The difficulty in this case, stems from the 'pollution' on the vertical sides due to the bottom boundary layer correctors.

From our point of view, the originality of this work emanate from the following aspects :

- the introduction of a general quarter-plane corrector ξ that reduces the oscillations of the periodic boundary layer approximations on a specific region of interest. Changing the type of boundary conditions implies only to change the boundary conditions of the quarter-plane corrector on a certain part of the microscopic domain.
- the analysis of decay properties of this new corrector: indeed we use techniques based on weighted Sobolev spaces to derive some of the estimates and we complete this description by integral representation and Fragmen-Lindelöf theory in order to derive sharper L^∞ bounds.
- we show new estimates based on duality on the traces and provide a weighted correspondence between macro and micro features of test functions of certain Sobolev spaces.

The error between the wall-law and the exact solution is evaluated on Ω^0, the smooth domain above the roughness, in the $L^2(\Omega^0)$ norm. This relies on very weak estimates [18] that moreover improve *a priori* estimates by a $\sqrt{\epsilon}$ factor. While this work focuses on the precise description of vertical boundary layer correctors in the *a priori* part, a second article extends our methods to the very weak context [16] in order to obtain optimal rates of convergence also for this step.

The paper is organized as follows : in section 2, we present the framework (including notations, domains characteristics and the toy PDE model under consideration), in section 3, we give a brief summary of what is already available from the periodic context [4] that should serve as a basis for what follows, in section 4 we present a microscopic vertical boundary layer and its careful analysis in terms of decay at infinity, such decay properties will be used in section 5 in the convergence proofs for the full boundary layer approximation as well as in the corresponding wall law analysis.

2 The framework

In this work, Ω^ϵ denotes the rough domain in $\mathbb{R}^2$ depicted in fig. 2, Ω^0 denotes the smooth one, Γ^ϵ is the rough boundary and Γ^0 (resp. Γ^1) the lower (resp. upper) smooth one (see fig 2). The rough boundary Γ^ϵ is described as a periodic repetition at the microscopic scale of a single boundary cell P^0. The latter can be parameterized as the graph of a Lipschitz function $f : [0, 2\pi[\to] - 1 : 0[$, the boundary is then defined as

$$P^0 = \{y \in [0, 2\pi]\times] - 1 : 0[\text{ s.t. } y_2 = f(y_1)\}. \quad (1)$$

Moreover we suppose that f is bounded and negative definite, i.e. there exists a positive constant δ such that $1 - \delta < f(y_1) < \delta$ for all $y_1 \in [0, 2\pi]$. The lower bound of f is arbitrary and it is useful only in order to define some weight function see section 4. We assume that the ratio between L (the width of Ω^0) and $2\pi\epsilon$ (the width of the periodic cell) is always a positive integer. We

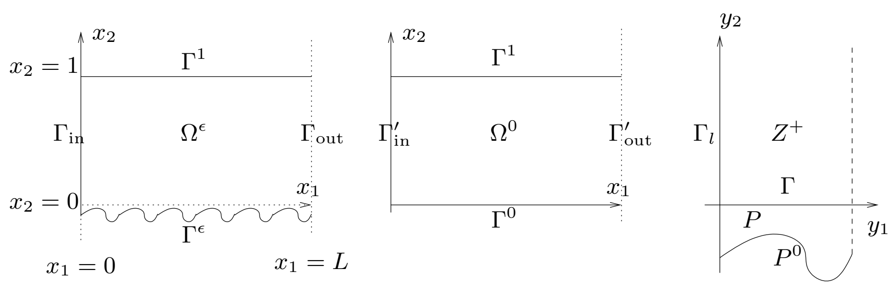

Figure 2: *Rough, smooth and cell domains*

consider a simplified setting that avoids theoretical difficulties and non-linear complications of the full Navier-Stokes equations. Starting from the Stokes system, we consider a Poisson problem for the axial component of the velocity. The axial component of the pressure gradient is assumed to reduce to a constant right hand side C. If we set periodic inflow and outflow boundary conditions, the simplified formulation reads : find u^ϵ such that

$$\begin{cases} -\Delta u^\epsilon_\# = C, \text{ in } \Omega^\epsilon, \\ u^\epsilon_\# = 0, \text{ on } \Gamma^\epsilon \cup \Gamma^1, \\ u^\epsilon_\# \text{ is } x_1 \text{ periodic.} \end{cases} \quad (2)$$

In section 3 we should give a brief summary of the framework already introduced in [4]. Nevertheless the main concern of this work is to consider the non periodic setting (see section 4) where we should consider an example of a more realistic inlet and outlet boundary conditions. Namely, we look for approximations of the problem: find u^ϵ such that

$$\begin{cases} -\Delta u^\epsilon = C, \text{ in } \Omega^\epsilon, \\ u^\epsilon = 0, \text{ on } \Gamma^\epsilon \cup \Gamma^1, \\ \dfrac{\partial u^\epsilon}{\partial \mathbf{n}} = 0, \text{ on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}. \end{cases} \tag{3}$$

In what follows, functions that do depend on $y = x/\epsilon$ should be indexed by an ϵ (e.g. $\mathcal{U}_\epsilon = \mathcal{U}_\epsilon(x, x/\epsilon)$).

3 Summary of the results obtained in the periodic case [4]

3.1 The cell problems

3.1.1 The first order cell problem

The rough boundary is periodic at the microscopic scale and this leads to solve the microscopic cell problem: find β s.t.

$$\begin{cases} -\Delta \beta = 0, \text{ in } Z^+ \cup \Gamma \cup P, \\ \beta = -y_2, \text{ on } P^0, \\ \beta \text{ is } y_1 - \text{periodic }. \end{cases} \tag{4}$$

We define the microscopic average along the fictitious interface Γ : $\overline{\beta} = \frac{1}{2\pi}\int_0^{2\pi} \beta(y_1, 0) dy_1$. As $Z^+ \cup \Gamma \cup P$ is unbounded in the y_2 direction, we define also

$$D^{1,2} = \{v \in L^1_{\text{loc}}(Z^+ \cup \Gamma \cup P)/\, Dv \in L^2(Z^+ \cup \Gamma \cup P)^2,\, v \text{ is } y_1 - \text{periodic } \},$$

then one has the result :

Theorem 3.1. *Suppose that P^0 is sufficiently smooth (f is Lipschitz) and does not intersect Γ. Let β be a solution of (4), then it belongs to $D^{1,2}$. Moreover, there exists a unique periodic solution $\eta \in H^{\frac{1}{2}}(\Gamma)$, of the problem*

$$< S\eta, \mu >=< 1, \mu >, \quad \forall \mu \in H^{\frac{1}{2}}(\Gamma),$$

where $<,>$ is the $(H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma))$ duality bracket, and S the inverse of the Steklov-Poincaré operator. One has the correspondence between β and the interface solution η :

$$\beta = H_{Z^+}\eta + H_P\eta,$$

where $H_{Z^+}\eta$ (resp. $H_P\eta$) is the y_1-periodic harmonic extension of η on Z^+ (resp. P). The solution in Z^+ can be written explicitly as a power series of Fourier coefficients of η and reads :

$$H_{Z^+}\eta = \beta(y) = \sum_{k=-\infty}^{\infty} \eta_k e^{iky_1 - |k|y_2}, \quad \forall y \in Z^+, \quad \eta_k = \int_0^{2\pi} \eta(y_1) e^{-iky_1} dy_1,$$

In the macroscopic domain Ω^0 this representation formula gives

$$\left\| \beta\left(\frac{\cdot}{\epsilon}\right) - \overline{\beta} \right\|_{L^2(\Omega^0)} \leq K\sqrt{\epsilon} \|\eta\|_{H^{\frac{1}{2}}(\Gamma)}. \tag{5}$$

3.1.2 The second order cell problem

The second order error on Γ^ϵ should be corrected thanks to a new cell problem : find $\gamma \in D^{1,2}$ solving

$$\begin{cases} -\Delta\tau = 0, \quad \forall y \in Z^+ \cup \Gamma \cup P, \\ \tau = -y_2^2, \quad \forall y_2 \in P^0, \\ \tau \text{ periodic in } y_1. \end{cases} \tag{6}$$

Again, the horizontal average is denoted $\overline{\tau}$. In the same way as for the first order cell problem, one can obtain a similar result:

Proposition 1. *Let P^0 be smooth enough and do not intersect Γ. Then there exists a unique solution τ of* (6) *in $D^{1,2}(Z^+ \cup \Gamma \cup P)$.*

3.2 Standard averaged wall laws

3.2.1 A first order approximation

Using the averaged value $\overline{\beta}$ defined above, one can construct a first order approximation u^1 defined on the smooth interior domain Ω^0 that solves :

$$\begin{cases} -\Delta u^1 = C, \quad \forall x \in \Omega^0, \\ u^1 = \epsilon\overline{\beta}\dfrac{\partial u^1}{\partial x_2}, \quad \forall x \in \Gamma^0, \quad u^1 = 0, \quad \forall x \in \Gamma^1, \\ u^1 \text{ is } x_1 - \text{periodic on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \end{cases} \tag{7}$$

whose explicit solution reads :

$$u^1(x) = -\frac{C}{2}\left(x_2^2 - \frac{x_2}{1+\epsilon\overline{\beta}} - \frac{\epsilon\overline{\beta}}{1+\epsilon\overline{\beta}}\right). \tag{8}$$

Under the hypotheses of theorem 3.1, one derives error estimates for the first order wall law

$$\left\|u^\epsilon_\# - u^1\right\|_{L^2(\Omega^0)} \leq K\epsilon^{\frac{3}{2}}.$$

3.2.2 A second order approximation

In the same way one should derive second order averaged wall law u^2 satisfying the boundary value problem :

$$\begin{cases} -\Delta u^2 = C, \quad \forall x \in \Omega^0, \\ u^2 = \epsilon\overline{\beta}\dfrac{\partial u^2}{\partial x_2} + \dfrac{\epsilon^2}{2}\overline{\tau}\dfrac{\partial^2 u^2}{\partial x_2^2}, \quad \forall x \in \Gamma^0, \\ u^2 = 0, \quad \forall x \in \Gamma^1, u^2 \text{ is } x_1 - \text{periodic on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \end{cases} \tag{9}$$

whose solution exists, is unique [4] and writes :

$$u^2(x) = -\frac{C}{2}\left(x_2^2 - \frac{x_2(1+\epsilon^2\overline{\tau})}{1+\epsilon\overline{\beta}} - \frac{\epsilon(\overline{\beta}-\epsilon\overline{\tau})}{1+\epsilon\overline{\beta}}\right). \tag{10}$$

Now, error estimates do not provide second order accuracy, namely we only obtain

$$\left\|u^\epsilon_\# - u^2\right\|_{L^2(\Omega^0)} \leq K\epsilon^{\frac{3}{2}},$$

which essentially comes from the influence of microscopic oscillations that this averaged second order approximation neglects. Thanks to estimates (5), one sees easily that these oscillations account as $\epsilon^{\frac{3}{2}}$ if not included in the wall law approximation.

3.3 Compact form of the full boundary layer ansatz

Usually in the presentation of wall laws, one first introduces the full boundary layer approximation. This approximation is an asymptotic expansion defined on the whole rough domain Ω^ϵ. In a further step one averages this approximation in the axial direction over a fast horizontal period and derives in a second step the corresponding standard wall law.

Thanks to various considerations already exposed in [4], the authors showed that actually a reverse relationship could be defined that expresses the full boundary layer approximations as functions of the wall laws. Obviously this works because the wall laws (defined only on Ω^0) are explicit and thus easy to extend to the whole domain Ω^ϵ. Indeed we re-define

$$u^1(x) = \frac{C}{2}\left((1-x_2)x_2\chi_{[\Omega^0]} + x_2\chi_{[\Omega^\epsilon\setminus\Omega^0]}\right) - \frac{\epsilon\beta}{1+\epsilon\overline{\beta}}(1-x_2), \quad \forall x \in \Omega^\epsilon \tag{11}$$

while we simply extend u^2 using the formula (10) over the whole domain. This leads to write :

$$\begin{aligned} u^{1,\infty}_{\#} &= u^1 + \epsilon\frac{\partial u^1}{\partial x_2}(x_1,0)\left(\beta\left(\frac{x}{\epsilon}\right) - \overline{\beta}\right), \\ u^{2,\infty}_{\#} &= u^2 + \epsilon\frac{\partial u^2}{\partial x_2}(x_1,0)\left(\beta\left(\frac{x}{\epsilon}\right) - \overline{\beta}\right) + \frac{\epsilon^2}{2}\frac{\partial^2 u^2}{\partial x_2^2}(x_1,0)\left(\tau\left(\frac{x}{\epsilon}\right) - \overline{\tau}\right). \end{aligned} \tag{12}$$

For these first order and second order full boundary layer approximations one can set the error estimates [4]:

$$\left\|u^\epsilon_{\#} - u^{1,\infty}_\epsilon\right\|_{L^2(\Omega^0)} \le K\epsilon^{\frac{3}{2}}, \quad \left\|u^\epsilon_{\#} - u^{2,\infty}_\epsilon\right\|_{L^2(\Omega^0)} \le Ke^{-\frac{1}{\epsilon}}.$$

Note that the second order full boundary layer approximation is very close to the exact solution in the periodic case, an important step that this work was aiming to reach is to show how far this can be extended to a more realistic boundary conditions considered in (3). Actually, convergence rates provided hereafter and in [16] show that only first order accuracy can be achieved trough the addition of a vertical boundary layer (see below). For this reason we study in the rest of this paper only the first order full boundary layer and its corresponding wall law.

4 The non periodic case: a vertical corrector

The purpose of what follows is to extend above results to the practical case of (3). We should show a general method to handle such a problem. It is inspired in a part from the homogenization framework already presented in [19, 17] for a periodic media in all directions. The approach below uses some arguments exposed in [3] for another setting.

4.1 Microscopic decay estimates

In what follows we mainly need to correct oscillations of the normal derivative of the first order boundary layer corrector β on the inlet and outlet $\Gamma_{\text{in}}\cup\Gamma_{\text{out}}$. For this sake, we define the notations $\Pi := \cup_{k=0}^{+\infty}[Z^+\cup\Gamma\cup P + 2\pi k\mathbf{e}_1]$, the vertical boundary will be denoted $E := \{0\}\times]f(0),+\infty[$ and the bottom $B := \{y \in P^0 \pm 2k\pi\mathbf{e}_1\}$ (cf. fig 3). In what follows we should denote $\Pi' := \mathbb{R}^2_+$, $B' := \mathbb{R}_+\times\{0\}$ and $E' := \{0\}\times\mathbb{R}_+$.
On this domain, we introduce the problem: find ξ such that

$$\begin{cases} -\Delta\xi = 0, \quad \text{in } \Pi, \\ \dfrac{\partial\xi}{\partial\mathbf{n}}(0,y_2) = \dfrac{\partial\beta}{\partial\mathbf{n}}(0,y_2), \quad \text{on } E, \\ \xi = 0, \quad \text{on } B. \end{cases} \tag{13}$$

We define the standard weighted Sobolev spaces : for any given integers (n,p) and a real α set

$$W^{n,p}_\alpha(\Omega) := \left\{ v \in \mathcal{D}'(\Omega) \,/\, |D^\lambda v|(1+\rho^2)^{\frac{\alpha+|\lambda|-n}{2}} \in L^p(\Omega),\, 0 \le |\lambda| \le n \right\}$$

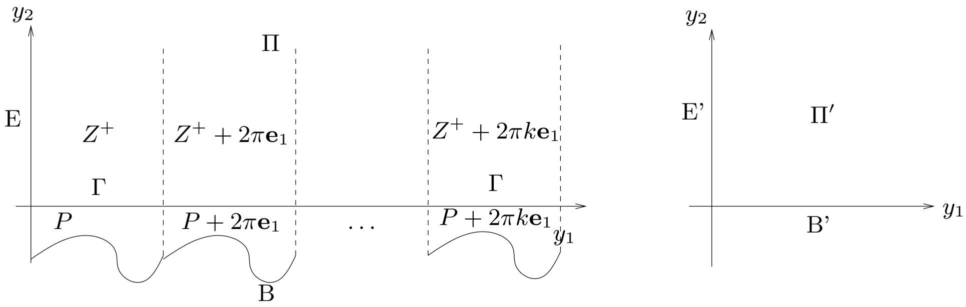

Figure 3: *Semi infinite microscopic domains:* Π*, the rough quarter-plane and* Π′*, the smooth one*

where $\rho := \sqrt{y_1^2 + (y_2+1)^2}$. In what follows we should distinguish between properties depending on ρ which is a distance to a point exterior to the domain Π and $r = \sqrt{y_1^2 + y_2^2}$ the distance to the interior point $(0,0)$. These weighted Sobolev spaces are Banach spaces for the norm

$$\|\xi\|_{W_\alpha^{m,p}(\Omega)} := \left(\sum_{0 \le |\lambda| \le m} \left\| (1+\rho^2)^{\frac{\alpha - m + |\lambda|}{2}} D^\lambda u \right\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}},$$

the semi-norm being

$$|\xi|_{W_\alpha^{m,p}(\Omega)} := \left(\sum_{|\lambda| = m} \left\| (1+\rho^2)^{\frac{\alpha - m + |\lambda|}{2}} D^\lambda u \right\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}.$$

We refer to [9, 15, 1] for detailed study of these spaces. We introduce a specific subspace

$$\dot{W}_\alpha^{p,n}(\Pi) = \{ v \in W_\alpha^{p,n}(\Pi) \text{ s.t. } v \equiv 0 \text{ on B} \}.$$

We begin by some important properties satisfied by ξ that will be used to prove convergence theorems 5.1, 5.2 and 5.3. Such estimates will be obtained by a careful study of the weighted Sobolev properties of ξ as well as its integral representation through a specific Green function.

Theorem 4.1. *Under the hypotheses of theorem 3.1, there exists ξ, a unique solution of problem* (13)*. Moreover $\xi \in \dot{W}_\alpha^{1,2}(\Pi)$ with $\alpha \in]-\alpha_0, \alpha_0[$ where $\alpha_0 := (\sqrt{2}/\pi)$ and*

$$|\xi(y)| \le \frac{K}{\rho(y)^{1-\frac{1}{2M}}}, \forall y \in \mathbb{R}_+^2 \text{ s.t. } \rho > 1, \qquad \int_0^\infty \left| \frac{\partial \xi}{\partial y_1}(y_1, y_2) \right|^2 dy_1 \le \frac{K}{y_2^{1+2\alpha}}, \forall y_2 \in \mathbb{R}_+.$$

where M is a positive constant such that $M < 1/(1-2\alpha) \sim 10$.

The proof follows as a consequence of every result claimed until the end of subsection 4.1.

Lemma 4.1. *In $\dot{W}_\alpha^{1,2}(\Pi)$ the semi-norm is a norm, moreover one has*

$$\|\xi\|_{W_\alpha^{1,2}(\Pi)} \le \frac{1}{2\alpha_0} |\xi|_{W_\alpha^{1,2}(\Pi)}, \quad \forall \alpha \in \mathbb{R}.$$

On the vertical boundary E one has the continuity of the trace operator

$$\|\xi\|_{W_\alpha^{\frac{1}{2},2}(E)} \le K \|\xi\|_{W_\alpha^{1,2}(\Pi)}, \quad \forall \alpha \in \mathbb{R},$$

the weighted trace norm being defined as

$$W_0^{\frac{1}{2},2}(\partial\Pi) = \left\{ u \in \mathcal{D}'(\partial\Pi) \text{ s.t. } \frac{u}{(1+\rho^2)^{\frac{1}{4}}} \in L^2(\partial\Pi), \int_{\partial\Pi_l^2} \frac{|u(y) - y(y')|^2}{|y-y'|^2} ds(y) ds(y') < +\infty \right\},$$

and

$$u \in W_\alpha^{\frac{1}{2},2}(\partial\Pi) \iff (1+\rho^2)^{\frac{\alpha}{2}} u \in W_0^{\frac{1}{2},2}(\partial\Pi).$$

The proof is omitted: the homogeneous Dirichlet condition on B allows to establish Poincaré Wirtinger estimates ([6], vol. I page 56) in a quarter-plane containing Π. Nevertheless, similar arguments are also used in the proof of lemma 4.4.

Lemma 4.2. *The normal derivative* $g := \frac{\partial\beta}{\partial y_1}(0, y_2)$ *is a linear form on* $\dot{W}_\alpha^{1,2}(\Pi)$ *for every* $\alpha \in \mathbb{R}$.

Proof. In Z^+, the upper part of the cell domain, the harmonic decomposition of β allows to characterize its normal derivative explicitly on E'. Indeed

$$g = \Re\left\{ \sum_{k=-\infty}^{+\infty} ik\eta_k e^{-|k|y_2} \right\} \chi_{[E']} + g_- =: g_+ + g_-$$

where g_- is a function whose support is located in $y_2 \in [f(0), 0]$. One has for the upper part

$$\int_{E'} g_+^2 y_2^\alpha dy_2 \leq K \|\eta\|^2_{H^{\frac{1}{2}}(\Gamma)}, \quad \forall \alpha \in \mathbb{R},$$

thus g_+ is in the weighted L^2 space for any power of $(1+\rho^2)^{\frac{1}{2}}$, it is a linear form on $W_{-\alpha}^{\frac{1}{2},2}(E)$. For g_-, we have no explicit formulation. We analyze the problem (4) but restricted to the bounded sublayer P. We define β_- to be harmonic in P satisfying $\beta_- = \eta$ on Γ, where η is the trace on the fictitious interface obtained in theorem 3.1 and $\beta_- = -y_2$ on P^0. Note that thanks to standard regularity results $\eta \in C^2(\Gamma)$ because Γ is strictly included in $Z^+ \cup \Gamma \cup P$, [7]. So β solves a Dirichlet y_1-periodic problem in P with regular data. As the boundary is Lipschitz $\beta \in H^1(P)$ and so on the compact interface $E_c := \{0\} \times [f(0), 0]$, $\partial_{\mathbf{n}}\beta$ is a linear form on $H^{\frac{1}{2}}$ functions. Then because E_c is compact :

$$\int_{Ec} \frac{\partial\beta}{\partial\mathbf{n}} v dy_2 \leq \left\| \frac{\partial\beta}{\partial\mathbf{n}} \right\|_{H^{-\frac{1}{2}}(E_c)} \|v\|_{H^{\frac{1}{2}}(E_c)} \leq K\|v\|_{W_\alpha^{\frac{1}{2},2}(E_c)}, \quad \forall \alpha \in \mathbb{R}$$

□

Lemma 4.3. *If* $\alpha \in]-\alpha_0 : \alpha_0[$ *there exists* $\xi \in \dot{W}_\alpha^{1,2}(\Pi)$ *a unique solution of problem* (13).

Proof. The weak formulation of problem (13) reads

$$(\nabla\xi, \nabla v)_\Pi = (g, v)_E, \quad \forall\, v \in C^\infty(\Pi),$$

leading to check hypothesis of the abstract inf-sup extension of the Lax-Milgram theorem [18, 2], for

$$a(u, v) = \int_{\Pi_+} \nabla u \cdot \nabla v\, dy, \quad l(v) = \int_{f(0)}^{+\infty} \frac{\partial\beta}{\partial\mathbf{n}} v\, dy_2.$$

By lemma 4.2, l is a linear form on $W_\alpha^{1,2}(\Pi)$. It remains to prove the inf-sup like condition on the bilinear form a. For this purpose we set $v = u\rho^{2\alpha}$ and we look for a lower estimate of $a(u, v)$.

$$a(u, u\rho^{2\alpha}) = \int_{\Pi_+} \nabla u \cdot \nabla \left(u\rho^{2\alpha}\right) dy = |u|^2_{W_\alpha^{1,2}(\Pi)} + 2\alpha \int_{\Pi_+} \rho^{2\alpha-1} u \nabla u \cdot \nabla\rho\, dy$$

Using Hölder estimates one has

$$\int_{\Pi_+} \rho^{2\alpha-1} u\nabla u \cdot \nabla\rho dy \leq \left(\int_{\Pi_+} \rho^{2\alpha} \left(\frac{u}{\rho}\right)^2 dy \right)^{\frac{1}{2}} \left(\int_{\Pi_+} \rho^{2\alpha} |\nabla u|^2 dy \right)^{\frac{1}{2}} \leq \frac{1}{2\alpha_0} |u|^2_{W_\alpha^{1,2}(\Pi)}$$

In this way one gets

$$a(u, u\rho^{2\alpha}) \geq (1 - \frac{\alpha}{\alpha_0}) |u|^2_{W_\alpha^{1,2}(\Pi)}$$

and if $\alpha < \alpha_0$ the inf-sup condition is fulfilled, the rest of the proof is standard and left to the reader [2]. □

Thanks to the Poincaré inequality in $\Pi \setminus \Pi'$ with $\alpha = 0$, we have

Corollary 4.1. *If a function ξ belongs to $\dot{W}_0^{1,2}(\Pi)$ it satisfies $\xi \in L^2(B')$.*

To characterize the weighted behavior of ξ on B' we set $\omega_\alpha(y_1) = (y_1^2+1)^{\frac{2\alpha-1}{2}} y_1$ and we give

Lemma 4.4. *If ξ in $\dot{W}_\alpha^{1,2}(\Pi)$ then $\xi \in L^2(B',\omega_\alpha) := \{u \in \mathcal{D}'(B')$ s.t. $\int_{B'} \xi^2 \omega_\alpha dy_1 < \infty\}$.*

Proof. Π is contained in a set $\mathbb{R}_+ \times \{-1,+\infty\}$. We map the latter with cylindrical coordinates (ρ,θ). Every function of $\dot{W}_0^{1,2}(\Pi)$, extended by zero on the complementary set of Π, belongs to the space of functions vanishing on the half-line $\theta = 0$. Using Wirtinger estimates, one has for every such a function.

$$\int_1^{+\infty} \xi^2 \left(\rho, \arcsin\left(\frac{1}{\rho}\right)\right) \rho^{2\alpha} d\rho \leq \int_1^\infty \int_0^{\frac{\pi}{2}} \rho^{2\alpha-1} \arcsin\left(\frac{1}{\rho}\right) \left|\frac{\partial \xi}{\partial \theta}\right|^2 \rho d\theta d\rho \leq K \|\xi\|^2_{W_\alpha^{1,2}(\Pi)}$$

because on B', $\rho d\rho = y_1 dy_1$, one gets the desired result. □

In order to derive local and global L^∞ estimates we introduce in this part a representation formula of ξ on Π'. As long as we use the representation formula below, x will be the symmetric variable to the integration variable y. Until the end of proposition 2 both x and y are microscopic variables living in Π.

Lemma 4.5. *The solution of problem (13) satisfies $\xi(y) \leq K\rho^{-1+\frac{1}{2M}}$ for every $y \in \Pi'$ such that $\rho(y) \geq 1$. The constant M can be chosen such that $M < 1/(1-2\alpha) \sim 10$.*

Proof. We set the representation formula

$$\xi(x) = \int_{E'} \Gamma_x g(y_2) dy_2 + \int_{B'} \frac{\partial \Gamma_x}{\partial \mathbf{n}} \xi(y_1,0) dy_1 =: N(x) + D(x), \quad \forall x \in \Pi', \tag{14}$$

where the Green function for the quarter-plane is

$$\Gamma_x(y) = \frac{1}{4\pi}\left(\ln|x-y| + \ln|x^*-y| - \ln|x_*-y| - \ln|\overline{x}-y|\right),$$

with $x = (x_1,x_2), x^* = (-x_1,x_2), x_* = (x_1,-x_2), \overline{x} = (-x_1,-x_2)$.

The Neumann part $N(x)$**.** We make the change of variables $x = (r\cos\vartheta, r\sin\vartheta)$ which gives

$$\begin{aligned} N &:= \lim_{m\to\infty} \sum_{k=0}^m \eta_k N_k = \lim_{m\to\infty} \sum_{k=0}^m \frac{\eta_k}{2\pi} \int_0^\infty k e^{-ky_2} \left(\ln(x_1^2 + (y_2-x_2)^2) - \ln(x_1^2+(y_2+x_2)^2)\right) dy_2 \\ &= \lim_{m\to\infty} \frac{1}{2\pi} \sum_{k=1}^m \eta_k \int_0^\infty k e^{-ky_2} \left(\ln\left(1 - \frac{2sy_2}{r} + \left(\frac{y_2}{r}\right)^2\right) - \ln\left(1 + \frac{2sy_2}{r} + \left(\frac{y_2}{r}\right)^2\right)\right) dy_2, \end{aligned}$$

where $c = \cos\vartheta, s = \sin\vartheta$. Now we perform the second change of variables $t_k = e^{-ky_2}$ and get

$$N_k \leq \frac{1}{\pi}\int_0^1 \ln\left(1 - \frac{2s\ln t_k}{r} + \left(\frac{\ln t_k}{r}\right)^2\right) dt_k \leq \frac{1}{\pi}\int_0^1 \ln\left(\left(1 - \frac{\ln t}{r}\right)^2\right) dt$$

The last rhs is independent of k; one easily estimates it using the change of variables $y = -\ln t/r$, indeed:

$$\int_0^1 \ln\left(\left(1-\frac{\ln t}{r}\right)^2\right) dt = \int_0^\infty \ln(1+y) e^{-ry} r dy = \int_0^\infty \frac{e^{-ry}}{1+y} dy \leq \frac{1}{r}$$

Now because the fictitious interface Γ is strictly included in $Z^+ \cup \Gamma \cup P$, $\beta \in H^2_{\text{loc}}(Z^+ \cup \Gamma \cup P)$ and thus $\|\eta\|_{H^1(\Gamma} := \sum_{k=1}^\infty |\eta_k|^2 k^2 < +\infty$, one has for every finite m

$$\sum_{k=1}^m |\eta_k N_k| \leq \left(\sum_{k=1}^\infty |\eta_k|^2 k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty \frac{1}{k^2}\right)^{\frac{1}{2}} \leq C\|\eta\|_{H^1(\Gamma)} \frac{1}{r}$$

the estimate being uniform wrt m one has that $N \leq C\|\eta\|_{H^1(\Gamma)}/r$.

The Dirichlet part $D(x)$**.** We have, by the same change of variable as above $(x := r(\cos(\vartheta), \sin\vartheta) := r(c,s)$:

$$D(x) = -\frac{x_2}{2\pi}\int_0^\infty \left(\frac{1}{(y_1-x_1)^2+x_2^2} + \frac{1}{(y_1+x_1)^2+x_2^2}\right)\xi(y_1,0)dy_1$$
$$\leq \frac{s}{\pi r}\int_0^\infty \frac{1}{1-2c\frac{y_1}{r}+\left(\frac{y_1}{r}\right)^2}|\xi|dy_1 = \frac{s}{\pi r}\int_0^\infty \frac{1}{(1-c^2)+\left(c-\frac{y_1}{r}\right)^2}|\xi|dy_1,$$

where we suppose that $c < 1$. We divide this integral in two parts, we set $m > 1$

$$D(x) \leq \frac{s}{\pi r}\left[\int_0^m \frac{1}{(1-c^2)+\left(c-\frac{y_1}{r}\right)^2}|\xi|dy_1 + \int_m^\infty \frac{1}{(1-c^2)+\left(c-\frac{y_1}{r}\right)^2}|\xi|dy_1\right] =: (I_1+I_2)(x).$$

For I_1 one uses the L^p_{loc} inclusions :

$$I_1 \leq \frac{s}{\pi r}\frac{K}{1-c^2}\|\xi(\cdot,0)\|_{L^1(0,m)} \leq \frac{2}{\pi x_2}K\|\xi\|_{L^2(B')},$$

while for I_2 one uses the weighted norm established in lemma 4.4

$$I_2 \leq \frac{s}{\pi r}\left(\int_m^\infty \left(\frac{1}{(1-c^2)+\left(c-\frac{y_1}{r}\right)^2}\right)^2 \frac{(y_1^2+1)^{\frac{1-2\alpha}{2}}}{y_1}dy_1\right)^{\frac{1}{2}} \|\xi\|_{L^2(B',\omega_\alpha)}$$
$$\leq \frac{2s}{\pi r}\left(\int_m^\infty \left(\frac{1}{(1-c^2)+\left(c-\frac{y_1}{r}\right)^2}\right)^2 y_1^{-2\alpha}dy_1\right)^{\frac{1}{2}} \|\xi\|_{L^2(B',\omega_\alpha)}$$
$$\leq \frac{2s}{\pi r}\left(\left(\int_m^\infty \left(\frac{1}{(1-c^2)+\left(c-\frac{y_1}{r}\right)^2}\right)^{2M} dy_1\right)^{\frac{1}{M}} \left(\int_m^\infty y_1^{-2\alpha M'}dy_1\right)^{\frac{1}{M'}}\right)^{\frac{1}{2}} \|\xi\|_{L^2(B',\omega_\alpha)},$$

where M and M' are Hölder conjugates. We choose $M'2\alpha > 1$ such that the weight contribution provided by ξ is integrable, this implies that $M < 1/(1-2\alpha) \sim 10$. One then recovers easily

$$I_2 \leq \frac{2sK}{\pi r}\left(\frac{\pi r}{(1-c^2)^{2M-\frac{1}{2}}}\right)^{\frac{1}{2M}} = \frac{2K}{(\pi x_2)^{1-\frac{1}{2M}}}.$$

We could shift the fictitious interface Γ to $\Gamma - \delta e_2$ and repeat again the same arguments because the rough boundary does not intersect it. Note that in this case we could establish again the explicit Fourier representation formula for β and its derivative as in theorem 3.1. Thus we can obtain that $\xi \leq c(x_2+\delta)^{1-1/(2M)}$ which shows that ξ is bounded in Π'. So that on E', one has $|\xi|\rho^{1-\frac{1}{2M}} = |\xi|(1+x_2)^{1-\frac{1}{2M}} \leq c'$. Here one applies the Fragmèn-Lindelöf technique (see [3], lemma 4.3, p.12). We restrict the domain to a sector, defining $\Pi_S = \Pi \cap S$, where $S = \{(\rho,\theta) \in [1,\infty] \times [0,\pi/2]\}$. On Π_S, we define $\varpi := -1 + 1/(2M)$ and $v := \rho^\varpi \sin(\varpi\theta)$ the latter is harmonic definite positive, we set $w = \xi/v$ which solves

$$\Delta w + \frac{2}{v}\nabla v \cdot \nabla w = 0, \text{ in } S_\Pi,$$

with $w = 0$ on $B_S = \partial S \cap B$, whereas w is bounded uniformly on $E_S := E \cap S$. Because by standard regularity arguments $\xi \in C^2(\Pi)$, w is also bounded when $\rho = 1$. Then by the Hopf maximum principle, we have

$$\sup_{\Pi_S}|w| \leq \sup_{\partial\Pi_S} w \leq K$$

which extends to the whole domain Π_S, the radial decay of ξ. □

Deriving the representation formula (14), one gets for all x strictly included in Π' that

$$\partial_{x_1}\xi(x) = \int_{E'} \partial_{x_1}\Gamma_x \, g dy_2 + \int_{B'} \frac{\partial}{\partial x_1}\frac{\partial \Gamma_x}{\partial y_2}\xi(y_1,0)dy_1 =: N_{x_1}(x) + D_{x_1}(x), \quad \forall x \in \Pi'$$

Lemma 4.6. *For any $x \in \Pi'$, the Neumann part of the normal derivative $\partial_{x_1}\xi$ satisfies $N_{x_1}(x) \leq Kr^{-2}$ for all x in Π'*

Proof. On E' the derivative wrt x_1 of the Green kernel reads

$$\partial_{x_1}\Gamma_x = x_1 \left(\frac{1}{x_1^2 + (x_2 - y_2)^2} - \frac{1}{x_1^2 + (x_2 + y_2)^2} \right)$$

thus using the cylindrical coordinates to express $x = (r\cos\vartheta, r\sin\vartheta) =: r(c,s)$ for $0 \leq s < 1$ one gets

$$\begin{aligned} N_{x_1}(x) &= \frac{c}{r}\int_0^\infty \left(\frac{1}{1 - 2s\frac{y_2}{r} + \left(\frac{y_2}{r}\right)^2} - \frac{1}{1 + 2s\frac{y_2}{r} + \left(\frac{y_2}{r}\right)^2} \right) g(y_2)dy_2 \\ &= \sum_k \frac{c}{r}\int_0^\infty \frac{4s\frac{y_2}{r}}{4s^2(1-s^2) + \left(1 - 2s^2 - \left(\frac{y_2}{r}\right)^2\right)^2} e^{-|k|y_2}dy_2 \\ &\leq 4\sum_k \frac{1}{x_1 x_2}\int_0^\infty y_2 e^{-|k|y_2}dy_1 \leq \frac{1}{x_1x_2}\sum_k \frac{4}{k^2}|\eta_k|^2 \leq \frac{4}{x_1x_2}\|\eta\|_{H^{-1}(\Gamma)}. \end{aligned} \tag{15}$$

This estimate is not optimal since it is singular near $x_1 = 0$ or $x_2 = 0$. But it provides useful decay estimates inside Π'.

It's easy to check that N_{x_1} is harmonic, $N_{x_1} = g$ on E', and that it vanishes on B'. Because on E' g is bounded, by the maximum principle N_{x_1} is bounded. We divide $\Pi' \setminus B(0,1)$ in three angular sectors :

$$S_i = \{(r,\vartheta) \text{ s.t. } r > 1, \, \vartheta \in [\vartheta_{i-1}, \vartheta_i]\}, \quad (\vartheta_i)_{i=0}^3 = \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$$

For S_1 and S_3 we define $v_i := \pm\rho^{-2}\cos(2\vartheta)$, $i = 1,2$ which is positive definite and harmonic, while for S_2, we set $v_i := \rho^{-2}\sin(2\vartheta)$, that shares the same properties. For each sector we define $w_i = N_{x_1}/v_i$, it solves

$$\Delta w_i + \frac{2}{v_i}\nabla v_i \cdot \nabla w_i = 0, \text{ in } S_i, \quad i = 1, \ldots, 3$$

The estimate (15) shows that on each interior boundary ∂S_i, w is bounded while on $E' \cup B'$ it is bounded by the boundary conditions that N_{x_1} satisfies. By the Hopf maximum principle [7], one shows that

$$\sup_{S_1} w_1 \leq \sup_{\vartheta=\frac{\pi}{6}} w_1 < \infty, \quad \sup_{S_2} w_2 \leq \sup_{\vartheta=\frac{\pi}{6},\vartheta=\frac{\pi}{3}} w_2 < \infty, \quad \sup_{S_3} w_3 \leq \sup_{\vartheta=\frac{\pi}{3},\vartheta=\frac{\pi}{2}} w_3 < \infty$$

which implies that $N_{x_1} \leq Kr^{-2}$ for every $y \in \Pi'_S := \Pi' \cap S$. □

In order to estimate D_{x_1} the latter term of the derivative, we introduce the lemma inspired by proofs of weighted Sobolev imbeddings in [13, 14].

Lemma 4.7. *If $\xi \in W^{1,2}_\alpha(\Pi)$ with $\alpha \in [0, 1/2[$ then its trace on a horizontal interface satisfies*

$$I(\xi) = \int_0^\infty \int_0^\infty \frac{|\xi(y_1 + h, 0) - \xi(y_1, 0)|^2}{h^{2-2\alpha}} dy_1 \, dh \leq \|\xi\|_{W^{1,2}_\alpha(\Pi)} \tag{16}$$

The proof follows ideas of theorem 2.4' in [14] p. 235, we give it for sake of self-containtness.

Proof. We make a change of variables $x_1 = r\cos\vartheta, x_2 = r\sin\vartheta, \vartheta = 0$, leading to rewrite I as

$$I = \int_0^\infty \int_0^\infty \frac{|\xi(r+h,0)-\xi(r,0)|^2}{h^{2-2\alpha}}\,dr\,dh,$$

note that the second space variable for ξ is now $\vartheta = 0$. We insert intermediate terms inside the domain, namely

$$I \le K\Bigg\{\int\int_0^\infty \frac{\left|\xi(r+h,0)-\xi\left(r+h,\operatorname{atan}\frac{h}{r}\right)\right|^2}{h^{2-2\alpha}} + \frac{\left|\xi\left(r+h,\operatorname{atan}\frac{h}{r}\right)-\xi\left(r,\operatorname{atan}\frac{h}{r}\right)\right|^2}{h^{2-2\alpha}}dr\,dh$$

$$+\int_0^\infty\int_0^\infty \frac{\left|\xi\left(r,\operatorname{atan}\frac{h}{r}\right)-\xi\left(r,0\right)\right|^2}{h^{2-2\alpha}}dr\,dh\Bigg\} =: I_1 + I_2 + I_1'$$

Obviously the terms I_1 and I_1' are treated the same way. We make a change of variable $(r, h = r\tan\vartheta)$

$$I_1 = \int_0^{\frac{\pi}{2}}\int_0^\infty \frac{|\xi(r(1+\tan\vartheta),0)-\xi(r(1+\tan\vartheta),\vartheta)|^2}{(r\tan\vartheta)^{2-2\alpha}}\frac{r}{1+\vartheta^2}dr\,d\vartheta.$$

In order to eliminate the dependence on ϑ in the first variable of ξ, we then make the change of variable $(r = \tilde{r}/(1+\tan\vartheta), \vartheta)$ which gives

$$I_1 = \int_0^{\frac{\pi}{2}}\int_0^\infty \frac{|\xi(\tilde{r},0)-\xi(\tilde{r},\vartheta)|^2}{\tilde{r}^{2-2\alpha}}\tilde{r}d\tilde{r}\frac{(1+\tan\vartheta)^{2\alpha}}{\tan^{2-2\alpha}\vartheta(1+\vartheta^2)}\,d\vartheta \le \int_0^\infty\int_0^{\frac{\pi}{2}} \frac{|\xi(\tilde{r},0)-\xi(\tilde{r},\vartheta)|^2}{\tilde{r}^{2-2\alpha}}\tilde{r}d\tilde{r}\,\frac{d\vartheta}{\vartheta^{2-2\alpha}}.$$

We are in the hypotheses of the Hardy inequality (see for instance [13], p.203 estimate (7)), thus we have

$$I_1 \le \frac{4}{(1-2\alpha)^2}\int_0^\infty\int_0^{\frac{\pi}{2}}\left|\frac{\partial\xi}{\partial\vartheta}(\tilde{r},\vartheta)\right|^2\vartheta^{2\alpha}d\vartheta\tilde{r}^{2\alpha-1}d\tilde{r} \le K\int_0^\infty \tilde{r}^{2\alpha}\int_0^{\frac{\pi}{2}}\frac{1}{\tilde{r}^2}\left|\frac{\partial\xi}{\partial\vartheta}(\tilde{r},\vartheta)\right|^2 d\vartheta\tilde{r}d\tilde{r}$$
$$\le \|\xi\|^2_{W^{1,2}_\alpha(\Pi)},$$

in the last estimate we used that in Π', the distance to the fixed point $(0,-1)$ can be estimated as $\rho^{2\alpha} := (y_1^2+(y_2+1)^2)^\alpha \ge (y_1^2+y_2^2)^\alpha =: \tilde{r}^{2\alpha}$, this explains why we need a positive α in the hypotheses. In the same manner

$$\begin{aligned}
I_2 &\le \int_0^{\frac{\pi}{2}}\int_0^\infty \frac{|\xi(r(1+\tan\vartheta),\vartheta)-\xi(r,\vartheta)|^2}{(r\tan\vartheta)^{2-2\alpha}}r dr\,d\vartheta,\\
&= \int_0^{\frac{\pi}{2}}\int_0^\infty\left|\int_0^{r\tan\theta}\frac{\partial\xi}{\partial r}(r+s,\vartheta)ds\right|^2 r^{2\alpha-1}dr\frac{d\vartheta}{\tan^{2-2\alpha}\vartheta},\\
&\le \int_0^{\frac{\pi}{2}}\left\{\int_0^{\tan\vartheta}\left[\int_0^\infty\left|\frac{\partial\xi}{\partial r}\right|^2(r(1+\sigma),\vartheta)r^{1+2\alpha}dr\right]^{\frac{1}{2}}d\sigma\right\}^2\frac{d\vartheta}{\tan^{2-2\alpha}\vartheta},
\end{aligned}$$

where we made the change of variables $s = rt$ and applied the generalized Minkowski inequality ([13], p.203 estimate (6)). Now we set $\tilde{r} = r(1+t)$ inside the most interior integral above

$$I_2 \le \int_0^{\frac{\pi}{2}}\left\{\int_0^{\tan\vartheta}\frac{dt}{(1+t)^{1+\alpha}}\left[\int_0^\infty\left|\frac{\partial\xi}{\partial r}\right|^2(\tilde{r},\vartheta)\tilde{r}^{1+2\alpha}d\tilde{r}\right]^{\frac{1}{2}}\right\}^2\frac{d\vartheta}{\tan^{2-2\alpha}\vartheta}.$$

Now, we have separated the integrals in t and r, the part depending on t is easy to integrate. Thus we obtain

$$I_2 \le \int_0^{\frac{\pi}{2}}\mathcal{S}(\vartheta)\int_0^\infty\left|\frac{\partial\xi}{\partial r}\right|^2(\tilde{r},\vartheta)\tilde{r}^{1+2\alpha}d\tilde{r}d\vartheta,\ \text{where } \mathcal{S}(\vartheta) = \frac{1}{\tan^{2-2\alpha}\vartheta}\left[\frac{1}{(1+\tan\vartheta)^\alpha}-1\right]^2$$

Distinguishing whether $\tan\vartheta$ is greater or not than 1, it is possible to show that $\mathcal{S}$ is uniformly bounded wrt ϑ. This gives the desired result. □

We follow similar arguments as in the proof of theorem 8.20 p. 144 in [12] to claim:

Proposition 2. *Set ξ a function belonging to $W^{1,2}_\alpha(\Pi)$, and*

$$D_{x_1}(x) := \int_0^\infty G(x,y_1)\xi(y_1,0)dy_1, \quad \forall x \in \Pi', \text{ where } G(x,y_1) = \left. \frac{\partial}{\partial x_1}\frac{\partial \Gamma_x}{\partial y_2}\right|_{y\in B'},$$

then it satisfies for every fixed positive h

$$\int_0^\infty |D_{x_1}(x_1,h)|^2 dx_1 \leq \frac{K}{h^{1+2\alpha}},$$

where the constant K is independent on h.

Proof. We recall that

$$G := -x_2 \left(\frac{x_1 - y_1}{((x_1-y_1)^2 + x_2^2)^2} + \frac{x_1+y_1}{((x_1+y_1)^2+x_2^2)^2} \right).$$

Because $\int_0^\infty G(x,y_1)dy_1 = 0$ for every $x \in \Pi'$ we have

$$D_{x_1}(x) = \int_0^\infty G(x,y_1)\left(\xi(y_1,0) - \xi(x_1,0)\right)dy_1, \quad \forall x \in \Pi',$$

we underline that $G(x,\cdot)$ is evaluated at $x \in \Pi'$ while $\xi(x_1,0)$ is taken on B'. By Hölder estimates in y_1 with $p=2, p'=2$, we have :

$$|D_{x_1}|^2 \leq \int_0^\infty G^2|y_1-x_1|^{2-2\alpha}dy_1 \int_0^\infty \frac{|\xi(y_1,0)-\xi(x_1,0)|^2}{|y_1-x_1|^{2-2\alpha}}dy_1 \tag{17}$$

integrating in x_1 and using Hölder estimates with $p=\infty, p'=1$ then

$$I_3 := \int_0^\infty |D_{x_1}|^2 dx_1 \leq \sup_{x_1\in\mathbb{R}_+} \int_{\mathbb{R}_+} G^2|y_1-x_1|^{2-2\alpha}\,dy_1 \int_0^\infty \int_0^\infty \frac{|\xi(y_1,0)-\xi(x_1,0)|^2}{|y_1-x_1|^{2-2\alpha}}dy_1dx_1$$

Thanks to proposition 4.7 we estimate the last integral in the rhs above

$$I_3 \leq K \sup_{x_1\in\mathbb{R}_+} I_4(x_1,x_2)\|\xi\|_{W^{1,2}_\alpha(\Pi)}, \text{ where } I_4(x_1,x_2) := \int_{\mathbb{R}_+} G^2|y_1-x_1|^{2-2\alpha}\,dy_1$$

Considering I_4 one has

$$\begin{aligned} I_4(x) &\leq K\int_0^\infty \frac{x_2^2(y_1-x_1)^{4-2\alpha}}{((x_1-y_1)^2+x_2^2)^4}dy_1 + \frac{x_2^2(y_1+x_1)^2(y_1-x_1)^{2-2\alpha}}{((x_1+y_1)^2+x_2^2)^4}dy_1 \\ &\leq K\int_0^\infty \frac{x_2^2(y_1-x_1)^{4-2\alpha}}{((x_1-y_1)^2+x_2^2)^4}dy_1 + \frac{x_2^2(y_1+x_1)^{4-2\alpha}}{((x_1+y_1)^2+x_2^2)^4}dy_1 =: I_5 + I_6 \end{aligned}$$

Both terms in the last rhs are treated the same, namely

$$I_5 \leq x_2^{2+5-2\alpha-8}\int_{-\infty}^\infty \frac{z^{4-2\alpha}}{(z^2+1)^4}dz \leq \frac{K}{x_2^{1+2\alpha}}$$

which ends the proof. □

Remark 4.1. *This is one of the key point estimates of the paper. One could think of using weighted properties of ξ of lemma 4.4 instead of the fractional Sobolev norm introduced from proposition 4.7, in the Hölder estimates* (17)*. This implies to transfer the x_1-integral on G, then it seems impossible to conclude because*

$$\int_0^\infty \int_0^\infty G^2\omega_\alpha(y_1)dy_1dx_1 = \infty,$$

which is easy to show if one performs the change of variables $z_1 = y_1 - x_1, y_1 = y_1$.

In this part we study the convergence properties of the normal derivative of ξ on vertical interfaces far from E. For this sake we call

$$\Pi_l = \{y \in \Pi, \text{ s.t. } y_1 > l\}, \quad E_l = \{y_1 = l,\, y_2 \in [f(0), +\infty[\}, \quad B_l = \{y \in B, y_1 > l\},$$

here we redefine the weighted trace spaces of Sobolev type

$$W^{\frac{1}{2},2}_{0,\sigma}(\partial\Pi_l) = \left\{ u \in \mathcal{D}'(\partial\Pi_l) \text{ s.t. } \frac{u}{(1+\sigma^2)^{\frac{1}{4}}} \in L^2(\partial\Pi_l), \int_{\partial\Pi_l^2} \frac{|u(y) - y(y')|^2}{|y-y'|^2} ds(y) ds(y') < +\infty \right\}$$

where we define $\sigma := |y - (0, f(0))| = \sqrt{y_1^2 + (y_2 - f(0))^2}$. Note that the weight is a distance to the fixed point $(0, f(0))$ independent on l.

Proposition 3. *Suppose that $v \in W^{\frac{1}{2},2}_{0,\sigma}(\partial\Pi_l)$ with $v = 0$ on B_l then there exists an extension denoted $R(v) \in \mathcal{D}'(\Pi_l)$ s.t.*

$$|\nabla R(v)|_{L^2(\Pi_l)} \leq K \|v\|_{W^{\frac{1}{2},2}_{0,\sigma}(\partial\Pi_l)},$$

where K depends only on $\|f'\|_\infty$.

Proof. We lift the domain in a first step in order to transform Π_l in a quarter-plane $\hat{\Pi}_l$.

$$y = \varphi(Y) := \begin{pmatrix} Y_1 \\ Y_2 + f(Y_1) \end{pmatrix}, \quad Y \in \hat{\Pi}_l := (\mathbb{R}_+)^2,$$

if we set $\hat{v}(Y_2) = v(Y_2 + f(0)) = v(y_2)$ then the L^2 part of the weighted norm above reads

$$\int_{E_l} \frac{v^2}{(1+\sigma^2)^{\frac{1}{2}}} dy_2 = \int_{E_l} \frac{v^2}{(1 + l^2 + (y_2 - f(0))^2)^{\frac{1}{2}}} dy_2 = \int_{\{l\}\times\mathbb{R}_+} \frac{\hat{v}^2}{(1 + l^2 + Y_2^2)^{\frac{1}{2}}} dY_2$$
$$= \int_{\{l\}\times\mathbb{R}_+} \frac{\hat{v}^2}{(1+\hat{\sigma}^2)^{\frac{1}{2}}} dY_2,$$

where $\hat{\sigma}^2 = Y_1^2 + Y_2^2$. If $v \in W^{\frac{1}{2},2}_{0,\sigma}(\partial\Pi_l)$ and $v = 0$ on B_l we know that ([8], p.43 theorem 1.5.2.3)

$$\int_0^\delta |\hat{v}(Y_2)|^2 \frac{dY_2}{Y_2} < +\infty,$$

which authorizes us to extend v by zero on $\hat{E}_l := \{Y_1 = l\} \times \mathbb{R}$, this extension still belongs to $W^{\frac{1}{2},2}_{0,\hat{\sigma}}(\hat{E}_l)$. Arguments above allow obviously to write for every v vanishing on B and $\hat{v}$ defined above

$$\|v\|_{W^{\frac{1}{2},2}_{0,\sigma}(\partial\Pi_l)} = \|\hat{v}\|_{W^{\frac{1}{2},2}_{0,\hat{\sigma}}(\hat{E}_l)}.$$

Here we use trace theorems II.1 and II.2 of Hanouzet [9], they follow exactly the same in our case except that the weight is not a distance to a point on the boundary (as in [9]) but it is a distance to a point exterior to the domain. So in order to define an extension ([9] p. 249), we set

$$\begin{cases} V(Y) = \displaystyle\int_{|s|<1} \mathcal{K}(s) \hat{v}(Y_1 s + Y_2) ds, \quad s \in \mathbb{R}, \quad \forall Y \in \hat{\Pi}_l, \\ \Psi(Y) = \Phi\left(\dfrac{Y_1 - l}{(1 + Y_2^2 + l^2)^{\frac{1}{2}}} \right), \end{cases}$$

where Φ is a cut-off function such that

$$\text{Supp}\Phi \in [0 : 1/4[, \quad \Phi(0) = 1, \quad \Phi \in C^\infty([0, 1/4[),$$

and $\mathcal{K}$ is a regularizing kernel i.e. $\mathcal{K} \in C_0^\infty(]-1:1[)$ and $\int_{\mathbb{R}} \mathcal{K}(s)ds = 1$. Then the extension in the quarter-plane domain reads

$$w(Y) = (\Psi V)(Y_1, Y_2) - (\Psi V)(Y_1, -Y_2), \quad Y \in \hat{\Pi}_l,$$

which allows to have $w(Y_1, 0) = 0$ for all $Y_1 \in \mathbb{R}_+$. According to theorems II.1 and II.2 in [9], one then gets

$$\left\| \frac{w}{(1+\hat{\sigma}^2)^{\frac{1}{2}}} \right\|_{L^2(\hat{\Pi}_l)} \leq K \|\hat{v}\|_{W_{0,\sigma}^{\frac{1}{2},2}(\hat{E}_l)}, \quad \|\nabla w\|_{L^2(\hat{\Pi}_l)} \leq K \|\hat{v}\|_{W_{0,\sigma}^{\frac{1}{2},2}(\hat{E}_l)}.$$

Turning back to our starting domain Π_l, we set

$$R(v) = w(\varphi^{-1}(y)) = w(y_1, y_2 - f(y_1)), \quad \forall y \in \Pi_l.$$

We focus on the properties of the gradient

$$\int_{\Pi_l} (A\nabla_y R(v), \nabla_y R(v))dy = \int_{\hat{\Pi}_l} |\nabla_Y w|^2 dY, \text{ where } A = \begin{pmatrix} 1 & f' \\ f' & 1+(f')^2 \end{pmatrix},$$

but the eigenvalues of A are

$$\lambda_\pm = \frac{2+(f')^2 \pm \sqrt{2+(f')^2}|f'|}{2},$$

the lowest eigenvalue is positive and tends to zero as $|f'|$ increases. The boundary is Lipschitz so that $\|f'\|_\infty$ is bounded. Thus there exists a minimum value of λ_-. All this guarantees the existence of a constant $\delta'(\|f'\|_\infty) > 0$ such that

$$\delta' \int_{\Pi_l} |\nabla_y R(v)|^2 \, dy \leq K \|\hat{v}\|_{W_{0,\hat{\sigma}}^{\frac{1}{2},2}(\hat{E}_l)},$$

which ends the proof. □

Thanks to the existence of a lift $R(v)$, we are able to estimate a sort of weak weighted Sobolev norm for the normal derivative on vertical interfaces located at $y_1 = L/\epsilon$.

Proposition 4. *If $v \in W_{0,\sigma}^{\frac{1}{2},2}(\partial\Pi_{\frac{L}{\epsilon}})$ and $v = 0$ on $B_{\frac{L}{\epsilon}}$ then one has*

$$\int_{E_{\frac{L}{\epsilon}}} \frac{\partial \xi}{\partial \mathbf{n}} \left(\frac{L}{\epsilon}, y_2 \right) v(y_2) dy_2 \leq K \epsilon^\alpha \|v\|_{W_{0,\sigma}^{\frac{1}{2},2}(\partial\Pi_{\frac{L}{\epsilon}})}$$

Proof. The function v given in the hypotheses belongs to the adequate spaces in order to apply proposition 3, thus there exists a lift $R(v) \in W_0^{1,2}(\Pi_{\frac{L}{\epsilon}})$ s.t. $R(v) = v$ on $\partial\Pi_{\frac{L}{\epsilon}}$. Because ξ is harmonic and belongs to $W_0^{1,2}(\Pi)$, for any $\varphi \in \mathcal{D}(\overline{\Pi_{\frac{L}{\epsilon}}})$ and $\varphi_{|B_{\frac{L}{\epsilon}}} = 0$, one writes the variational form:

$$\int_{\Pi_{\frac{L}{\epsilon}}} \nabla\xi \cdot \nabla\varphi dy = \int_{\partial\Pi_{\frac{L}{\epsilon}}} \frac{\partial \xi}{\partial \mathbf{n}} \varphi d\sigma(y) = \int_{E_{\frac{L}{\epsilon}}} \frac{\partial \xi}{\partial \mathbf{n}} \varphi dy_2$$

Then by density and continuity arguments one extends this formula to every test functions in $\varphi \in W_{0,\sigma}^{1,2}(\Pi_{\frac{L}{\epsilon}})$ such that $\varphi = 0$ on $B_{\frac{L}{\epsilon}}$. As the specific lift $R(v)$ belongs to this space one has

$$\begin{aligned} \int_{E_{\frac{L}{\epsilon}}} \frac{\partial \xi}{\partial \mathbf{n}} v dy_2 = \int_{\Pi_{\frac{L}{\epsilon}}} \nabla\xi \nabla R(v) dy \leq \left(\sup_{\Pi_{\frac{L}{\epsilon}}} \frac{1}{\rho^{2\alpha}} \int_{\Pi_l} |\nabla\xi|^2 \rho^{2\alpha} dy \right)^{\frac{1}{2}} \|\nabla R(v)\|_{L^2(\Pi_{\frac{L}{\epsilon}})} \\ \leq K\epsilon^\alpha \|\xi\|_{W_\alpha^{1,2}(\Pi_{\frac{L}{\epsilon}})} \|\nabla R(v)\|_{L^2(\Pi_{\frac{L}{\epsilon}})} \leq K'\epsilon^\alpha \|v\|_{W_{0,\sigma}^{\frac{1}{2},2}(\partial\Pi_{\frac{L}{\epsilon}})} \end{aligned}$$

which ends the proof. □

4.2 Test functions: from macro to micro and vice-versa

We suppose that $v \in H^1_D(\Omega^\epsilon) := \{u \in H^1(\Omega^\epsilon), u = 0 \text{ on } \Gamma^\epsilon \cup \Gamma^1\}$ then $\gamma(v) \in H^{\frac{1}{2}}(\partial\Omega^\epsilon)$ which implies that $v \in H^{\frac{1}{2}}(\Gamma_{\text{in}} \cup \Gamma_{\text{out}})$ and that for any corner

$$\int_0^\delta |v(x(t))|^2 \frac{dt}{t} < \infty,$$

where $x(t) \in \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$ is a mapping of the neighborhood of the corners. To the trace of v on Γ_{in} or Γ_{out}, we associate a trace of a function defined on $\partial\Pi_{\frac{L}{\epsilon}}$ which is zero on $B_{\frac{L}{\epsilon}}$ s.t.

$$\tilde{v}\left(\frac{L}{\epsilon}, y_2\right) := v(0, \epsilon y_2) = v(0, x_2), \quad \forall x_2 \in [\epsilon f(0), 1] \quad \text{and } \tilde{v}\left(\frac{L}{\epsilon}, y_2\right) := 0, \quad y_2 > \frac{1}{\epsilon},$$

then one has the following connexion between the macroscopic trace norm and the microscopic weighted one.

Proposition 5. *Under the hypotheses above on functions v and $\tilde{v}$,*

$$\|v\|_{H^{\frac{1}{2}}(\Gamma_{\text{in}} \cup \Gamma^\epsilon \cup \Gamma^1)} \sim \|\tilde{v}\|_{W^{\frac{1}{2},2}_{0,\sigma}(\partial\Pi_{\frac{L}{\epsilon}})},$$

if $v = 0$ on $\Gamma^\epsilon \cup \Gamma^1$ (resp. $\tilde{v} = 0$ on $B_{\frac{L}{\epsilon}}$).

Proof. Thanks to the change of variables $x_2 = \epsilon y_2$ we have that

$$\int_{\epsilon f(0)}^1 v^2(0, x_2) dx_2 = \epsilon \int_{f(0)}^{\frac{1}{\epsilon}} \tilde{v}^2\left(\frac{L}{\epsilon}, y_2\right) dy_2 \leq K\epsilon \sup_{E_{\frac{L}{\epsilon}}} (1+\sigma^2)^{\frac{1}{2}} \int_{E_{\frac{L}{\epsilon}}} \frac{\tilde{v}^2}{(1+\sigma^2)^{\frac{1}{2}}} dy_2,$$
$$\leq K\epsilon \frac{L}{\epsilon} \|\tilde{v}\|^2_{W^{\frac{1}{2},2}_{0,\sigma}(\partial\Pi_{\frac{L}{\epsilon}})}.$$

Conversely

$$\int_{E_{\frac{L}{\epsilon}}} \frac{\tilde{v}^2}{(1+\sigma^2)^{\frac{1}{2}}} dy_2 = \int_{f(0)}^{\frac{1}{\epsilon}} \frac{\tilde{v}^2}{(1+\sigma^2)^{\frac{1}{2}}} dy_2 \leq \sup_{E_{\frac{L}{\epsilon}}} \frac{1}{(1+\sigma^2)^{\frac{1}{2}}} \int_{f(0)}^{\frac{1}{\epsilon}} \tilde{v}^2 dy_2$$
$$\leq K\epsilon \|\tilde{v}\|^2_{L^2(f(0),\frac{1}{\epsilon})} = K\|v\|^2_{L^2(\Gamma_{\text{in}})}.$$

For the semi-norm the same change of variable provides an equality due to the homogeneity in ϵ i.e.

$$|v|^2_{H^{\frac{1}{2}}(\Gamma_{\text{in}})} = \int\int_{\Gamma^2_{\text{out}}} \frac{|v(x_2) - v(x_2)|^2}{|x_2 - x_2'|^2} dx_2 dx_2' = |\tilde{v}|^2_{W^{\frac{1}{2},2}_{0,\sigma}(E_{\frac{L}{\epsilon}})}$$

□

Remark 4.2. *We insist on the fact that one can associate traces of v either from Γ_{in} or Γ_{out} to $\tilde{v}$, the weight that one gains in the microscopic norm comes from the scaling from macro to micro and not from the vertical position of the macroscopic interface wrt the origin of the domain Ω^ϵ.*

5 A new proof of convergence for standard averaged wall laws

5.1 The full first order boundary layer approximation: error estimates

The periodic boundary layer approximations given in (12) introduce some microscopic oscillations on the inlet and outlet boundaries $\Gamma_{\text{in}} \cup \Gamma_{\text{out}}$. We define a new full boundary layer approximation

$$u^{1,\infty}_\epsilon = u^1 + \epsilon \frac{\partial u^1}{\partial x_2}(x_1, 0) \left(\beta - \overline{\beta} - \xi_{\text{in}} - \xi_{\text{out}}\right)\left(\frac{x}{\epsilon}\right) \tag{18}$$

where we define

$$\xi_{\text{in}}\left(\frac{x}{\epsilon}\right)=\xi\left(\frac{x}{\epsilon}\right),\quad \xi_{\text{out}}\left(\frac{x}{\epsilon}\right)=\tilde{\xi}\left(\frac{x_1-L}{\epsilon},\frac{x_2}{\epsilon}\right),$$

and ξ is the solution of problem (13), and $\tilde{\xi}$ solves the symmetric problem for Γ_{out}:

$$\begin{cases} -\Delta\tilde{\xi}=0, \quad \text{in } \Pi_- := \cup_{k=1}^{\infty}\{Z^+\cup\Gamma\cup P-2\pi k e_1\} \\ \dfrac{\partial\tilde{\xi}}{\partial\mathbf{n}}=\dfrac{\partial\beta}{\partial\mathbf{n}}, \text{ on } E \\ \tilde{\xi}=0, \text{ on } B_- := \cup_{k=1}^{\infty}\{P^0-2\pi k e_1\} \end{cases}$$

Every result shown for ξ in sections above holds equally for $\tilde{\xi}$. One easily checks that

$$\left.\frac{\partial\xi_{\text{in}}}{\partial\mathbf{n}}\right|_{\Gamma_{\text{out}}}=\frac{1}{\epsilon}\frac{\partial\xi}{\partial\mathbf{n}}\left(\frac{L}{\epsilon},\frac{x_2}{\epsilon}\right) \text{ and } \frac{\partial\xi_{\text{out}}}{\partial\mathbf{n}}|_{\Gamma_{\text{in}}}=\frac{1}{\epsilon}\frac{\partial\tilde{\xi}}{\partial\mathbf{n}}\left(-\frac{L}{\epsilon},\frac{x_2}{\epsilon}\right)$$

We estimate the error of this new boundary layer approximation. We denote $r_\epsilon^{1,\infty}:=u^\epsilon-u_\epsilon^{1,\infty}$, it solves

$$\begin{cases} -\Delta r_\epsilon^{1,\infty}=C\chi_{[\Omega^\epsilon\setminus\Omega^0]}, \text{ on } \Omega^\epsilon, \\ \dfrac{\partial r_\epsilon}{\partial\mathbf{n}}^{1,\infty}=\dfrac{\partial\xi}{\partial\mathbf{n}}\left(\dfrac{L}{\epsilon},\dfrac{x_2}{\epsilon}\right) \text{ on } \Gamma_{\text{out}}, \dfrac{\partial r_\epsilon}{\partial\mathbf{n}}^{1,\infty}=\dfrac{\partial\tilde{\xi}}{\partial\mathbf{n}}\left(\dfrac{L}{\epsilon},\dfrac{x_2}{\epsilon}\right) \text{ on } \Gamma_{\text{in}}, \\ r_\epsilon^{1,\infty}=\epsilon\dfrac{\partial u^1}{\partial x_2}(x_1,0)\left(\beta-\overline{\beta}-\xi_{\text{in}}-\xi_{\text{out}}\right)\left(\dfrac{x_1}{\epsilon},\dfrac{1}{\epsilon}\right)=:b\left(\dfrac{x_1}{\epsilon},\dfrac{1}{\epsilon}\right) \text{ on } \Gamma^1, r_\epsilon^{1,\infty}=0, \text{ on } \Gamma^\epsilon \end{cases} \tag{19}$$

As $u_\epsilon^{1,\infty}$ is only a first order approximation, a second order error remains in $\Omega^\epsilon\setminus\Omega^0$. This explains the constant source term on the rhs of the first equation in the system above. We then have

Theorem 5.1. *Under the hypotheses of theorem 3.1, $r_\epsilon^{1,\infty}$ satisfies*

$$\left\|r_\epsilon^{1,\infty}\right\|_{H^1(\Omega^\epsilon)}\leq\epsilon$$

Proof. We separate various sources of errors, we set r^1 the solution of the Neumann part of the errors, it solves :

$$\begin{cases} -\Delta r_1=0, \text{ in } \Omega^\epsilon, \\ \dfrac{\partial r_1}{\partial\mathbf{n}}=\dfrac{\partial\xi}{\partial\mathbf{n}}\left(\dfrac{L}{\epsilon},\dfrac{x_2}{\epsilon}\right) \text{ on } \Gamma_{\text{out}}, \dfrac{\partial r_1}{\partial\mathbf{n}}=\dfrac{\partial\tilde{\xi}}{\partial\mathbf{n}}\left(\dfrac{L}{\epsilon},\dfrac{x_2}{\epsilon}\right) \text{ on } \Gamma_{\text{in}}, \\ r_1=0 \quad \text{on } \Gamma^\epsilon\cup\Gamma^1, \end{cases} \tag{20}$$

then the rest r_2 satisfies

$$\begin{cases} -\Delta r_2=C\chi_{[\Omega^\epsilon\setminus\Omega^0]}, \text{ in } \Omega^\epsilon, \\ \dfrac{\partial r_2}{\partial\mathbf{n}}=0 \text{ on } \Gamma_{\text{in}}\cup\Gamma_{\text{out}}, \\ r_2=b\left(\dfrac{x_1}{\epsilon},\dfrac{1}{\epsilon}\right) \quad \text{on } \Gamma^1, r_2=0 \text{ on } \Gamma^\epsilon, \end{cases} \tag{21}$$

which is the Dirichlet part of the errors and should be evaluated in a second step thanks to appropriate extensions and lifts.

The Neumann part The variational form of the problem (20) reads

$$\int_{\Omega^\epsilon} \nabla r_1 \cdot \nabla v \, dx = \int_{\Gamma_{\rm out}} \frac{\partial r_1}{\partial \mathbf{n}} \gamma(v) dx_2, \quad \forall v \in H^1_D(\Omega^\epsilon),$$

dividing this expression by $\|\nabla v\|_{L^2(\Omega^\epsilon)}$ we first obtain the equivalence :

$$\sup_{v \in H^1_D(\Omega^\epsilon)} \frac{\int_{\Omega^\epsilon} \nabla r_1 \cdot \nabla v \, dx}{\|\nabla v\|_{L^2(\Omega^\epsilon)}} \equiv \|\nabla r_1\|_{L^2(\Omega^\epsilon)}.$$

Indeed, by Cauchy-Schwartz one has easily that the L^2 norm is greater than the supremum while a specific choice of $v = r_1$ gives the reverse estimate. Thanks to this, one has

$$\|\nabla r_1\|_{L^2(\Omega^\epsilon)} = \sup_{v \in H^1_D(\Omega^\epsilon)} \frac{\int_{\Gamma_{\rm out}} \frac{\partial r_1}{\partial \mathbf{n}} \gamma(v) dx_2}{\|\nabla v\|_{L^2(\Omega^\epsilon)}}.$$

We underline that we kept the properties of the traces of $H^1_D(\Omega^\epsilon)$ functions inside the sup that we aim to evaluate. This norm is lower that the simple $H^{-\frac{1}{2}}(\Gamma_{\rm out})$ which authorizes different behaviors of test functions near the corners of $\Gamma_{\rm out}$ ([8], p 43, thm 1.5.2.3).

Now the integral in the rhs of the last expression reads in fact :

$$\int_{\Gamma_{\rm out}} \frac{\partial r_1}{\partial \mathbf{n}} \gamma(v) dx_2 = \int_{\Gamma_{\rm out}} \frac{\partial \xi}{\partial \mathbf{n}} \left(\frac{L}{\epsilon}, \frac{x_2}{\epsilon} \right) \gamma(v) dx_2 = \epsilon \int_{f(0)}^{\frac{1}{\epsilon}} \frac{\partial \xi}{\partial \mathbf{n}} \left(\frac{L}{\epsilon}, y_2 \right) \gamma(\tilde{v}) dy_2,$$

where we constructed $\tilde{v}$ as in section 4.2 i.e. $\tilde{v}$ has the same trace as v but $\tilde{v}$ is expressed as a microscopic trace function. Thanks to proposition 5 the corresponding microscopic trace $\tilde{v}$ belongs to $W^{\frac{1}{2},2}_{0,\sigma}(\partial \Pi_{\frac{L}{\epsilon}})$ and propositions 4 and 5 give that

$$\int_{f(0)}^{\frac{1}{\epsilon}} \frac{\partial \xi}{\partial \mathbf{n}} \left(\frac{L}{\epsilon}, y_2 \right) \gamma(\tilde{v}) dy_2 \leq \epsilon^\alpha \|\tilde{v}\|_{W^{\frac{1}{2},2}_{0,\sigma}(\partial \Pi_{\frac{L}{\epsilon}})} \leq K \epsilon^\alpha \|v\|_{H^{\frac{1}{2}}(\partial \Omega^\epsilon)} \leq K' \epsilon^\alpha \|v\|_{H^1(\Omega^\epsilon)}.$$

The same analysis and convergence rates hold on $\Gamma_{\rm in}$ with the normal derivative of $\xi_{\rm out}$. All together one obtains

$$\|\nabla r_1\|_{L^2(\Omega^\epsilon)} \leq \epsilon^{1+\alpha}.$$

The Dirichlet part We should lift b, the non homogeneous Dirichlet boundary conditions on Γ^1 defined in (19), so we set

$$\varsigma := b \left(\frac{x_1}{\epsilon}, \frac{1}{\epsilon} \right) x_2^2 \chi_{[\Omega^0]}, \text{ and } \tilde{r}_2 := r_2 - \varsigma.$$

Standard *a priori* estimates give

$$\|\nabla \tilde{r}_2\|_{L^2(\Omega^\epsilon)} \leq \|\nabla \varsigma\|_{L^2(\Omega^\epsilon)} + \epsilon,$$

where the ϵ in the last rhs comes when estimating the constant source term C localized in $\Omega^\epsilon \setminus \Omega^0$, indeed :

$$(C, v)_{\Omega^\epsilon \setminus \Omega^0} \leq \|C\|_{L^2(\Omega^\epsilon \setminus \Omega^0)} \|v\|_{L^2(\Omega^\epsilon \setminus \Omega^0)} \leq \sqrt{\epsilon} \|C\|_{L^2(\Omega^\epsilon \setminus \Omega^0)} \|\nabla v\|_{L^2(\Omega^\epsilon \setminus \Omega^0)} \leq \epsilon C \|\nabla v\|_{L^2(\Omega^\epsilon)}$$

thanks to a Poincaré inequality in the sub-layer. Hereafter we estimate the gradient of the lift,

$$\begin{aligned} \|\nabla \varsigma\|_{L^2(\Omega^\epsilon)} \leq & \epsilon K \left\| \beta \left(\frac{\cdot}{\epsilon}, \frac{1}{\epsilon} \right) - \overline{\beta} \right\|_{L^2(\Gamma^1)} + K \left\| \partial_{x_1} \beta \left(\frac{\cdot}{\epsilon}, \frac{1}{\epsilon} \right) \right\|_{L^2(\Gamma^1)} \\ & + \epsilon \|\xi_{\rm in}\|_{L^2(\Gamma^1)} + \epsilon \|\partial_{x_1} \xi_{\rm in}\|_{L^2(\Gamma^1)} + \epsilon \|\xi_{\rm out}\|_{L^2(\Gamma^1)} + \epsilon \|\partial_{x_1} \xi_{\rm out}\|_{L^2(\Gamma^1)} \\ & \leq K \epsilon^{\frac{3}{2}} + 2 \left[\epsilon \left\| \xi \left(\frac{\cdot}{\epsilon}, \frac{1}{\epsilon} \right) \right\|_{L^2(\Gamma^1)} + \left\| \frac{\partial \xi}{\partial y_1} \left(\frac{\cdot}{\epsilon}, \frac{1}{\epsilon} \right) \right\|_{L^2(\Gamma^1)} \right] \\ & \leq K \left[\epsilon^{\frac{3}{2}} + \epsilon^{2 - \frac{1}{2M}} + \epsilon^{1+\alpha} \right] \leq K \epsilon^{1+\alpha} \end{aligned}$$

where we used the second estimate of theorem 4.1 describing the decay properties of ξ. This ends the proof: the main error is still made when linearizing the Poiseuille profile in $\Omega^\epsilon \setminus \Omega^0$. As $r_\epsilon^{1,\infty} := r_1 + r_2$ one gets the desired estimate. $\square$

Here comes the error estimate in the L^2 norm that add approximately an $\sqrt{\epsilon}$ factor to the *a priori* estimates above.

Theorem 5.2. *Under the hypotheses of theorem 3.1, $r_\epsilon^{1,\infty}$, the solution of system* (19) *satisfies*

$$\left\| r_\epsilon^{1,\infty} \right\|_{L^2(\Omega^0)} \leq K \epsilon^{1+\alpha},$$

where the constant K is independent on ϵ and $\alpha < \sqrt{2}/\pi \sim 0.45$.

Proof. We define v a regular solution on the "smooth" domain Ω^0 (in the sense: not rough, in particular, Ω^0 is a rectangle) of the problem

$$\begin{cases} -\Delta v = F, \text{ in } \Omega^0, \\ \dfrac{\partial v}{\partial \mathbf{n}} = 0, \text{ on } {\Gamma_{\text{in}}}' \cup \Gamma'_{\text{out}} \\ v = 0, \text{ on } \Gamma^0 \cup \Gamma^1 \end{cases}$$

where the function F is in $L^2(\Omega^0)$. Thanks to [8] theorem 4.3.1.4 p. 198, one has that there are no singularities near the corners i.e. $v \in H^2(\Omega^0) \cap H^1_D(\Omega^0)$ and

$$\|v\|_{H^2(\Omega^0)} \leq K \|F\|_{L^2(\Omega^0)}.$$

Testing $r_\epsilon^{1,\infty}$ against F, one gets

$$(r_\epsilon^{1,\infty}, F)_{\Omega^0} = \left\langle \frac{\partial r_\epsilon^{1,\infty}}{\partial \mathbf{n}}, v \right\rangle - \left(r_\epsilon^{1,\infty}, \frac{\partial v}{\partial \mathbf{n}} \right)_{\Gamma^1 \cup \Gamma^\epsilon}$$

where the brackets stand for the duality pairing between $H^{-1}({\Gamma_{\text{in}}}' \cup \Gamma'_{\text{out}})$ and $H^1_0({\Gamma_{\text{in}}}' \cup \Gamma'_{\text{out}})$, whereas the left bracket denotes the standard $L^2(\Gamma^0 \cup \Gamma^1)$ scalar product. This leads to write :

$$\left\| r_\epsilon^{1,\infty} \right\|_{L^2(\Omega^0)} \leq \left\| \frac{\partial r_\epsilon^{1,\infty}}{\partial \mathbf{n}} \right\|_{H^{-1}({\Gamma_{\text{in}}}' \cup \Gamma'_{\text{out}})} + \left\| r_\epsilon^{1,\infty} \right\|_{L^2(\Gamma^0 \cup \Gamma^1)} \quad =: I_1 + I_2$$

the latter term of the rhs is classically estimated through Poincaré on the sublayer and the *a priori* estimates above for the Γ^0 part :

$$\left\| r_\epsilon^{1,\infty} \right\|_{L^2(\Gamma^0)} \leq \sqrt{\epsilon} \left\| r_\epsilon^{1,\infty} \right\|_{H^1(\Omega^\epsilon \setminus \Omega^0)} \leq \sqrt{\epsilon} \left\| r_\epsilon^{1,\infty} \right\|_{H^1(\Omega^\epsilon)} \leq \epsilon^{\frac{3}{2}}.$$

On Γ^1 there is an exponentially small contribution of the periodic boundary layer and an almost ϵ^2 term coming from the vertical correctors ξ_{in} and ξ_{out} :

$$\left\| r_\epsilon^{1,\infty} \right\|_{L^2(\Gamma^1)} \leq K \epsilon \left\| \xi \left(\frac{\cdot}{\epsilon}, \frac{1}{\epsilon} \right) \right\|_{L^2(0,L)} \leq \epsilon^{2 - \frac{1}{2M}}.$$

I_1 follows using the same arguments as in the proof of the *a priori* estimates. The astuteness resides in the fact that

$$I_1 \leq \sup_{v \in H_0^{\frac{1}{2}}(\Gamma_{\text{out}})} \frac{\left\langle \frac{\partial r_\epsilon^{1,\infty}}{\partial \mathbf{n}}, v \right\rangle}{\|v\|_{H^{\frac{1}{2}}(\Gamma_{\text{out}})}} \leq K \epsilon^{1+\alpha}$$

Indeed $H^1_0(\Gamma'_{\text{out}})$ functions when extended by zero on Γ_{out} are a particular subset of $H^{\frac{1}{2}}(\Gamma_{\text{out}})$ functions vanishing on $\partial\Gamma_{\text{out}}$. At this point one uses the same estimates as in the previous proof to obtain the last term in the rhs.

$\square$

5.2 The standard averaged wall law: new error estimates

We use the full boundary layer approximation above as an intermediate step to prove error estimates for the wall law. We denote $r^1_\epsilon := u^\epsilon - u^1$.

Theorem 5.3. *Under the hypotheses of theorem 3.1, one has*

$$\left\| r^1_\epsilon \right\|_{L^2(\Omega^0)} \leq \epsilon^{1+\alpha}$$

Proof. We insert the full boundary layer approximation between u^ϵ and u^1

$$\begin{aligned} r^1_\epsilon &:= u^\epsilon - u^1 = u^\epsilon - u^{1,\infty}_\epsilon + u^{1,\infty}_\epsilon - u^1 \\ &= r^{1,\infty}_\epsilon + \epsilon \frac{\partial u^1}{\partial x_2}(x_1, 0) \left(\beta - \overline{\beta} - \xi_{\text{in}} + \xi_{\text{out}} \right) \left(\frac{x}{\epsilon} \right) =: r^{1,\infty}_\epsilon + \frac{\partial u^1}{\partial x_2}(x_1, 0) I_1 \end{aligned}$$

We evaluate the $L^2(\Omega^0)$ norm of I_1

$$I_1 \leq K \left\{ \epsilon \left\| \beta \left(\frac{\cdot}{\epsilon} \right) - \overline{\beta} \right\|_{L^2(\Omega^0)} + \epsilon \left\| \xi \left(\frac{\cdot}{\epsilon} \right) \right\|_{L^2(\Omega^0)} \right\} \leq K \left\{ \epsilon^{\frac{3}{2}} + \epsilon \left\| \xi \left(\frac{\cdot}{\epsilon} \right) \right\|_{L^2(\Omega^0)} \right\}$$

while the first term is classical and the estimate comes from (5), for what concerns the second term, we use the L^∞ estimates of theorem 4.1 and get

$$\begin{aligned} \int_{\Omega^0} \xi^2 \left(\frac{x}{\epsilon} \right) dx &= \epsilon^2 \int_0^{\frac{L}{\epsilon}} \int_0^{\frac{1}{\epsilon}} \xi^2 dy \leq \epsilon^2 \int_0^{\frac{L}{\epsilon}} \int_0^{\frac{1}{\epsilon}} \frac{1}{\rho^{2-\frac{1}{M}}} dy \\ &\leq \epsilon^2 \sup_{\left[0, \frac{L}{\epsilon}\right] \times \left[0, \frac{1}{\epsilon}\right]} \rho^{\frac{1}{M} + \delta''} \int_{\left[0, \frac{L}{\epsilon}\right] \times \left[0, \frac{1}{\epsilon}\right]} \frac{1}{\rho^{2+\delta''}} dy \leq K \epsilon^{2 - \frac{1}{M} - \delta''} \end{aligned}$$

where δ'' is a positive constant as small as desired. This ends the proof. □

6 Conclusion

In this work we established error estimates for a new boundary layer approximation and for the standard wall law with respect to the exact solution of a rough problem set with non periodic lateral boundary conditions. The final order of approximation is of $\epsilon^{1+\alpha}$ where $1+\alpha \sim 1.45$ which is compatible and comparable to results obtained in the periodic case (see [5, 4] and references there in).

Establishing estimates in the spirit of very weak solution [18] but in the weighted context improves the $L^2(\Omega^0)$ estimates but requires an extra amount of work not presented here. This is done in [16], we perform also a numerical validation illustrating the accuracy of our theoretical results.

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