Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
J. Differential Equations 202 (2004) 183–207
Journal of Differential Equations
http://www.elsevier.com/locate/jde

A discrete BGK approximation for strongly degenerate parabolic problems with boundary conditions

F.R. Guarguaglini,[a,*] V. Milišić,[b] and A. Terracina[c]

[a] *Dipartimento di Matematica Pura e Applicata, Universitá degli Studi di L'Aquila, Via Vetoio, Coppito I-67100 (L'Aquila), Italy*
[b] *Modelling and Scientific Computing Chair, Ecole Polytechnique Fédérale de Lausanne, SB/IACS/CMCS, 1015 Lausanne, Switzerland*
[c] *Dipartimento di Matematica Universitá di Roma "La Sapienza", Piazzale Aldo Moro n. 5, Roma I-00185, Italy*

Received July 1, 2002; revised March 3, 2004

Abstract

We consider a class of BGK systems with a finite number of velocities, depending on a positive relaxation parameter, that approximate strongly degenerate hyperbolic–parabolic equations with initial boundary conditions. We prove a priori estimates for the solutions of the systems, showing that these functions converge towards the entropy solutions of strongly degenerate problems when the relaxation parameter goes to zero.

Keywords: Entropy solutions; Strongly degenerate parabolic equations; Singular perturbation problems; BGK models

1. Introduction

In this paper we are interested in the approximation of the following parabolic equation:

$$\partial_t u + \partial_x A(u) = \partial_{xx}[B(u)], \quad (x,t) \in \mathbb{R}^+ \times (0,T), \tag{1.1}$$

*Corresponding author. SB/IACS/CMCS, Ecole Polytechnique Federale de Lausanne, Lausanne 1015, France. Fax: +41-21-693-4303.

E-mail addresses: guarguag@univaq.it (F.R. Guarguaglini), Vuk.Milisic@EPFL.ch (V. Milišić), terracin@mat.uniroma1.it (A. Terracina).

0022-0396/$ - see front matter
doi:10.1016/j.jde.2004.03.008

with initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^+ \tag{1.2}$$

and boundary condition

$$u(0,t) = a_0(t) \quad t \in (0,T), \tag{1.3}$$

here $u = u(x,t) \in \mathbb{R}$ with $(x,t) \in \mathbb{R}^+ \times (0,T)$.

We assume that A, B are locally lipschitz functions, u_0 and a_0 are BV functions and the function $B(u)$ is not decreasing.

This assumption allows the diffusion function B to be constant for some intervals of the state function u; for these values the problem is completely hyperbolic and it is necessary to give an entropy formulation for it (see [20]). Obviously the situation is different for the solutions that take values in the intervals in which B' is strictly separate from zero; in such case the problem is purely parabolic and the solution is classic. In general the equation is of parabolic–hyperbolic type; such phenomena appears in many application models, (see [5–7]).

One needs a formulation that considers both the features of the problem. In our case there are also difficulties related to the boundary condition, in fact it is well known that for first-order hyperbolic problems this condition is not achieved in a classical sense and should be interpreted as a consistent condition (see [12,16]).

An entropy formulation for problem (1.1)–(1.3) was given in [13] in the multi-dimensional case for homogeneous boundary data, proving that this is well-posed. More recently, an entropy formulation for general boundary data (see Definition 1.1) was given in (see [24,26]), as while existence and uniqueness results were proved.

In particular in [24] was proved that the entropy solution of (1.1)–(1.3) can be obtained as limit of solutions of regularized equations of non-degenerate parabolic type (the diffusion function B is approximated by functions B^ε strictly increasing). Other results can be found in [7], for one-dimensional case with homogeneous boundary data, while in [5,6] some application models with different boundary value problems are examined.

In this paper we are interested in the approximation of the above problem by means of a sequence of semilinear systems of conservation laws with source and initial-boundary conditions.

We introduce the following system:

$$\partial_t f_k^\varepsilon + \gamma_k^\varepsilon \partial_x f_k^\varepsilon = \frac{1}{\varepsilon}(M_k(u^\varepsilon, \varepsilon) - f_k^\varepsilon) \quad \text{in } \mathbb{R}^+ \times (0,T), \quad k = 1, \dots, N \tag{1.4}$$

with $\gamma_k^\varepsilon = \lambda_k + \frac{\theta_k}{\sqrt{\varepsilon}}$, $\varepsilon > 0$ and

$$u^\varepsilon(x,t) = \sum_{i=1}^{N} f_i^\varepsilon(x,t), \tag{1.5}$$

where the functions M_i (called Maxwellian functions) satisfy the following properties:

(M_1) $\sum_{i=1}^{N} M_i(w,\varepsilon) = w$ for all $\varepsilon \in]0,1]$ and for all $w \in I$,
(M_2) $\sum_{i=1}^{N} \gamma_i^{\varepsilon} M_i(w,\varepsilon) = A(w)$ for all $\varepsilon \in]0,1]$ and for all $w \in I$,
(M_3) $\sum_{i=1}^{N} \theta_i^2 M_i(w,0) = B(w)$ for all $w \in I$,
(M_4) $M_i(w,\varepsilon) \to M_i(w,0)$, when $\varepsilon \to 0$, uniformly for w in I,
(M_5) $M_i'(\cdot,\varepsilon) \geqslant 0$ in I for all $\varepsilon \in]0,1]$.

Here I is an interval of $\mathbb{R}$.

These properties assure that systems (1.4) approximate problem (1.1). In fact it is easy to see, formally, that if the sequence $\{u^{\varepsilon}\}$ converges to some limit function u strongly in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^+))$, the function u is a weak solution of Eq. (1.1) [11].

In particular condition (M_5) is a stability condition: it is crucial in proving comparison results for the solutions of system (1.4) and compactness properties for the sequence $\{f_i^{\varepsilon}\}$. In fact we will show that, under the stability condition (M_5), our approximation (1.4)–(1.7) keeps the monotonicity properties which are typical of Eq. (1.1).

We complement the system by the initial conditions

$$f_i^{\varepsilon}(x,0) = M_i(u_0(x),0), \quad i = 1,\dots,N \tag{1.6}$$

and the boundary conditions for the functions satisfying Eq. (1.4) as entering characteristics,

$$f_i^{\varepsilon}(0,t) = M_i(a_0(t),0) \quad \text{for } i \in \{1,\dots,N\} \text{ such that } \gamma_i^{\varepsilon} > 0. \tag{1.7}$$

System (1.4), (1.6), (1.7) can be considered as a discrete BGK model [3]. These kind of models were introduced in the kinetic theory of the gases in order to simplify the Boltzmann equation [8,14,15]. In the BGK model the collision term is replaced by the quantity $Q(f) = \frac{1}{\varepsilon}(M(f) - f)$, where M is a Maxwellian distribution and ε is the mean free path of the molecules.

In particular Euler and incompressible Navier–Stokes equations can be obtained by the BGK equation introducing suitable scaling for the x, t variables and letting the parameter ε go to zero [15,14,34].

This paper was inspired by the results in [11,31] and [18]. The first paper considers the approximation of the Cauchy problem for Eq. (1.1) in the multidimensional case, with a BGK model similar to that in (1.4); in the second one, the authors prove the convergence of a similar relaxation system, with two transport coefficients independent from ε, towards an initial-boundary value problem for a conservation law; in the last paper the authors consider a particular system (1.4) with three velocities that converges to a weakly parabolic problem with initial and boundary condition (diffusion B is not allowed to be constant in an interval).

BGK approximations to initial-boundary value problems for conservation laws are also studied in [27]; here the author introduces a new technique to estimate the boundary terms which allows to consider a general BGK model, without constraints on the number of velocities. This technique will be crucial in the present paper to treat strongly degenerate parabolic problems.

We recall that this kind of discrete velocities approximations was introduced in [30] for conservation laws in $\mathbb{R}^d \times [0, T]$. The study of diffusive limit of Cauchy problems for discrete velocity hyperbolic systems can be found in [21,25,23,22,34] and in [19,29,17,4] for numerical results. Finally, relaxation approximations for boundary value problems are studied in [1,2,33,32,36,35] (besides [31]) in the setting of hyperbolic equations.

Let us state more precisely the convergence result proved in this paper.

We consider the following definition for entropy solution of strongly degenerate problem (1.1)–(1.3) (see [24,26])

Definition 1.1. Let $T>0$, $u_0 \in BV_{\mathrm{loc}}(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $a_0 \in BV((0,T)) \cap C((0,T))$. A function $u \in L^\infty(\Omega \times (0,T))$ is said to be an entropy solution of problem (1.1)–(1.3) if and only if

(i) (*regularity*):

$$\partial_x B(u) \in L^2(\mathbb{R}^+ \times (0,T)),$$

and $B(u)_{|_{\{0\}\times(0,T)}} = B(a_0)$;

(ii) (*entropy condition*): let $K_x^\pm(u,c) = H^\pm(u-c)(A(u) - A(c) - \partial_x B(u))$ then

$$\int_{\mathbb{R}^+\times(0,T)} \{[u-c]_\pm \phi_t + K_x^\pm(u,c)\phi_x\}\, dx\, dt + \int_0^\infty [u_0 - c]_\pm\, dx \geqslant 0, \tag{1.8}$$

for every $\phi \in H^1(\mathbb{R}^+ \times (0,T))$, $\phi \geqslant 0$, such that $\phi H^\pm(a_0 - c)_{|_{\{0\}\times(0,T)}} = 0$. Here $H^\pm(s)$ are the Heaviside functions, $\frac{\mathrm{sgn}(s)\pm 1}{2}$ and $[s]_\pm$ denote, respectively, positive and negative part of s.

We have the following definition for the weak solution to the relaxation system (1.4).

Definition 1.2. Let $u_0 \in BV(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $a_0 \in BV((0,T)) \cap C((0,T))$. The N-ple $(f_1^\varepsilon, \dots, f_N^\varepsilon) \in (BV(\mathbb{R}^+ \times (0,T)) \cap L^\infty(\mathbb{R}^+ \times (0,T)))^N$ is a weak solution to problem (1.4), (1.6), (1.7) if and only if

(i) for every $\phi \in C_0^1(\mathbb{R}^+ \times (0,T))$

$$\int_0^T \int_{\mathbb{R}^+} f_i^\varepsilon(\phi_t + \gamma_i^\varepsilon \phi_x) + \frac{1}{\varepsilon}(M_i(u^\varepsilon, \varepsilon) - f_i^\varepsilon)\phi\, dx\, dt = 0, \quad i = 1, \dots, N,$$

(ii) $f_i^\varepsilon(x,0) = M_i(u_0(x), 0)$ for almost every $x \in \mathbb{R}^+, i = 1, \dots, N$,

(iii) $f_i^\varepsilon(0,t) = M_i(a_0(t), 0)$ for almost every $t \in (0,T)$, for $i \in \{1, \dots, N\}$ such that $\gamma_i^\varepsilon > 0$.

In this paper we prove comparison and stability results with respect to the data for the weak solutions of problem (1.4)–(1.7), moreover we prove a priori estimates for it.

Then we prove the main result.

Theorem 1.1. *Let* $u_0 \in BV(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ *and* $a_0 \in BV((0,T)) \cap C((0,T))$, *such that* $u_0(\cdot) \in I$ *and* $a_0(\cdot) \in I$. *Let* $(f_1^\varepsilon, \ldots, f_N^\varepsilon)$ *be the solution of problem* (1.4), (1.6), (1.7) *in* $(\mathbb{R}^+ \times (0,T])$, $u^\varepsilon = \sum_{i=1}^N f_i^\varepsilon$ *and let* u *be the solution of the problem* (1.1)–(1.3) *in* $\mathbb{R}^+ \times (0,T))$. *Then*

$$\lim_{\varepsilon \to 0} u^\varepsilon = u \quad \text{in } C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^+)).$$

We show in the last section that for every given interval I one can construct a set of Maxwellian functions verifying conditions (M_1)–(M_5).

The main difficulty is to treat the boundary terms that appear in this problem, in particular we have to verify that the weak boundary condition proposed in Definition (1.1) is achieved for the limit of solutions of the relaxation approximation (1.4), (1.6), (1.7).

Theorem (1.1) gives the first result of convergence for BGK approximation of problem (1.1)–(1.3), also providing another proof of existence. The results proved in this paper can be useful to implement numerical schemes for the approximation of the solution of the strongly degenerate parabolic boundary value problem [28].

The paper is organized in three further sections. In the next section we state comparison results for problems (1.4),(1.6), (1.7), existence and uniqueness theorem. Moreover, we prove a set of a priori estimates which ensure relatively compactness in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^+))$ for the sequences $\{f_i^\varepsilon\}$ and $\{u^\varepsilon\}$. Section 3 is devoted to prove that the limit of the sequence $\{u^\varepsilon\}$ is in fact the unique entropy solution of problem (1.1)–(1.3). In the last section we consider some examples of BGK systems of the form (1.4) that approximate (1.1)–(1.3).

2. A priori estimates

The goal of this section is to establish stability, comparison, existence and uniqueness results and some a priori estimates for the solutions of the system (1.4). Most of the proofs are the extension of those in [18] for a particular choice of the Maxwellian functions, to a general BGK system of type (1.4). On the contrary, the result in Proposition 2.4 is proved by means of a different technique which does not involve BV estimates for the traces $f_i^\varepsilon(0,t)$ [18]; in [18] the proofs of such boundary estimates for i such that $\theta_i = 0$ are based on the assumption of parabolicity for Eq. (1.1).

We will make use of the following lemma [20].

Lemma 2.1. *Let u be a weak solution to the linear Cauchy problem*

$$\partial_t u + \theta \partial_x u = g(x,t),$$

$$u(x,0) = u_0(x),$$

for $x \in \mathbb{R}^+$ and $t \in (0,T)$ $(T>0)$, and v the solution of the above problem with a function h in place of g and the initial data v_0 in place of u_0. Then for every $\phi \in C_0^1(\mathbb{R}^+ \times (0,T))$, with $\phi \geqslant 0$, there holds

$$\int_0^T \int_{\mathbb{R}^+} [u-v]_+ \partial_t \phi + \theta [u-v]_+ \partial_x \phi \, dx \, dt \geqslant - \int_0^T \int_{\mathbb{R}^+} H(u-v)(g-h)\phi \, dx \, dt, \tag{2.1}$$

where H is the Heaviside function.

In the following $f_i^\varepsilon(0,t), u^\varepsilon(0,t)$ denote the traces of the functions $f_i^\varepsilon(x,t), u^\varepsilon(x,t)$ on the boundary $x=0$.

We assume ε varying in a suitable small interval $(0,\bar{\varepsilon})$ where the Maxwellian functions are increasing in u and $\operatorname{sgn} \gamma_i^\varepsilon = \operatorname{sgn} \theta_i$ for all i such that $\theta_i \neq 0$. Moreover we set $Z^+ = \{i = 1, \dots, N : \theta_i > 0\}$, $Z^- = \{i = 1, \dots, N : \theta_i < 0\}$, $Z_0^+ = \{i = 1, \dots, N : \gamma_i^\varepsilon \geqslant 0, \theta_i = 0\}$, $Z_0^- = \{i = 1, \dots, N : \gamma_i^\varepsilon < 0, \theta_i = 0\}$.

Proposition 2.1. *Assume that $u_0, \overline{u_0} \in BV(\mathbb{R}^+)$, $a_0, \overline{a_0} \in BV((0,T))$. Let $(f_i^\varepsilon)_{i=1,\dots,N}$ be a solution of problem (1.4)–(1.7) and $u^\varepsilon \in I$ for almost every $(x,t) \in \mathbb{R}^+ \times (0,T)$. Let $(\bar{f}_i)_{i=1,\dots,N}$ be another solution corresponding to the initial-boundary conditions $(\overline{u_0}, \overline{a_0})$ and $\overline{u^\varepsilon} = \sum_{i=1}^N \overline{f_i^\varepsilon} \in I$ for almost every $(x,t) \in \mathbb{R}^+ \times (0,T)$. Let $\hat{\gamma} = \max\{|\gamma_i^\varepsilon| : \gamma_i^\varepsilon < 0, i = 1, \dots, N\}$. Then for every $K \in \mathbb{R}^+$ and for almost every $t \in (0,T)$ the following inequalities hold*:

$$\int_0^K \sum_{i=1}^N [f_i^\varepsilon(x,t) - \overline{f_i^\varepsilon}(x,t)]_+ \, dx \leqslant \int_0^{K+\hat{\gamma}t} [u_0(x) - \overline{u_0}(x)]_+ \, dx + \max_{\{\gamma_i^\varepsilon : i=1,\dots,N\}} \int_0^t [a_0(s) - \overline{a_0}(s)]_+ \, ds, \tag{2.2}$$

$$\sum_{Z^- \cup Z_0^-} |\gamma_i^\varepsilon| \int_0^t [f_i^\varepsilon(0,s) - \overline{f_i^\varepsilon}(0,s)]_+ \, ds \leqslant \int_0^{\hat{\gamma}t} [u_0(x) - \overline{u_0}(x)]_+ \, dx + \max_{\{\gamma_i^\varepsilon : i=1,\dots,N\}} \int_0^t [a_0(s) - \overline{a_0}(s)]_+ \, ds. \tag{2.3}$$

Proof. Setting $w_i = f_i^\varepsilon - \overline{f_i^\varepsilon}$ and using inequality (2.1) we obtain

$$\int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^N [w_i]_+ (\phi_t + \gamma_i^\varepsilon \phi_x)\, dx\, dt \geqslant -\frac{1}{\varepsilon} \int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^N H(w_i)(M_i(u^\varepsilon, \varepsilon) - M_i(\overline{u^\varepsilon}, \varepsilon) - w_i)\phi\, dx\, dt,$$

for every $\phi \in C_0^1(\mathbb{R}^+ \times (0,T))$, $\phi \geqslant 0$. If $H(w_i) = 1$ for $i = 1, \dots, N$ or $H(w_i) = 0$ for $i = 1, \dots, N$ then the right-hand side of the above inequality is zero. In order to prove that it is always nonnegative set $J_+ = \{i : H(w_i) = 1\}$; then, thanks to $(\mathbf{M}_1)$ and $(\mathbf{M}_5)$

$$\sum_{J_+} (M_i(u^\varepsilon, \varepsilon) - M_i(\overline{u^\varepsilon}, \varepsilon) - w_i) \leqslant \left(\sum_{J_+} M_j'(\eta, \varepsilon) - 1 \right) \sum_{J_+} w_j \leqslant 0,$$

where η is a suitable intermediate value.

Hence

$$\int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^N [f_i^\varepsilon - \overline{f_i^\varepsilon}]_+ (\phi_t + \gamma_i^\varepsilon \phi_x)\, dx\, dt \geqslant 0.$$

Now, choosing a sequence of test functions approximating the characteristic function of the set $\{(x,s) \in \mathbb{R}^+ \times (0,t) : 0 \leqslant x \leqslant K + \hat{\gamma}(t-s)\}$ we obtain

$$\begin{aligned} \int_0^K \sum_{i=1}^N [f_i^\varepsilon(x,t) - \overline{f_i^\varepsilon}(x,t)]_+\, dx &\leqslant \int_0^{K+\hat{\gamma}t} \sum_{i=1}^N [M_i(u_0, \varepsilon) - M_i(\overline{u_0}, \varepsilon)]_+\, dx + \int_0^t \sum_{i=1}^N \gamma_i^\varepsilon [f_i^\varepsilon(0,s) - \overline{f_i^\varepsilon}(0,s)]_+\, ds \\ &\quad + \int_0^t \sum_{i=1}^N (-\hat{\gamma} - \gamma_i^\varepsilon)[f_i^\varepsilon(K + \hat{\gamma}(t-s), s) - \overline{f_i^\varepsilon}(K + \hat{\gamma}(t-s), s)]_+\, ds \end{aligned}$$

and (2.2) follows since $-\hat{\gamma} - \gamma_i^\varepsilon \leqslant 0$ for $i = 1, \dots, N$.

In order to prove inequality (2.3) we introduce a sequence of test functions approximating the characteristic function of the set $\{(x,s) \in \mathbb{R}^+ \times (0,t) : 0 \leqslant x \leqslant \hat{\gamma}(t-s)\}$ and, with the same technique used in the first part of the proof, we obtain the claim. $\square$

As a consequence of the previous proposition we have the following two results.

Corollary 2.1. *Assume that $u_0, \overline{u_0} \in BV(\mathbb{R}^+)$, $a_0, \overline{a_0} \in BV((0,T))$. Let $(f_i^\varepsilon)_{i=1,\dots,N}$ be a solution of problem* (1.4)–(1.7) *and $u^\varepsilon \in I$ for almost every $(x,t) \in \mathbb{R}^+ \times (0,T)$. Let $(\bar{f}_i)_{i=1,\dots,N}$ be another solution corresponding to the initial-boundary conditions $(\overline{u_0}, \overline{a_0})$*

and $\overline{u^\varepsilon} = \sum_{i=1}^N \overline{f_i^\varepsilon} \in I$ *for almost every* $(x,t) \in \mathbb{R}^+ \times (0,T)$. *Assume that*

$$u_0 \leqslant \bar{u}_0 \quad \textit{almost everywhere in} \quad \mathbb{R}^+;$$

$$a_0 \leqslant \bar{a}_0 \quad \textit{almost everywhere in } (0,T).$$

Then

$$f_i^\varepsilon \leqslant \bar{f}_i^\varepsilon \quad \textit{almost everywhere in } \mathbb{R}^+ \times (0,T) \textit{ for } i = 1,\dots,N.$$

Corollary 2.2. *Assume that* $u_0 \in BV(\mathbb{R}^+)$, $a_0 \in BV((0,T))$ *and* $u_0, a_0 \in I$. *Let* $(f_i^\varepsilon)_{i=1,\dots,N}$ *be a solution of problem* (1.4)–(1.7) *and* $u^\varepsilon \in I$ *for almost every* $(x,t) \in \mathbb{R}^+ \times (0,T)$. *Then for almost every* $(x,t) \in \mathbb{R}^+ \times (0,T)$

$$M_i(\min\{essinf(u_0), essinf(a_0)\}, \varepsilon) \leqslant f_i^\varepsilon \leqslant M_i(\max\{esssup(u_0), esssup(a_0)\}, \varepsilon)$$

for $i = 1,\dots,N$.

Proof. Observe that for every $p \in \mathbb{R}$ the vector $(M_1(p,\varepsilon),\dots,M_N(p,\varepsilon))$ is a weak solution to the system (1.4)–(1.7) with initial and boundary data identically equal to p. Then the claim follows by Corollary 2.1. □

In the following we set

$$I \coloneqq \{u \in \mathbb{R}; \min\{\mathrm{essinf}(u_0), \mathrm{essinf}(a_0)\} \leqslant u \leqslant \max\{\mathrm{esssup}(u_0), \mathrm{esssup}(a_0)\}\}.$$

By using standard regularity properties, Corollary 2.2 and proceeding as in [30] we obtain the following result for the solutions of the problem (1.4)–(1.7) (see [27]).

Theorem 2.1. *Let* $u_0 \in BV(\mathbb{R}^+), a_0 \in BV((0,T))$. *Then there exists a unique global weak solution* $(f_i^\varepsilon)_{i=1,\dots,N}$ *to problem* (1.4)–(1.7) *and* $u^\varepsilon \in I$. *Moreover, when the data* u_0, a_0 *are in the class* C^k $(k \geqslant 1)$ *with* $u_0(0) = a_0(0)$, ${u_0}'(0) = {a_0}'(0) = 0$ *then* $f_i^\varepsilon \in C^1(\mathbb{R}^+ \times [0,T)) \cap C^k(\mathbb{R}^+ \times (0,T) \backslash \cup_{Z^+ \cup Z_0^+} \Gamma_j)$, $i = 1,\dots,N$, *where* $\Gamma_j \coloneqq \{(x,t) \in \mathbb{R}^+ \times [0,T) : x = \gamma_j^\varepsilon t\}$.

Now we prove some estimates for the solutions of problem (1.4)–(1.7) in the case of smooth data, which can be extended to the general BV case thanks to Proposition 2.1.

Proposition 2.2. *Let* $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ *and* $a_0 \in C^2([0,T)) \cap BV((0,T))$ *with* $u_0(0) = a_0(0)$, ${u_0}'(0) = {a_0}'(0) = 0$. *Let* $(f_i^\varepsilon)_{i=1,\dots,N}$ *be the solution of problem* (1.4)–(1.7). *Then for every* $t \in [0,T)$ *the following estimate holds* *for* $i = 1, \dots, N$

$$\int_{\mathbb{R}^+} |\partial_t f_i^\varepsilon(x,t)|\,dx \leqslant \max\{|\gamma_i^\varepsilon| : i = 1, \dots, N\} TV(u_0) + \max\{\gamma_i^\varepsilon : i = 1, \dots, N\} TV(a_0). \tag{2.4}$$

Proof. We fix $L, t \in \mathbb{R}^+$ and we set $p_i = \partial_t f_i^\varepsilon$. Now we first differentiate the equations with respect to t and multiply each equation by the corresponding $sgn(p_i)$, then taking the sum for $i = 1, \dots, N$ and integrating on the domain $D = \{(x,s) \in \mathbb{R}^+ \times (0,t) : 0 \leqslant x \leqslant L + \hat{\gamma}(t-s)\}$, where $\hat{\gamma}$ is defined as in Proposition 2.1, we have

$$\int\int_D \partial_t \left(\sum_{i=1}^N |p_i(x,t)| \right) + \partial_x \left(\sum_{i=1}^N \gamma_i^\varepsilon |p_i(x,t)| \right) dx\,dt = \frac{1}{\varepsilon} \int\int_D F(x,t)\,dx\,dt,$$

where

$$F(x,t) := \sum_{i=1}^N \left(\sum_{j=1}^N M_j{}'(u^\varepsilon(x,t)) sgn(p_j) - sgn(p_i) \right) p_i(x,t).$$

Since $F(x,t) \leqslant 0$ (thanks to condition (M_1) and (M_5)), using the divergence theorem on the domain D we obtain

$$\int_0^L \sum_{i=1}^N |p_i(x,s)|\,dx \leqslant \int_0^{L+\hat{\gamma}t} \sum_{i=1}^N |p_i(x,0)|\,dx + \int_0^t \sum_{i=1}^N \gamma_i^\varepsilon |p_i(0,s)|\,ds;$$

then, using the initial and the boundary conditions, we have the claim. $\square$

The estimate proved in the above proposition is not uniform in ε, due to the expressions of γ_i^ε, however it allows to obtain the following uniform estimate for the BV norm of the traces of f_i^ε on the boundary, for each index i such that $\theta_i < 0$.

Proposition 2.3. *Let* $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ *and* $a_0 \in C^2([0,T)) \cap BV((0,T))$ *with* $u_0(0) = a_0(0)$, $u_0{}'(0) = a_0{}'(0) = 0$. *Let* $(f_i^\varepsilon)_{i=1,\dots,N}$ *be the solution of problem* (1.4)–(1.7). *Then there exists a constant* $K_1 = K_1(\lambda_i, \theta_i, TV(a_0), TV(u_0))$, *not depending on* ε, *such that for every* $h > 0$ *small enough*

$$\int_0^{T-h} \sum_{Z^-} |f_i^\varepsilon(0,t+h) - f_i^\varepsilon(0,t)|\,dt \leqslant K_1 h. \tag{2.5}$$

Proof. For $i = 1, \dots, N$ we set $w_i(x,t) = f_i^\varepsilon(x, t+h) - f_i^\varepsilon(x,t)$ and

$$C_i(x,t) := \frac{M_i(u^\varepsilon(x,t+h)) - M_i(u^\varepsilon(x,t))}{u^\varepsilon(x,t+h) - u^\varepsilon(x,t)}.$$

We consider now the equations verified by the functions w_i for $i = 1, \dots, N$

$$\partial_t w_i(x,t) + \gamma_i^\varepsilon \partial_x w_i(x,t) = \frac{1}{\varepsilon}\left((C_i(x,t) - 1) w_i(x,t) + C_i(x,t) \sum_{j \neq i} w_j(x,t) \right)$$

and we multiply each of them by the corresponding $sgn(w_i)$; summing up the equations and integrating on the domain $D := \{(x,t) \in \mathbb{R}^+ \times (0,T) : 0 < x < \hat{\gamma}(T-t)\}$ we obtain the equality

$$\int\int_D \partial_t \left(\sum_{i=1}^N |w_i| \right) + \partial_x \left(\sum_{i=1}^N \gamma_i^\varepsilon |w_i| \right) dx\,dt = \frac{1}{\varepsilon} \int\int_D F\,dx\,dt,$$

where

$$F(x,t) := \sum_{i=1}^N \left(\sum_{j=1}^N C_j(x,t) sgn(w_j) - sgn(w_i) \right) w_i(x,t).$$

We observe that the conditions (M_1) and (M_5) imply that $0 \leqslant C_i(x,t) \leqslant 1$ for $i = 1, \dots, N$ and $\sum_{i=1}^N C_i(x,t) = 1$ and then that $F(x,t) \leqslant 0$; therefore, using the divergence theorem on the domain D we obtain

$$\begin{aligned}
&\sum_{Z^- \cup Z_0^-} |\gamma_i^\varepsilon| \int_0^t |f_i^\varepsilon(0,t+h) - f_i^\varepsilon(0,t)|\,dt \\
&\quad \leqslant \int_0^{\hat{\gamma}t} \sum_{i=1}^N |f_i^\varepsilon(x,h) - f_i^\varepsilon(x,0)|\,dx + \sum_{Z^+ \cup Z_0^+} \gamma_i^\varepsilon \int_0^t |M_i(a_0(t+h)) - M_i(a_0(t))|\,dt
\end{aligned}$$

and then

$$\begin{aligned}
&\sum_{Z^-} \int_0^t |f_i^\varepsilon(0,t+h) - f_i^\varepsilon(0,t)|\,dt \\
&\quad \leqslant \frac{\sqrt{\varepsilon}}{\min\{\sqrt{\varepsilon}\lambda_i + \theta_i : i \in Z^-\}} \int_0^{\hat{\gamma}t} \sum_{i=1}^N |f_i^\varepsilon(x,h) - f_i^\varepsilon(x,0)|\,dx \\
&\qquad + \frac{\max\{\sqrt{\varepsilon}\lambda_i + \theta_i : i \in Z^+ \cup Z_0^+\}}{\min\{\sqrt{\varepsilon}\lambda_i + \theta_i : i \in Z^-\}} TV(a_0)h.
\end{aligned}$$

The claim follows using (2.4). $\square$

Next lemma gives an estimate for the spatial derivatives of the functions f_i^ε on the boundary, in terms of the BV-norm of $a_0(t)$. The result is crucial to prove a further uniform estimate holding for sequence $\{f_i^\varepsilon\}$, which will be useful to derive compactness properties .

Lemma 2.2 (Vuk Milišić [27]). *Let* $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ *and* $a_0 \in C^2([0,T)) \cap BV((0,T))$ *with* $u_0(0) = a_0(0)$, $u_0'(0) = a_0'(0) = 0$. *Let* $(f_i^\varepsilon)_{i=1,\dots,N}$ *be the solution of problem* (1.4)–(1.7). *Then*

$$\sum_{i=1}^{N} \int_0^T \gamma_i^\varepsilon |\partial_x f_i^\varepsilon(t,0)|\, dt \leqslant \int_0^T |a_0'(t)|\, dt. \tag{2.6}$$

Proof. Using the equations for f_i^ε we obtain the following inequality:

$$\begin{aligned}\sum_{i=1}^{N} \gamma_i^\varepsilon |\partial_x f_i^\varepsilon(0,t)| \leqslant & \sum_{i\in Z^+\cup Z_0^+} \left(\frac{|M_i(u^\varepsilon(0,t)) - M_i(a_0(t)|}{\varepsilon} + |\partial_t M_i(a_0(t))| \right) \\ & - \sum_{i\in Z^-\cup Z_0^-} \left| \frac{M_i(u^\varepsilon(0,t)) - f_i^\varepsilon(0,t)}{\varepsilon} - \partial_t f_i^\varepsilon(0,t) \right|\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\sum_{i=1}^{N} \gamma_i^\varepsilon |\partial_x f_i^\varepsilon(0,t)| \leqslant & \frac{|u^\varepsilon(0,t) - a_0(t)|}{\varepsilon} + |a_0'(t)| \\ & - \sum_{i\in Z^-\cup Z_0^-} \left(\frac{|M_i(u^\varepsilon(0,t) - M_i(a_0(t))|}{\varepsilon} + |M_i'(a_0(t))| \right. \\ & \left. + \left| \frac{M_i(u^\varepsilon(0,t) - f_i^\varepsilon(0,t)}{\varepsilon} - \partial_t f_i^\varepsilon(0,t) \right| \right).\end{aligned}$$

Now, it is readily seen that

$$\begin{aligned}\sum_{i=1}^{N} \gamma_i^\varepsilon |\partial_x f_i^\varepsilon(0,t)| \leqslant & \frac{|u^\varepsilon(0,t) - a_0(t)|}{\varepsilon} + |a_0'(t)| \\ & - \left| \sum_{i\in Z^-\cup Z_0^-} \left(\frac{M_i(a_0(t)) - f_i^\varepsilon(0,t)}{\varepsilon} + \partial_t (M_i(a_0(t)) - f_i^\varepsilon(0,t)) \right) \right|\end{aligned}$$

and using that $u^\varepsilon(0,t) = \sum_{Z^+\cup Z_0^+} M_i(a_0(t)) + \sum_{Z^-\cup Z_0^-} f_i^\varepsilon(0,t)$ we derive the following inequality:

$$\sum_{i=1}^{N} \gamma_i^\varepsilon |\partial_x f_i^\varepsilon(0,t)| \leqslant \frac{|u^\varepsilon(0,t) - a_0(t)|}{\varepsilon} + |a_0'(t)| - \left| \frac{a_0(t) - u^\varepsilon(0,t)}{\varepsilon} + \partial_t (a_0(t) - u^\varepsilon(0,t)) \right|.$$

Thanks to convexity property of the absolute value function we obtain

$$\begin{aligned}\sum_{i=1}^{N}\gamma_i^{\varepsilon}|\partial_x f_i^{\varepsilon}(0,t)| &\leqslant -\operatorname{sgn}\left(\frac{a_0(t)-u^{\varepsilon}(0,t)}{\varepsilon}\right)\partial_t(a_0(t)-u^{\varepsilon}(0,t))+|{a_0}'(t)|\\ &= -\partial_t(|a_0(t)-u^{\varepsilon}(0,t)|)+|{a_0}'(t)|.\end{aligned}$$

Integrating on $(0,T)$ the above estimate we obtain

$$\sum_{i=1}^{N}\int_0^T \gamma_i^{\varepsilon}|\partial_x f_i^{\varepsilon}(0,t)|\,dt \leqslant \int_0^T -(\partial_t(|a_0(t)-u^{\varepsilon}(0,t)|)+|{a_0}'(t)|)\,dt$$

and the claim follows.

Now we use the previous lemma to estimate the spatial derivatives of f_i^{ε} in the interior domain.

Proposition 2.4. *Let* $u_0\in C^2(\overline{\mathbb{R}^+})\cap BV(\mathbb{R}^+)$ *and* $a_0\in C^2([0,T))\cap BV((0,T))$ *with* $u_0(0)=a_0(0)$, ${u_0}'(0)={a_0}'(0)=0$. *Let* $(f_i^{\varepsilon})_{i=1,\dots,N}$ *be the solution of problem* (1.4)–(1.7). *Then there exists a constant* $K_2=K_2(TV(a_0),TV(u_0))$, *not depending on* ε, *such that for every* $L>0$ *and for every* $h>0$ *small enough*

$$\int_0^L |\partial_x f_i^{\varepsilon}(x,t)|\,dx \leqslant K_2, \quad \textit{for } i=1,\dots,N. \tag{2.7}$$

Proof. We fix $L,t>0$ and we set $m_i=\partial_x f_i^{\varepsilon}$; applying the technique of the previous proofs we establish that

$$\int_0^L\sum_{i=1}^{N}|m_i(x,t)|\,dx \leqslant \int_0^{L+\hat{\gamma}t}\sum_{i=1}^{N}|m_i(x,0)|\,dx+\int_0^t\sum_{i=1}^{N}\gamma_i^{\varepsilon}|m_i(0,s)|\,ds.$$

The last integral can be estimate using Lemma 2.2 and the proof is complete. □

Now we state a proposition which gives an estimate of the deviation from the equilibrium in the L^1 norm (see [11] for the proof). The result will be crucial in the proof of the last uniform estimate in this section and in the proof of consistency of our relaxation approximation in the next section.

Proposition 2.5. *Let* $u_0\in C^2(\overline{\mathbb{R}^+})\cap BV(\mathbb{R}^+)$ *and* $a_0\in C^2([0,T))\cap BV((0,T))$ *with* $u_0(0)=a_0(0)$, ${u_0}'(0)={a_0}'(0)=0$. *Let* $(f_i^{\varepsilon})_{i=1,\dots,N}$ *be the solution of problem* (1.4)–(1.7). *Then there exists a constant* $K_3=K_3(\lambda_i,\theta_i,TV(a_0),TV(u_0))$, *not depending on* ε, *such that for every* $L>0$

$$\sum_{i=1}^{N}\int_0^L |f_i^{\varepsilon}(x,t)-M_i(u^{\varepsilon},\varepsilon)|\,dx \leqslant \sqrt{\varepsilon}K_3.$$

The following proposition completes the set of estimates necessary to prove the convergence result in the next section.

Proposition 2.6. *Let* $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ *and* $a_0 \in C^2([0,T)) \cap BV((0,T))$ *with* $u_0(0) = a_0(0)$, $u_0{}'(0) = a_0{}'(0) = 0$. *Let* $(f_i^\varepsilon)_{i=1,\dots,N}$ *be the solution of problem* (1.4)–(1.7). *Then for every* $L > 0$ *there exist a positive constant* h_0 *and a continuous nondecreasing function* $w : [0, h_0] \to \mathbb{R}$, *not depending on* ε, *with* $w(0) = 0$, *such that, for* $t \in (0, T-h)$ *and* $h \in (0, h_0)$

$$\int_0^L |u^\varepsilon(x, t+h) - u^\varepsilon(x,t)|\, dx \leqslant w(h).$$

Proof. We use Lemma 5 in [20]; in view of Proposition 2.4 we must only prove that there exists a constant C_L such that

$$\left| \int_0^L \partial_t u^\varepsilon(x,t)\varphi(x)\, dx \right| \leqslant C_L ||\varphi||_{C^1} \quad \text{for every function } \varphi \in C_0^1((0,L)).$$

Using the equations (1.4) for f_i^ε and condition $(\mathbf{M}_2)$ we establish that

$$\left| \int_0^t \partial_t u^\varepsilon(x,t)\varphi(x)\, dx \right| \leqslant ||\varphi||_{C^1} \int_0^L \max\{|\gamma_i^\varepsilon|\} \sum_{i=1}^N |f_i^\varepsilon(x,t) - M_i(u^\varepsilon, \varepsilon)|\, dx + ||\varphi||_{C^1} \int_0^L |A(u^\varepsilon)|\, dx$$

and using Proposition 2.5 we conclude the proof. □

Now the above propositions in combination with the comparison results (2.2)–(2.3) imply the following theorem.

Theorem 2.2. *Let* $L > 0$, $u_0 \in BV(\mathbb{R}^+)$ *and* $a_0 \in BV((0,T))$. *Let* $(f_i^\varepsilon)_{i=1,\dots,N}$ *be the solution of problem* (1.4)–(1.7) *in* $(\mathbb{R}^+ \times (0,T))$. *Then for every* $L, h > 0$ *there exist two positive constants* $h_0, C_1 = C_1(\lambda_i, \theta_i, TV(a_0), TV(u_0), L)$ *and a continuous nondecreasing function* $w : [0, h_0] \to \mathbb{R}$, *not depending on* ε, *with* $w(0) = 0$, *such that*

$$TV(f_i^\varepsilon(\cdot,t), (0,L)) \leqslant C_1 \quad \textit{for } t \in (0,T], i = 1, \dots, N,$$

$$\int_0^L |f_i^\varepsilon(x,t) - M_i(u^\varepsilon(x,t), \varepsilon)|\, dx \leqslant C_1\sqrt{\varepsilon} \quad \textit{for } t \in (0,T], i = 1, \dots, N,$$

$$\sum_{Z^+} \int_0^{T-h} |f_i^\varepsilon(0, t+h) - f_i^\varepsilon(0,t)|\, dt \leqslant C_1 h,$$

$$\int_0^L |u^\varepsilon(x, t+h) - u^\varepsilon(x,t)|\, dx \leqslant w(h) \quad \textit{for } t \in (0, T-h), h \in (0, h_0).$$

Using the above results we can extract a subsequence from $\{u^\varepsilon\}$ converging to some limit function u. In the next section we will show that such a limit is the solution of problem (1.4)–(1.7), concluding the proof of Theorem 1.1 stated in the Introduction.

3. Proof of Theorem 1.1

The last step in the proof of Theorem 1.1 is to prove that the function u obtained as limit of a subsequence of $\{u^\varepsilon\}$ verifies conditions (i) and (ii) of Definition 1.1.

As in [11] we are going to use the convex kinetic entropies $H^\varepsilon_{\eta,k}$ associated to each macroscopic convex entropy η of (1.1). According to [9], they can be obtained as

$$\begin{aligned} H^\varepsilon_{k,\eta}(f^\varepsilon_k) = & \int_{\mathbb{R}} \tfrac{1}{2}(|f^\varepsilon_k - M^\varepsilon_k(c)| - |M^\varepsilon_k(c)|)\eta''(c)\,dc \\ & + \tfrac{1}{2} f^\varepsilon_k(\eta'(-\infty) + \eta'(\infty)) \end{aligned}$$

which are convex in f^ε_k and have a Lipschitz constant independent of ε.

As in [11] we make first a simplifying assumption: we assume that

$$M^\varepsilon_k(\cdot) \text{ is strictly increasing in } I = \{u \in \mathbb{R} : |u| \leqslant \mu_\infty\}, \tag{3.1}$$

where $\mu_\infty = \max(||u_0||_{L^\infty(\mathbb{R}^+)}, ||a_0||_{L^\infty(\mathbb{R}^+)})$ and let us consider only entropies $\eta \in C^2$. Now for such C^2 entropies, (3.1) ensures that $H^\varepsilon_{k,\eta} \in C^1([\mathrm{M}^\varepsilon_k(-\mu_\infty), \mathrm{M}^\varepsilon_k(-\mu_\infty)])$. As in [11], using (M_1)–(M_4), we have the following relationships:

$$(H^\varepsilon_{k,\eta}(M^\varepsilon_k(w))' = \eta'(w), \ \ \forall w \in I, \tag{3.2}$$

$$\begin{aligned} & \sum_k H^\varepsilon_{k,\eta}(M^\varepsilon_k(w)) = \eta(w) - \eta(0), \\ & \sum_k \gamma^\varepsilon_k H^\varepsilon_{k,\eta}(M^\varepsilon_k(w)) = G_\eta(w) \quad \text{for } {G_\eta}' = \eta' A', \ G_\eta(0) = 0, \\ & \sum_k \theta^2_k H^\varepsilon_{k,\eta}(M^\varepsilon_k(w)) = B_\eta(w) + o(1) \quad \text{for } {B_\eta}' = \eta' B', \ B_\eta(0) = 0. \end{aligned} \tag{3.3}$$

In the following we set $Q = \mathbb{R}^+ \times (0, T)$.

The first result we need is an $L^2(Q)$-estimate of the quantities $(M_i(u^\varepsilon) - f^\varepsilon_i)$. In order to do this we will prove some preliminary results.

Let us multiply (1.4) by ${H_k}'(f^\varepsilon_k)$, which is possible since we suppose $H_k \in C^1$. We get

$$\partial_t H_k(f^\varepsilon_k) + \partial_x \gamma^\varepsilon_k H_k(f^\varepsilon_k) = {H_k}'(f^\varepsilon_k)\frac{M^\varepsilon_k(f^\varepsilon_k) - f^\varepsilon_k}{\varepsilon}. \tag{3.4}$$

If we choose η such that $\eta(0) = 0$, $\eta'(0) = 0$, then $H_k(M^\varepsilon_k(0)) = 0, {H_k}'(M^\varepsilon_k(0)) = 0$, and thus $H_k \geqslant 0$. Integrating over a semi cone and letting it tend to infinity after summing up with respect to k we have

$$\begin{aligned}
&\sum_k \int_0^L H_k(f_k^\varepsilon)(x,T)\,dx + \sum_k \int_Q [H_k{}'(M_k^\varepsilon(u^\varepsilon)) - H_k{}'(f_k^\varepsilon)]\frac{M_k^\varepsilon(f_k^\varepsilon) - f_k^\varepsilon}{\varepsilon}\,dQ \\
&\quad \leqslant \sum_k \int_{\mathbb{R}^+} H_k(M_k(u_0))\,dx + \sum_k \int_0^T \gamma_k^\varepsilon H_k(f_k^\varepsilon(0,t))\,dt \\
&\quad = \int_{\mathbb{R}^+} \eta(u_0)\,dx + \sum_k \int_0^T \gamma_k^\varepsilon H_k(f_k^\varepsilon(0,t))\,dt. \qquad (3.5)
\end{aligned}$$

In the following lemma we find an upper bound for the last boundary term.

Lemma 3.1. *Assume hypothesis* (3.1). *For any C^2 convex entropy $\eta(\cdot)$, on the boundary we have the following estimate*:

$$\int_0^T \sum_k \gamma_k^\varepsilon H_{\eta,k}(f_k^\varepsilon)(0,t)\,dt \leqslant C,$$

where C is independent of ε.

Proof. Observe that it is not restrictive to assume that the data and the solution of (1.4) are regular since using the stability results of Proposition 2.1 we can treat general data. Imposing boundary condition on the entering characteristics we have for any entropy functions

$$\begin{aligned}
\sum_k \gamma_k^\varepsilon H_{\eta,k}(f_k^\varepsilon) &= \sum_{\gamma_k^\varepsilon > 0} \gamma_k^\varepsilon H_{\eta,k}(M_k^\varepsilon(a_0)) + \sum_{\gamma_k^\varepsilon < 0} \gamma_k^\varepsilon H_{\eta,k}(f_k^\varepsilon) \\
&= G_\eta(a_0) + \sum_{\gamma_k^\varepsilon < 0} \gamma_k^\varepsilon (H_{\eta,k}(f_k^\varepsilon) - H_{\eta,k}(M_k^\varepsilon(a_0))) \\
&\leqslant G_\eta(a_0) + \sum_{\gamma_k^\varepsilon < 0} \gamma_k^\varepsilon H'_{\eta,k}(M_k^\varepsilon(a_0))(f_k^\varepsilon - M_k^\varepsilon(a_0)) \\
&= G_\eta(a_0) + \eta'(a_0) \sum_{\gamma_k^\varepsilon < 0} \gamma_k^\varepsilon (f_k^\varepsilon - M_k^\varepsilon(a_0)) \\
&= G_\eta(a_0) + \eta'(a_0)\left[\sum_k \gamma_k^\varepsilon f_k^\varepsilon - A(a_0)\right]. \qquad (3.6)
\end{aligned}$$

The boundary term that we want to control can then be written

$$\int_0^T \eta'(a_0(t))v^\varepsilon(t,0)\,dt, \qquad (3.7)$$

where

$$v^\varepsilon = \sum_k \gamma_k^\varepsilon f_k^\varepsilon$$

and v^ε satisfies in the interior of the domain

$$\partial_t u^\varepsilon + \partial_x v^\varepsilon = 0. \tag{3.8}$$

Let us introduce a function ρ such that

$$\rho \in C^2(\mathbb{R}^+), \;\; \rho(x) = 0, \forall x > 1,$$

$$\rho(0) = 1 \quad \text{and } |\rho'(\cdot)| \leqslant c.$$

We aim to control

$$\begin{aligned}\int_0^T \eta'(a_0) v^\varepsilon \, dt &= -\int_0^T \int_0^1 \partial_x(\eta'(a_0)v^\varepsilon)\rho \, dx \, dt - \int_0^T \int_0^1 \eta'(a_0) v^\varepsilon \rho' \, dt \\ &= I_1 + I_2.\end{aligned}$$

Let us examine the first term I_1. Using (3.8) we have

$$\begin{aligned} I_1 &= \int_0^T \int_0^1 \eta'(a_0)\partial_t u^\varepsilon \rho \, dx \, dt \\ &= \int_0^T \int_0^1 (\partial_t(\eta'(a_0)u^\varepsilon) - (\partial_t \eta'(a_0))u^\varepsilon)\rho \, dx \, dt \\ &= \int_0^1 (\eta'(a_0(T))u^\varepsilon(x,T) - \eta'(a_0(0))u^\varepsilon(x,0))\rho \, dx - \int_0^T \int_0^1 (\partial_t \eta'(a_0))u^\varepsilon \rho \, dx \, dt. \end{aligned}$$

So that we can bound I_1 by

$$|I_1| \leqslant C(||a_0||_{L^\infty(0,T)} ||u^\varepsilon||_{L^\infty((0,T)\times\mathbb{R}^+)} + ||u^\varepsilon||_{L^\infty} BV(a_0))$$

independently of ε

For the last term it is enough to write

$$
\begin{aligned}
|I_2| &\leqslant \left| \int_0^T \int_0^1 \eta'(a_0) A(u^\varepsilon) \rho' \, dx \, dt \right| + \left| \int_0^T \int_0^1 \eta'(a_0)(v^\varepsilon - A(u^\varepsilon)) \rho' \, dx \, dt \right| \\
&\leqslant C \|a_0\|_{L^\infty(0,T)} (\|A(u^\varepsilon)\|_{L^1(0,T\times(0,1))} + \frac{\max_i\{|\theta_i|\}}{\sqrt{\varepsilon}} \\
&\quad \times \int_0^T \int_0^1 \sum_i |f_i^\varepsilon(t,x) - M_i(u^\varepsilon, \varepsilon)| \, dx) \\
&\leqslant C.
\end{aligned}
$$

So that again this is independent of ε, and C only depends of the data (u_0, a_0) and Lipshitz constant of $\eta'(\cdot)$. $\square$

Now we can derive the desired L^2 estimate.

Lemma 3.2. *There exists a positive constant C such that, for all $\varepsilon > 0$ and for all $k = 1, \ldots, N$*

$$
\int_0^T \int_{\mathbb{R}^+} [M_k^\varepsilon(f_k^\varepsilon) - f_k^\varepsilon]^2 \, dx \leqslant \varepsilon C.
$$

Proof. Using the previous lemma, one has for any C^2 convex entropy that

$$
\sum_k \int_0^T \int_{\mathbb{R}^+} [H_{k,\eta}^{\varepsilon}{}'(M_k^\varepsilon(w)) - H_{k,\eta}^{\varepsilon}{}'(f_k^\varepsilon)] \frac{M_k^\varepsilon(f_k^\varepsilon) - f_k^\varepsilon}{\varepsilon} \, dx \, dt \leqslant C.
$$

Now we choose $\eta(u) = \frac{u^2}{2}$. Again using the method of [11], one has that

$$
[f_k^\varepsilon - M_k^\varepsilon(u^\varepsilon)]^2 \leqslant [H_k'(f_k^\varepsilon) - H_k'(M_k^\varepsilon(u^\varepsilon))](f_k^\varepsilon - M_k^\varepsilon(f_k^\varepsilon)) \tag{3.9}
$$

which ends the proof. $\square$

Remark 1. In [11] the authors show that assumption (3.1) can be removed, so that $H_{k,\eta}^\varepsilon$ is not C^1. The main point is to replace $H_{k,\eta}^{\varepsilon}{}'(M_k^\varepsilon(w))$ with $\eta'(w)$, since, in this case, equality (3.2) does not make sense for some values of w. Then, inequality (3.5) and Lemmas 3.1–3.2 are still true.

The previous lemma is crucial for establishing convergence when ε goes to zero, for interior regions as for boundary terms.

An immediate consequence is that the first part of condition (i) in Definition (1.1) is verified by the limit function u, as the following lemma shows.

Lemma 3.3. *Let u be the limit of a subsequence of* $\{u^\varepsilon\}$. *Then*

$$\partial_x B(u) \in L^2(\mathbb{R}^+ \times (0,T)).$$

Proof. Using the equations for f_k^ε we have, for all $\phi \in C_0^\infty(\mathbb{R}^+ \times (0,T))$

$$\int_Q \gamma_k\,(f_k^\varepsilon - M_k^\varepsilon(u^\varepsilon))\phi\, dQ = \varepsilon \int_Q \gamma_k f_k^\varepsilon \phi_t + \varepsilon \int_Q \gamma_k^2 f_k^\varepsilon \phi_x\, dQ \ \forall k \in \{1, \dots, N\}$$

and then

$$\begin{aligned}&\int_Q \gamma_k (f_k^\varepsilon - M_k^\varepsilon(u^\varepsilon))\phi\, dQ\\ &\quad = \varepsilon \int_Q \gamma_k f_k^\varepsilon \phi_t + \varepsilon \int_Q \gamma_k^2 (f_k^\varepsilon - M_k^\varepsilon(u^\varepsilon))\phi_x\, dQ + \varepsilon \int_Q \gamma_k^2 M_k^\varepsilon(u^\varepsilon)\phi_x\, dQ.\end{aligned}$$

Summing over k and letting $\varepsilon \to 0$ we have

$$\lim_{\varepsilon\to 0} \sum_{k=1}^N \int_Q \gamma_k (f_k^\varepsilon - M_k^\varepsilon(u^\varepsilon))\phi\, dQ = \int_Q B(u)\phi_x\, dQ,$$

where we used Proposition 2.5, condition (M_4) and the immediate estimate

$$\left|\varepsilon \int_Q \gamma_k^\varepsilon f_k \phi_t\, dQ\right| \leqslant C||f_k||_{L^\infty(Q)} ||\phi_t||_{L^1(Q)} \sqrt{\varepsilon}.$$

On the other hand we know that $\sum_{k=1}^N \gamma_k (f_k^\varepsilon - M_k^\varepsilon(u^\varepsilon))$ is bounded in $L^2(Q)$; hence

$$\sum_{k=1}^N \gamma_k (f_k^\varepsilon - M_k^\varepsilon(u^\varepsilon)) \rightharpoonup \partial_x B(u) \ \text{ in } \ L^2(Q) - \textit{weak}$$

and

$$\partial_x B(u) \in L^2(Q). \qquad \square$$

In the following lemma, using again the L^2 estimate of Lemma 3.2, we obtain the important information that when the velocity $\theta_k \neq 0$, the trace of the solution $f_k^\varepsilon(0,t)$ tends to the equilibrium state $M_k(a_0(t))$ when ε goes to zero. This is not the case for the hyperbolic equations (see [27]).

We recall that, thanks to Theorem 2.2, for $k \in Z^+ \cup Z^-$ there exists a subsequence of $\{f_k^\varepsilon(0,t)\}$ converging to some function $w_k(t)$ in $L^1((0,T))$.

Lemma 3.4. *For all* $k \in Z^+ \cup Z^-$

$$w_k(t) = M_k(u(0,t)) \quad t \in (0,T).$$

Proof. For all $\phi \in C_0^\infty([0,+\infty) \times (0,T))$ we have

$$\int_Q \gamma_k^\varepsilon (f_k^\varepsilon - M_k^\varepsilon(u^\varepsilon))\phi \, dQ$$
$$= \int_Q \varepsilon \gamma_k^\varepsilon f_k^\varepsilon \phi_t \, dQ + \int_Q \varepsilon \gamma_k^{\varepsilon 2} f_k \phi_x \, dQ + \int_0^T \varepsilon \gamma_k^{\varepsilon 2} f_k^\varepsilon(0,t)\phi(0,t)\, dt. \tag{3.10}$$

The first and the second term on the right-hand side can be treated as in the previous lemma, for ε going to zero.

For the last term, which gives information on the boundary, it can be easily proved that

$$\varepsilon \int_0^T \gamma_k^{\varepsilon 2} f_k \phi(t,0)\, dt \to \int_0^T \theta_k^2 w_k(t)\phi(t,0)\, dt.$$

For the source term part, one has again that $\gamma_k^\varepsilon(f_k^\varepsilon - M_k^\varepsilon(u^\varepsilon)) \in L^2(Q)$, then, up to extracting a subsequence, it converges in the weak topology. It follows that, letting ε go to zero in (3.10) we obtain

$$\int_Q h\phi \, dQ = \int_0^T \theta_k^2 w_k(t)\phi(0,t)\, dt + \int_Q \theta_k^2 M_k(u)\phi_x \, dQ, \tag{3.11}$$

where h is a suitable function in $L^2(Q)$. Choosing as text function $\phi_\delta = \tau_\delta(x)\psi(t)$, where $\tau_\delta(x) = \max(0, 1 - \frac{x}{\delta})$ and $\psi \in C_0^\infty((0,T))$, since

$$\lim_{\delta \to 0} ||\phi_\delta||_{L^2(Q)} = 0,$$

the left-hand side of (3.11) vanishes when δ goes to zero. So the claim follows letting δ go to zero in (3.11). $\square$

The result stated in the previous lemma allows to prove that u satisfies the boundary condition in Definition 1.1.

Lemma 3.5. *For* $t \in (0,T)$

$$B(u(0,t)) = B(a_0(t)).$$

Proof. From condition (M_4) we know that $B(u) = \sum_{i=1}^N \theta_k^2 M_k(u,0)$, so the result is proved if we show that

$$M_k(a_0) = M_k(u) \quad \forall k \in Z^+ \cup Z^-. \tag{3.12}$$

Condition (M_2) implies that, for $\varepsilon = 0$ the Maxwellian functions must satisfy the following property:

$$\sum_k \theta_k M_k(u) = 0 \quad \forall u \in I. \tag{3.13}$$

But, due to the convergence results obtained previously and the boundary condition imposed on entering characteristics ($\theta_k > 0$), one has that on the boundary

$$M_k(a_0) = M_k(u) \quad \forall k \in Z^+$$

so that, in fact, we have the following constraint:

$$\sum_{k \in Z^-} \theta_k M_k(u) + \sum_{k \in Z^+} \theta_k M_k(a_0) = 0.$$

Using again the same property (3.13) on a_0, one has

$$\sum_{\theta_k < 0} \theta_k (M_k(u) - M_k(a_0)) = 0.$$

The monotonicity property of all the Maxwellian functions implies that all the terms in the above sum have to be equal to zero. Therefore (3.12) holds and the claim follows. $\square$

If the Maxwellian functions are strictly increasing the previous lemma tells us that $u(t,0) = a_0(t)$, which corresponds to the expected regularity for non-degenerate parabolic case.

Moreover, if $\theta_k \neq 0$ for every $k = 1, \dots, N$, Lemma 3.4 implies that $u(t,0) = a_0(t)$. This shows that in the strongly degenerate case, when the boundary condition is not achieved in the classical sense, one of the θ_k must vanish.

Equality (3.12) obtained in the proof of the above lemma shows that for all $k \in Z^+ \cup Z^-$ the functions $w_k(t)$ are in fact the limits of the whole sequences $\{f_k^\varepsilon(0,t)\}$.

In the remaining of this section we prove that the function u verifies the entropy condition (ii) of Definition 1.1. After this proof, thanks to uniqueness result for entropy solutions of Eq. (1.1), we will be able to state that the whole sequence $\{u^\varepsilon\}$ converges in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^+))$ to u, solution of problem (1.1)–(1.3). In order to establish this result we have to work with entropies that are only lipschitz continuous, more precisely we are interested in the entropies $\eta_c^\pm(s) = [s-c]_\pm$ and the associated kinetic entropies $H_{k,c}^{\pm,\varepsilon}(s) = [s - M_k^\varepsilon(c)]_\pm$. It is a simple check to prove that the equalities (3.3) are verified for these entropies (see [11]). In such a case, we set $A_{\eta_c^\pm}(s) = H^\pm(s-c)(A(s) - A(c))$ and $B_{\eta_c^\pm}(s) = H^\pm(s-c)(B(s) - B(c))$.

Proposition 3.1. *The function* $u \in C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^+)) \cap L^\infty((0,T) \times \mathbb{R}^+)$ *verifies condition* (ii) *of Definition* 1.1.

Proof. Let us consider the entropy $\eta_c^+(s)$. In order to simplify the notation in this proof we put $H_k(s) = H_{k,c}^{+,\varepsilon}(s)$.

In [11] the authors observe that the multiplication of Eq. (1.4) with the multivalued function $H_k{}'(s) = H^+(s - M_k^\varepsilon(c))$ is well defined, and gives again (3.4), (see [10, Theorem 3.1]). Let us sum these equalities, multiply with a test function $\phi \in C_0^1(\mathbb{R}^+ \times [0, T))$, $\phi \geqslant 0$ and integrate in Q.

Let us note that

$$\sum_k H^+(f_k^\varepsilon - M_k^\varepsilon(c))(f_k^\varepsilon - M_k^\varepsilon(u^\varepsilon)) \\ = \sum_k (H^+(f_k^\varepsilon - M_k^\varepsilon(c)) - H^+(u^\varepsilon - c))(f_k^\varepsilon - M_k^\varepsilon(u^\varepsilon)) \geqslant 0. \tag{3.14}$$

In fact the equality in (3.13) is obvious and the inequality can be proved cheking as in [11] that every term of the sum is positive.

We obtain

$$-\int_Q \sum_k H_k(f_k^\varepsilon)\phi_t \, dQ - \int_Q \sum_k \gamma_k^\varepsilon H_k(f_k^\varepsilon)\phi_x \, dQ - \int_{\mathbb{R}^+} \eta_c^+(u_0)\phi(x, 0) \, dx \\ \leqslant \int_{\mathbb{R}^+} \sum_k \gamma_k^\varepsilon H_k(f_k^\varepsilon)\phi(0, t) \, dt.$$

Proceeding as in the proof of Lemma 3.1 (see (3.6)) it can be observed that

$$\sum_k \gamma_k^\varepsilon H_k(f_k^\varepsilon) \leqslant H^+(a_0 - c)(A(a_0) - A(c)) \\ + H^+(a_0 - c) \sum_k \gamma_k^\varepsilon (f_k^\varepsilon - M_k^\varepsilon(a_0)). \tag{3.15}$$

Let us consider a test function ϕ such that $H^+(a_0 - c)\phi(t, 0) = 0$, then for such test function we obtain, using inequality (3.15)

$$\int_Q \sum_k H_k(f_k^\varepsilon)\phi_t \, dQ + \int_Q \sum_k \gamma_k^\varepsilon H_k(f_k^\varepsilon)\phi_x \, dQ \\ + \int_{\mathbb{R}^+} [u_0 - c]_+ \phi(x, 0) \, dx \geqslant 0. \tag{3.16}$$

The idea is to pass to the limit when ε goes to zero in the inequality (3.16). It is easy to verify that the first and the last term in inequality (3.16) converge to the corresponding term of (1.8). We want to investigate the second term. To this aim we prove that for every $k = 1, \ldots, N$

$$H_k(f_k^\varepsilon) = H_k(M_k^\varepsilon(u^\varepsilon)) - (H_k{}')(f_k^\varepsilon)(M_k^\varepsilon(u^\varepsilon) - f_k^\varepsilon) + o(\sqrt{\varepsilon}) \quad \text{in } L^1_{\text{loc}}(Q). \tag{3.17}$$

As in [11] we have that

$$\begin{aligned}0 &\leqslant H_k(M_k^\varepsilon(u^\varepsilon)) - H_k(f_k^\varepsilon) - H_k{}'(f_k^\varepsilon)(M_k^\varepsilon(u^\varepsilon) - f_k^\varepsilon)\\ &\leqslant (\eta_c^{+\prime}(u^\varepsilon) - H_k{}'(f_k^\varepsilon))(M_k^\varepsilon(u^\varepsilon) - f_k^\varepsilon)\\ &\leqslant \sum_i (\eta_c^{+\prime}(u^\varepsilon) - H_i{}'(f_i^\varepsilon))(M_i^\varepsilon(u^\varepsilon) - f_i^\varepsilon).\end{aligned} \tag{3.18}$$

Proceeding as (3.16) and Lemma 3.1 we have

$$\begin{aligned}&\int_Q \sum_i (\eta_c{}'^{+}(u^\varepsilon) - H_i{}'(f_i^\varepsilon))(M_i^\varepsilon(u^\varepsilon) - f_i^\varepsilon)\, dQ\\ &\quad\leqslant \varepsilon\Bigg(\int_{\mathbb{R}^+} \eta_c^+(u^0)\, dx + \int_0^T H^+(a_0 - c)(A(a_0) - A(c))\\ &\qquad + H^+(a_0 - c)\sum_i \gamma_k^\varepsilon(f_i^\varepsilon - M_i^\varepsilon(a_0))\, dt\Bigg).\end{aligned} \tag{3.19}$$

Using Lemma 3.4 and inequalities (3.18)–(3.19) we obtain (3.17). From (3.17) we deduce

$$\sum_k \gamma_k^\varepsilon H_k(f_k^\varepsilon) = H^+(u - c)(A(u) - A(c)) - \sum_k \gamma_k^\varepsilon H_k{}'(f_k^\varepsilon)(M_k^\varepsilon(u^\varepsilon) - f_k^\varepsilon) + \mathrm{o}(1) \quad \text{in } L^1_{\mathrm{loc}}(Q). \tag{3.20}$$

To complete the proof, using Eq. (1.4), we prove as in Lemma 3.3 that

$$\lim_{\varepsilon\to 0} \sum_k \gamma_k^\varepsilon H_k{}'(f_k^\varepsilon)(M_k^\varepsilon(u^\varepsilon) - f_k^\varepsilon) = (B_{\eta_c^+})_x = H^+(u - c)(B(u))_x \quad \text{in } L^2_{\mathrm{loc}}(Q) - \mathit{weak}. \tag{3.21}$$

Letting ε goes to zero in the inequality (3.16) and using (3.20)–(3.21) we obtain the result in the case of the entropy η_c^+. The proof for the entropy η_c^- follows exactly the same lines. □

4. Examples

In [10], the authors present several examples of BGK systems of type (1.4), showing that one can always define Maxwellian functions and fix parameters θ_i and λ_i in such a way conditions (M_1)–(M_5) are satisfied.

We already remarked that, in cases of strongly degeneracy involving boundary data, we are forced to set $\theta_i = 0$ for some i (at least one). Here we reconsider some examples in order to show that in fact it is possible to introduce BGK systems satisfying (M_1)–(M_5) with some θ_i equal to zero.

Models with three equations can be obtained only by considering Maxwellian functions depending on ε. In such a situation one can achieve compatibility conditions without using parameters λ_i. As an example we can consider a model presented in [18]: the parameters λ_i, θ_2 are imposed to be zero, $\theta_1 < 0$, $\theta_3 > 0$ and the Maxwellian functions are defined as follows

$$\begin{aligned} M_1(u,\varepsilon) &= \sqrt{\varepsilon}\left(\theta_3 u + \frac{A(u)}{\theta_1 - \theta_3}\right) + \frac{B(u)}{(\theta_1 - \theta_3)\theta_1}, \\ M_2(u,\varepsilon) &= [1 - \sqrt{\varepsilon}(\theta_3 - \theta_1)]u + \frac{B(u)}{\theta_1\theta_3}, \\ M_3(u,\varepsilon) &= \sqrt{\varepsilon}\left(-\theta_1 u - \frac{A(u)}{\theta_1 - \theta_3}\right) - \frac{B(u)}{(\theta_1 - \theta_3)\theta_3}. \end{aligned} \tag{4.1}$$

The conditions (M_1)–(M_4) can be easily verified. If we choose θ_1 and θ_3 in such a way that

$$\begin{cases} max\{B'(u) : u \in I\} < -\theta_1\theta_3 \\ \theta_1(\theta_3 - \theta_1) \leqslant A'(u) \leqslant \theta_3(\theta_3 - \theta_1) \quad u \in I \end{cases} \tag{4.2}$$

the stability condition (M_5) holds for $\varepsilon < \tilde{\varepsilon}$, $\tilde{\varepsilon}$ sufficiently small.

To construct systems with $N \geqslant 4$ we can use the general procedure proposed in [10] for d-dimensional cases, with Maxwellian functions not depending on ε.

In this case the compatibility condition (M_2) can be written

(M_{2a}) $\sum_{i=1}^{N} \lambda_i M_i(w,\varepsilon) = A(w)$ for all $\varepsilon \in]0,1]$ and for all $w \in I$;
(M_{2b}) $\sum_{i=1}^{N} \theta_i M_i(w,\varepsilon) = 0$ for all $\varepsilon \in]0,1]$ and for all $w \in I$.

We fix $\hat{N}$ such that $1 < \hat{N} < N-1$. Then we consider parameters λ_i satisfying conditions

$$\lambda_i \neq 0 \text{ for some } i, \quad \sum_{i=1}^{\hat{N}} \lambda_i = \sum_{i=\hat{N}+1}^{N} \lambda_i = 0.$$

While for parameters θ_i we set

$$\theta_i = 0 \text{ for } i = 1, \ldots, \hat{N}, \quad \sum_{i=\hat{N}+1}^{N} \theta_i = 0, \quad \sum_{i=\hat{N}+1}^{N} \theta_i^2 = \beta^2, \quad \beta \neq 0.$$

In this case, the Maxwellian function are defined as follows

$$M_i(u) = \left(\frac{u}{\hat{N}} + \frac{A(u)\lambda_i}{\sum_{i=1}^{\hat{N}} \lambda_i^2}\right)v_i + \frac{B(u)}{\beta^2}\mu_i,$$

where $v_i = 1$ for $i = 1, \ldots, \hat{N}$, $v_i = 0$ for $i = \hat{N}+1, \ldots, N$, $\mu_i = \frac{-N-\hat{N}}{\hat{N}}$ for $i = 1, \ldots, \hat{N}$, and $\mu_i = 1$ for $i = \hat{N}+1, \ldots, N$. These functions satisfy the compatibility conditions

(M_1)–(M_3). The monotonicity condition requires that

$$M_i'(u) = \frac{1}{\hat{N}} + \frac{A'(u)\lambda_i}{\sum_{i=1}^{\hat{N}} \lambda_i^2} - \frac{N-\hat{N}}{\hat{N}} \frac{B'(u)}{\beta^2} \geqslant 0 \quad i = 1, \dots, \hat{N};$$

the above inequalities hold if β and $|\lambda_i|$ are sufficiently large for any $i \in \{1, \dots, \hat{N}\}$.

References

[1] D. Aregba-Driollet, V. Milisic, Kinetic approximation of a boundary value problem for conservation laws, accepted for publishing in Numerische Matematik.
[2] D. Aregba-Driollet, R. Natalini, Convergence of relaxation schemes for conservation laws, Appl. Anal. 61 (1996) 163–193.
[3] D. Aregba-Driollet, R. Natalini, Discrete kinetic schemes for multidimensional conservation laws, SIAM J. Numer. Anal. 37 (2000) 1973–2004.
[4] D. Aregba-Driollet, R. Natalini, S.Q. Tang, Diffusive kinetic explicit schemes for nonlinear degenerate parabolic systems, Hyperbolic problems: theory, numerics, applications, Vol. I, II (Magdeburg, 2000), Internat. Ser. Numer. Math., 140, 141, Birkhäuser, Basel, (2001), 49–58.
[5] R. Bürger, K.H. Karlsen, A strongly degenerate convection–diffusion problem modeling centrifugation of flocculated suspensions, Hyperbolic problems: theory, numerics, applications, Vol. I, II (Magdeburg, 2000), Internat. Ser. Numer. Math., 140, 141, Birkhäuser, Basel, (2001), 207–216.
[6] R. Bürger, S. Evje, K.H. Karlsen, On strongly degenerate convection–diffusion problems modeling sedimentation–consolidation processes, J. Math. Anal. Appl. (2000) 517–556.
[7] P. Benilan, H. Touré, Sur l' équation génerale $u_t = \phi(u)_{xx} - \psi(u)_x + v$, C.R. Acad. Sci. Paris Sér. I Math. 299 (1984) 919–922.
[8] P.L. Bhatnagar, E.P. Gross, M. Krook, A model for collision processes in gases, Phys. Rev. 94 (1954).
[9] F. Bouchut, Construction of BGK models with a family of kinetic entropies for a given system of conservation laws, J. Statist. Phys. 95 (1999) 113–170.
[10] F. Bouchut, Renormalized solutions to the Vlasov equation with coefficients of bounded variation, Arch. Rational Mech. Anal. 157 (2001) 75–90.
[11] F. Bouchut, F.R. Guarguaglini, R. Natalini, Diffusive BGK approximations for nonlinear multidimensional parabolic equations, Indiana Univ. Math. J. 49 (2) (2000) 723–749.
[12] C. Bardos, A.Y. le Roux, J.-C. Nédélec, First order quasilinear equations with boundary conditions, Comm. Partial Differential Equations 4 (9) (1979) 1017–1034.
[13] J. Carrillo, Entropy solutions for nonlinear degenerate problems, Arch. Rational. Mech. Anal. 147 (4) (1999) 269–361.
[14] C. Cercignani, The Boltzmann Equation and Its Applications, Springer, New York, 1988.
[15] C. Cercignani, R. Illner, M. Pulvirenti, The Mathematical Theory of Diluite Gases, Springer, New York, 1994.
[16] F. Dubois, P. Le Floch, Boundary conditions for nonlinear hyperbolic systems of conservation laws, J. Differential Equations 71 (1988) 93–122.
[17] E. Gabetta, L. Pareschi, M. Ronconi, Central schemes for hydrodynamical limits of discrete-velocity kinetic equations, Proceedings of the fifth international workshop on mathematical aspects of fluid and plasma dynamics, transport theory statis, Phys 29 (2000) 465–477.
[18] F. Guarguaglini, A. Terracina, A BGK Approximation to nonlinear parabolic initial–boundary value problems, Asymptot. Anal. 28 (1) (2001) 75–89.
[19] S. Jin, L. Pareschi, G. Toscani, Diffusive relaxation schemes for multiscale discrete-velocity kinetic equations, SIAM J. Numer. Anal. 35 (1998) 2405–2439.
[20] S.N. Kružkov, First order quasilinear equations in several independent variables, Math. USSR Sb. 10 (1970) 217–243.

[21] T.G. Kurtz, Convergence of sequences of semigroups of nonlinear operators with an application to gas kinetic, Trans. Amer. Math. Soc. 186 (1973) 259–272.
[22] C. Lattanzio, R. Natalini, Convergence of diffusive BGK approximations for nonlinear strongly parabolic systems, Proc. Roy. Soc. Edinburg Sect. A 132 (2002) 341–358.
[23] P.L. Lions, G. Toscani, Diffusive limit for finite velocity Boltzmann kinetic models, Rev. Mater. Iberoamericana 13 (1997) 473–513.
[24] C. Mascia, A. Porretta, A. Terracina, Nonhomogeneous Dirichlet Problems for Degenerate Parabolic–Hyperbolic Equations, Arch. Rational Mech. Anal. 163 (2002) 87–124.
[25] H.P. McKean, The central limit theorem for Carleman's equation, Israel J. Math. 21 (1975) 54–92.
[26] A. Michel, J. Vovelle, A finite volume method for parabolic degenerate problems with general Dirichlet boundary conditions, accepted for publication in SIAM, Journal of Num. Anal.
[27] Vuk Milišić, Stability and convergence of discrete kinetic approximations to an initial-boundary value problem for conservation laws, Proceedings of the AMS 2001 131 (6) (2003) 1727–1737.
[28] Vuk Milišić, Kinetic schemes for strongly degenerate parabolic initial-boundary value problems, work in preparation.
[29] G. Naldi, L. Pareschi, Numerical schemes for kinetic equations in diffusive regimes, Appl. Math. Lett. 11 (1998) 29–35.
[30] R. Natalini, A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws, J. Differential Equations 148 (1998) 292–317.
[31] R. Natalini, A. Terracina, Convergence of a relaxation approximation to a boundary value problem for conservation laws, Commun. Partial Differential Equations 26 (2001) 1235–1252.
[32] S. Nishibata, The initial-boundary value problems for hyperbolic conservation laws with relaxation, J. Differential Equations 130 (1) (1996) 100–126.
[33] A. Nouri, A. Omrane, J.P. Vila, Boundary conditions for scalar conservation laws from a kinetic point of view, J. Statist. Phys. 94 (5–6) (1999) 779–804.
[34] T. Platkowski, R. Illner, Discrete velocity models of the Boltzmann equation: a survey on the mathematical aspects of the theory, SIAM Rev. 30 (1988) 213–255.
[35] W.C. Wang, Z. Xin, Asymptotic limit of initial-boundary value problems for conservation laws with relaxational extensions, Commun. Pure Appl. Math. 51 (5) (1998) 505–535.
[36] W.A. Yong, Boundary conditions for hyperbolic systems with stiff source terms, Indiana Univ. Math. J. 48 (1) (1999) 115–137.