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STABILITY AND CONVERGENCE OF DISCRETE KINETIC APPROXIMATIONS TO AN INITIAL-BOUNDARY VALUE PROBLEM FOR CONSERVATION LAWS

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ABSTRACT. We present some new convergence results for a discrete velocities BGK approximation to an initial boundary value problem for a single hyperbolic conservation law. In this paper we show stability and convergence toward a unique entropy solution in the general BV framework without any restriction either on the data of the limit problem or on the set of velocity of the BGK model.

1. INTRODUCTION

We study the initial-boundary value problem for a scalar conservation law

$$\begin{cases} \partial_t u + \partial_x F(u) = 0, \\ \\ u(0,x) = u^0(x) \end{cases} \tag{1.1}$$

where u is a scalar function of $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, F is a Lipschitz continuous function and the boundary condition is

$$u(t,0) = u_b(t). \tag{1.2}$$

Actually, this condition cannot be imposed for all $t > 0$, but it has to be intended in a generalized sense as a compatibility condition between the trace of the solution and the flux function F; see [3] and (2.1) below. To approximate the solution of (1.1) we choose a semi-linear hyperbolic system of BGK type

$$\begin{cases} \partial_t f_k + \lambda_k \partial_x f_k = \frac{1}{\epsilon}(M_k(u^\epsilon) - f_k), \quad k \in \{1, \ldots, N\}, \\ \\ f_k(0,x) = M_k(u^0(x)), \quad t = 0, \end{cases} \tag{1.3}$$

where λ_k are fixed velocities, u^ϵ is defined by $u^\epsilon = \sum_k = f_k$, and ϵ is a positive parameter. In addition, we impose a boundary condition for the positive speeds

$$f_k(t,0) = M_k(u_b(t)), \quad \text{if} \quad \lambda_k > 0. \tag{1.4}$$

It easy to see, at least at the formal level, that if the sequence u^ϵ converges (strongly) to a limit function u, then u is a weak solution of problem (1.1). The functions

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M_k are called Maxwellian functions and are the link between (1.3) and the scalar conservation law (1.1), since they are chosen to satisfy the following compatibility relationship with (1.1):

$$\begin{cases} \sum_k M_k(u) = u, \\ \sum_k \lambda_k M_k(u) = F(u). \end{cases} \tag{1.5}$$

To have stability with respect to the parameter ϵ, we introduce a monotonicity condition on the previous functions (as already seen in [19, 13] for the Cauchy problem) that provides all results of monotonicity on the corresponding solutions.

This kind of approximation was first introduced by S. Jin and Z. Xin in [9] for its specific features, namely the analytical simplicity, the meaningful physical interpretation and numerical flexibility. From the theoretical point of view, weak and smooth solutions were studied in [7, 13, 14] in the case of the Cauchy problem. Stability of the boundary layers was investigated in [16] and in [11]. For more general regular solutions the reader can refer to Yong's works; see [20]. The specific case of isentropic gas dynamics is studied in a recent work of [5], where the authors present a complete theory of convergence of a kinetic approximation with a continuous set of velocities toward the unique entropy solution using the general L^∞ framework. François James relaxed a model of chemical engineering with special reflux boundary conditions in [8] using compensated compactness. A particular case of the same model led, for the Cauchy problem, to another work, [6] using the BV theory.

For a continuous set of velocities, the weak entropy case was studied in [17] in several space dimensions, but with a restriction on fluxes (either concave or convex). In [18], Wei-Cheng Wang and Zhouping Xin gave an important contribution to the weak solution in the general $BV \cap L^\infty$ framework. In fact, the authors studied a relaxation 2×2 system in one space dimension, with a rather natural boundary condition

$$\sum_{k=1}^{2} f_k = u_b(t);$$

this result was obtained only for a small perturbation of a nontransonic constant state u^* (i.e., $F'(u^*) \neq 0$).

In [15], R. Natalini and A. Terracina studied the same 2×2 relaxation case but with a slightly different boundary condition:

$$f_2(t, 0) = M_2(u_b(t))$$

where $f_2(t, 0)$ is the positive velocity component. They proved the existence and uniqueness of the entropy solution when the relaxation parameter ϵ tends to zero, without any restriction either on the data, or on the flux F.

The direct extension of the techniques of [15] to the case of any set of velocity $(\lambda_k)_{1 \leq k \leq N}$, with $N > 2$ seems to be a difficult problem. On the other hand, this extension seems to be of interest for at least three reasons: the multidimensional case cannot be considered with a two velocities model, and, from a numerical point of view, more velocities can decrease the numerical diffusion, and, at last, every flux splitting scheme can be derived from a kinetic model with at least three velocities (see [4]).

All these considerations give motivation to establish stability and convergence results for a general velocity case.

The main difficulty lies in the estimates of the total variation in space. In [15], those estimates are established thanks to a bound on the total variation of the solution's trace. It is possible to show the same kind of result for any set of velocity, but for null-velocity components the total variation of the trace is not uniform in ϵ. In this paper, we show how to avoid the time dependency, and obtain uniform estimates. The key argument is the use of all the information available on the boundary for positive velocity components of (1.3). Note that all the previous arguments permitted to expand numerics of the Cauchy problem already studied in [2] to the case with boundary in [1].

The paper is organized as follows: first we recall all results that are in the background of the proofs below; later, in section 3, we establish a new way of obtaining the $BV \cap L^\infty(\mathbb{R}_+)$ bounds, and we end the paper by showing the consistency with the entropy condition, using some weak convergence arguments.

2. Preliminaries

Definition 2.1. Let $u^0 \in BV(\mathbb{R}^+)$ and $u_b \in BV(0,T)$ $(T > 0)$. We say that a function $u \in BV(\mathbb{R}^+ \times (t,0)$ is an entropy solution to the boundary value problem (1.1) if

i) for any $\phi \in C_0^1(\mathbb{R}^+ \times [0,T))$, with $\phi \geq 0$, and for any $k \in \mathbb{R}$,

$$\int_0^T \int_{\mathbb{R}^+} |u-k|\partial_t\phi + \operatorname{sgn}(u-k)(F(u) - F(k)\partial_x\phi \, dxdt$$
$$+ \int_{\mathbb{R}^+} |u^0 - k|\phi \, dx \geq 0;$$

ii) for almost any $t \in (0,T)$,

$$\max\{\operatorname{sgn}(u(t,0) - u_b(t))(F(u(t,0)) - F(k)) \,:\, k \in I(u_b(t), u(t,0))\} = 0. \tag{2.1}$$

Here BV stands for the space of the functions of bounded variation and $u = (t,0)$ for the trace of the function $u = u(t,x)$ on the boundary, which is well-defined for $u \in BV$; see [3].

We recall also some standard properties established for the relaxation problem with boundary from [15] that are easily extendable to our case. Considering problem (1.3), set for every bounded initial-boundary data u^0 and u_b,

$$I(u^0, u_b) = \left\{u \in \mathbb{R}; \beta_m(u^0,u_b) \leq u \leq \beta_M(u^0,u_b)\right\},$$

where

$$\begin{cases} \beta_m(u^0,u_b) = \min\{\operatorname{ess\,inf}(u^0), \operatorname{ess\,inf}(u_b)\}, \\ \beta_M(u^0,u_b) = \max\{\operatorname{ess\,sup}(u^0), \operatorname{ess\,sup}(u_b)\}. \end{cases}$$

Proposition 2.1. *Fix $\epsilon > 0$ and assume the Maxwellian functions that satisfy* (1.5) *and are nondecreasing on the interval I. Let $(f_k^\epsilon)_{1\leq k\leq N}$ and $(\overline{f}_k^\epsilon)_{1\leq k\leq N}$ be two weak solutions to problem* (1.3)*, with initial data (u^0, u_b) and $(\overline{u}^0, \overline{u}_b)$ respectively, and $u^\epsilon := \sum_k f_k^\epsilon, \overline{u}^\epsilon := \sum_k \overline{f_k^\epsilon}$ for almost every $(t,x) \in \mathbb{R}^+ \times (0,T)$. Then for every $K \in \mathbb{R}^+$, and almost every $t \in (0,T)$, the following inequalities hold true:*

$$\begin{aligned} \textstyle\int_0^K \sum_{k=1}^N [f_k^\epsilon(t,x) - \overline{f_k^\epsilon}(t,x)]_+ dx &\leq \textstyle\int_0^{K+\sigma t} [u^0(x) - \overline{u}^0(x)]_+ dx \\ &+ \textstyle\sum_{\lambda_k>0} \lambda_k \int_0^t [M_k(u_b(\tau)) - \overline{M}_k(u_b(\tau))]_+ d\tau. \end{aligned} \tag{2.2}$$

For negative velocity components one has

$$\begin{aligned}\sum_{\lambda_k<0} |\lambda_k| \int_0^t [f_k^\epsilon(0,\tau) - \overline{f}_k^\epsilon(0,\tau)]_+ \,d\tau \leq \int_0^{\sigma t} [u^0(x) - \overline{u}^0(x)]_+ \,dx \\ + \sum_{\lambda_k>0} \lambda_k \int_0^t [M_k(u_b(\tau)) - M_k(\overline{u}_b(\tau))]_+ \,d\tau\end{aligned} \tag{2.3}$$

where $\sigma = \max_k |\lambda_k|$.

Corollary 2.2. *With the same assumptions as in Proposition 2.1, let* $(f_k^\epsilon)_{1\leq k\leq N}$ *be a weak solution to problem* (1.3) *with initial data* (u^0, u_b). *Then, for almost every* $(t,x) \in \mathbb{R}^+ \times (0,T)$, *and for* $k = 1, \dots, N$,

$$M_k(\inf I(u^0,u_b)) \leq f_k^\epsilon \leq M_k(\sup I(u^0,u_b)).$$

We recall here a lemma describing the behavior of the TV semi-norm for truncated function of BV. We call χ_δ a monotone decreasing truncature function whose value is 1 in $x \in [0, \frac{\delta}{2}]$ and whose support is included in $[0,\delta]$, and c is a real constant.

Lemma 2.1. *If* u *is a* $BV(\mathbb{R}^+)$ *function and* u_δ *its truncature defined as* $u_\delta = u(1-\chi_\delta) + c\chi_\delta$, *we have*

$$\forall \gamma > 0 \;\; \exists \alpha_\gamma \;\; s.t. \;\; \forall\, \delta < \alpha_\gamma \quad TV(u_\delta) \leq TV(u) + |u(0^+) - c| + \gamma.$$

If $v \in \mathrm{BV}(\mathbb{R}^+)$, then

$$v^{\delta,\nu} = (v(1-\chi_\delta) + c\chi_\delta) * \phi_\nu \tag{2.4}$$

where ϕ_ν is the standard approximation of unity. Note that to obtain a C^∞ regularity in a right neighborhood of $x = 0$, we have to take $\nu < \frac{\delta}{2}$.

By using the propagation of singularity results (see [12]) and the approximation of the solutions of the problem (1.3) by truncature and regularization, we obtain the following result. The proof is omitted.

Proposition 2.3. *For some given functions* $u^0 \in BV(\mathbb{R}^+)$ *and* $u_b \in BV((0,T))$, *we take* $(u^0_{\delta,\nu}, u_b^{\delta,\nu})$ *as defined in* (2.4). *Also let the Maxwellian functions* M_k *satisfy* (1.5) *for* $u \in I = I(u^0,u_b)$, *and assume that they are nondecreasing on this interval. Then for every* $\epsilon > 0$ *there exists a unique vector solution* $f^\epsilon_{\delta,\nu}$ *to problem* (1.3), *and* $f^\epsilon_{\delta,\nu} \in (C^\infty((0,T)\times\mathbb{R}^+))^N$. *Moreover, if* f^ϵ *is a weak solution associated to the data* (u^0,u_b), *then we have that* $f^\epsilon_{\delta,\nu} \to f^\epsilon$ *in* $(L^1_{loc}((0,T)\times\mathbb{R}^+))^N$, *when* δ,ν *go to* 0.

3. Stability

The a priori estimates that will be obtained for regular solutions are extendable to the general BV case thanks to Proposition 2.3, so that we treat only the smooth case. First we need to establish a very basic lemma to extend properties of the collision already used in [15].

Lemma 3.1. *Suppose that the Maxwellian functions are nondecreasing and satisfy* (1.5). *We have*

$$\sum_k \mathrm{sgn}(\partial_x f_k)\partial_x(M_k(u) - f_k) \leq 0.$$

Proof. We write the previous expression as

$$\sum_k \mathrm{sgn}(\partial_x f_k)\partial_x(M_k(u) - f_k) = \sum_k \mathrm{sgn}(\partial_x f_k)M_k'(u)\sum_l \partial_x f_l - \sum_k |\partial_x f_k|.$$

Inverting the sums in the first term of the right-hand side, we have

$$\begin{aligned}\textstyle\sum_k \mathrm{sgn}(\partial_x f_k) \quad & \partial_x(M_k(u) - f_k) = \textstyle\sum_l \partial_x f_l \sum_k \mathrm{sgn}(\partial_x f_k)M_k'(u) - \sum_k |\partial_x f_k| \\ &= \textstyle\sum_l |\partial_x f_l| \sum_k \mathrm{sgn}(\partial_x f_l)\mathrm{sgn}(\partial_x f_k)M_k'(u) - \sum_k |\partial_x f_k| \\ &= \textstyle\sum_l |\partial_x f_l|(\sum_k \mathrm{sgn}(\partial_x f_l)\mathrm{sgn}(\partial_x f_k)M_k'(u) - 1);\end{aligned}$$

and because the M_k are nondecreasing and

$$\mathrm{sgn}(a)\mathrm{sgn}(b) \leq 1 \ \ \forall a, b \in \mathbb{R},$$

we have the desired result. □

Here we expose the key argument of this paper, in fact the following lemma shows that there is no need to use total variation in time to estimate the space total variation along the trace. Those estimates are obtained thanks to a very careful use of the information that entering characteristics provide on the boundary.

Lemma 3.2. *For any set of velocities one has*

$$\sum_k \int_0^t \lambda_k |\partial_x f_k|(\tau, 0)d\tau \leq \int_0^t |u_b'(\tau)|d\tau - |u(t,0) - u_b(t)| + |u(0,0) - u_b(0)|.$$

Proof. Using the equations of (1.3), we transform the space derivatives into the difference between the collision term and the time derivatives

$$\begin{aligned}\textstyle\sum_k \int_0^t \lambda_k |\partial_x f_k|(\tau,0)d\tau \quad \leq \quad & \textstyle\sum_{\lambda_k>0} \int_0^t |\frac{1}{\epsilon}(M_k(u) - M_k(u_b)) - \partial_t M_k(u_b)|d\tau \\ & \textstyle - \sum_{\lambda_k \leq 0} \int_0^t |\frac{1}{\epsilon}(M_k(u) - f_k) - \partial_t f_k|d\tau.\end{aligned}$$

By a simple triangular inequality this becomes

$$\begin{aligned}\textstyle\sum_k \int_0^t \lambda_k |\partial_x f_k|(\tau,0)d\tau \quad \leq \quad & \textstyle\int_0^t \sum_{\lambda_k>0} |\frac{1}{\epsilon}(M_k(u) - M_k(u_b))| + |\partial_t M_k(u_b)| \\ & \textstyle - \sum_{\lambda_k \leq 0} |\frac{1}{\epsilon}(M_k(u) - f_k) - \partial_t f_k|d\tau.\end{aligned}$$

We now see that the last three terms inside the integral can be rewritten as

$$\begin{aligned}\textstyle\sum_k \lambda_k |\partial_x f_k| \quad \leq \quad & \textstyle\frac{|u-u_b|}{\epsilon} + |u_b'| \\ & \textstyle - \left(\sum_{\lambda_k \leq 0} |\frac{M_k(u)-M_k(u_b)}{\epsilon}| + |\partial_t(M_k(u_b))| + |\frac{(M_k(u)-f_k)}{\epsilon} - \partial_t f_k|\right).\end{aligned}$$

Adding all terms in the last sum over nonpositive velocities, we get

$$\begin{aligned}\textstyle\sum_k \lambda_k |\partial_x f_k| \quad \leq \quad & \textstyle\frac{|u-u_b|}{\epsilon} + |u_b'| \\ & \textstyle - \sum_{\lambda_k \leq 0} |\frac{M_k(u_b)-f_k}{\epsilon} + \partial_t(M_k(u_b) - f_k)|,\end{aligned}$$

which becomes

$$\begin{aligned}\textstyle\sum_k \lambda_k |\partial_x f_k| \quad \leq \quad & \textstyle\frac{|u-u_b|}{\epsilon} + |u_b'| \\ & \textstyle -|\sum_{\lambda_k \leq 0} \frac{M_k(u_b)-f_k}{\epsilon} + \partial_t(M_k(u_b) - f_k)|.\end{aligned}$$

Now, using that $u = \sum_{\lambda_k > 0} M_k(u_b) + \sum_{\lambda_k \leq 0} f_k$, we obtain the following estimate:

$$\sum_k \lambda_k |\partial_x f_k| \leq \frac{|u_b - u|}{\epsilon} + |u_b'| - |\frac{u_b - u}{\epsilon} + \partial_t(u_b - u)|.$$

We use the convexity of the absolute value

$$\begin{aligned}
\textstyle\sum_k \lambda_k |\partial_x f_k| &\leq -\mathrm{sgn}(\tfrac{u_b - u}{\epsilon})\partial_t(u_b - u) + |u_b'| \\
&= -\mathrm{sgn}(u_b - u)\partial_t(u_b - u) + |u_b'| \\
&= -\partial_t |u - u_b| + |u_b'|,
\end{aligned}$$

so that the previous quantity integrated over $(0, t)$ can be estimated in fact by

$$\sum_k \int_0^t \lambda_k |\partial_x f_k|(\tau, 0) d\tau \leq -\int_0^t \partial_t |u_b - u| + |u_b'| d\tau,$$

which gives the desired result. □

Now we just use the results presented previously to end the estimates of BV-space bounds.

Proposition 3.1. *Suppose that the Maxwellian functions are nondecreasing, and that $u^0 \in BV(\mathbb{R}^+)$ and $u_b \in BV((0,T))$. So if $(f_k)_{1 \leq k \leq N}$ is a solution of* (1.3)*, it satisfies the estimate*

$$\sum_k TV(f_k(t, \cdot)) \leq TV(u^0, \mathbb{R}^+) + TV(u_b, (0,T)) + |u_b(0) - u^0(0)|.$$

Proof. For simplicity of notation we omit the parameter ϵ.

* Results on regularized data. As in Proposition 2.3, we approximate the data (u_b, u^0) by truncature and regularization (u_b^δ, u_δ^0), so that the solution $f^{\delta,\nu}$ of (1.3) is in $C^\infty((0,T) \times \mathbb{R}^+)$. We take the x-derivative of all the equations in (1.3) and then we multiply each one by $\mathrm{sgn}(\partial_x f_k^\delta)$. Summing up and integrating over $D_L^t = \{(x, \tau) \in \mathbb{R}^+ \times (0,T) \;:\; 0 < x < L + \sigma(t - \tau)\}$, yields

$$\begin{aligned}
\textstyle\sum_k \int_0^L |\partial_x f_k^{\delta,\nu}(t,x)| dx \quad &\leq \quad \textstyle\sum_k \int_0^{L+\sigma t} |\partial_x f_k^{\delta,\nu}(0,x)| dx + \sum_k \int_0^t \lambda_k |\partial_x f_k^{\delta,\nu}(s,0)|\, ds \\
&\quad + \tfrac{1}{\epsilon} \textstyle\sum_k \int_{D_L^t} \mathrm{sgn}(\partial_x f_k^{\delta,\nu}) \partial_x (M_k(u^{\delta,\nu}) - f_k^{\delta,\nu})\, dx\, dt.
\end{aligned}$$

The parameter σ is chosen to make a negative contribution on the boundary $(L + \sigma(t - \tau))$, that is, $\sigma = \max_k |\lambda_k|$. Now it is clear that by using Lemma 3.1 the last term is also negative; whereas by applying Lemma 3.2 the second one is bounded. Then we have

$$\sum_k TV(f_k^{\delta,\nu}) \leq TV(u_b^{\delta,\nu}) + TV(u_{\delta,\nu}^0).$$

* TV properties of the regularized data. We regularize and truncate as in (2.4). A very standard property of the BV functions allows us to write

$$TV(u_b^{\delta,\nu}) \leq TV(u_b^\delta)$$

and twice applying Lemma 2.1, we have that

$$\liminf_{\delta,\nu\to 0}\Big(TV(u_b^{\delta,\nu}) \;\; + TV(u^0_{\delta,\nu})\Big) \le \liminf_{\delta\to 0}\big(TV(u_b^{\delta}) + TV(u^0_{\delta})\big)$$

$$\le TV(u_b) + TV(u^0) + |u_b(0^+) - c| + |u^0(0^+) - c|.$$

* General $BV \cap L^\infty$ solution. Since $f^{\delta,\nu} \to f$ in $(L^1_{\rm loc}((0,T)\times\mathbb{R}^+))^N$ by Proposition 2.3, we have also that

$$\sum_k TV(f_k) \le \liminf_{\delta,\nu\to 0}\sum_k TV(f_k^{\delta,\nu}).$$

Thus we have the following estimate:

$$\sum_k TV(f_k) \le TV(u_b) + TV(u^0) + |u_b(0^+) - c| + |u^0(0^+) - c|.$$

To obtain an optimal estimate we apply the last result with the constant c, namely setting

$$c = \frac{u_b(0) + u^0(0)}{2}.$$

We obtain the desired result. □

Now that we have obtained the $BV \cap L^\infty$ bounds, the rest of the stability criteria follow exactly the same ideas already exposed in [15], so that we can claim

Theorem 3.1. *Let $u^0 \in BV(\mathbb{R}^+)$, $u_b \in BV(0,T)$. Then the solution $(f_k\epsilon)_{1\le k\le N}$ to problem* (1.3) *satisfies the following uniform estimates for every $t \in (0,T)$:*

$$TV(f_k^\epsilon(\cdot,t)) \le C,$$

$$\textstyle\sum_k \int_{\mathbb{R}^+} |f_k^\epsilon(t+h,x) - f_k^\epsilon(t,x)|\, dx \le C|h|,$$

$$\textstyle\int_0^L \sum_k |(M_k(u^\epsilon) - f_k)(t,x)|\, dx \le C\epsilon.$$

4. Entropy consistency

In this section we prove that the solution given by the system (1.3) converges when ϵ goes to zero towards the unique entropy solution of (1.1). With respect to the arguments used in [15], our proof does not use the strong convergence of the traces, which is actually much more difficult to obtain in the present multivelocities case.

To establish the kinetic entropy inequalities for fixed ϵ, we argue exactly as in [15], so we omit the proof.

Lemma 4.1. *Let $u^0 \in BV(\mathbb{R}^+)$, let $u_b \in BV(0,T)$, and let $(f_k^\epsilon)_{1\le k\le N}$ be the weak solution of problem* (1.3). *Then for every $\phi \in C^1_0(\mathbb{R}^+ \times [0,T))$, with $\phi \ge 0$, and for* *every $c \in \mathbb{R}$, it holds that*

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^+} \sum_k |f_k^\epsilon - M_k(c)|(\partial_t \phi + \lambda_k \partial_x \phi)\, dxdt \\
&+ \int_{\mathbb{R}^+} \sum_k |M_k(u_0) - M_k(c)|\phi(0,x)\, dx + \int_0^T \sum_k \lambda_k |f_k^\epsilon - M_k(c)|\phi(t,0)\, dt \qquad (4.1)\\
&\geq \tfrac{1}{\epsilon} \int_0^T \int_{\mathbb{R}^+} (\sum_k |f_k^\epsilon - M_k(c)| - |u-c|)\phi\, dxdt \geq 0.
\end{aligned}$$

To show the convergence towards the entropy solution, we need the following result.

Lemma 4.2. *If we call $S_k(f_k) = |f_k - M_k(c)|$ the microscopic entropies associated to the Kružkov entropy for* (1.1)*, we have*

$$\begin{aligned}
&\textstyle\sum_{\lambda_k>0} \lambda_k |M_k(u_b) - M_k(c)| + \sum_{\lambda_k<0} \lambda_k |f_k - M_k(c)| \qquad (4.2)\\
&\leq \operatorname{sgn}(u_b - c)[\textstyle\sum_{\lambda_k>0} \lambda_k M_k(u_b) + \sum_{\lambda_k<0} \lambda_k f_k - F(c)].
\end{aligned}$$

Proof. Using that

$$\sum_k \lambda_k |M_k(u_b) - M_k(c)| = \operatorname{sgn}(u_b - c)[F(u_b) - F(c)],$$

thanks to the compatibility relations (1.5), we see that

$$\begin{aligned}
&\textstyle\sum_{\lambda_k>0} \lambda_k |M_k(u_b) - M_k(c)| + \sum_{\lambda_k<0} \lambda_k |f_k - M_k(c)| \\
&= \operatorname{sgn}(u_b - c)[F(u_b) - F(c)] + \textstyle\sum_{\lambda_k<0} \lambda_k (|f_k - M_k(c)| - |M_k(u_b) - M_k(c)|).
\end{aligned}$$

Because of the convexity of the microscopic entropies, the last inequality becomes

$$\begin{aligned}
&\textstyle\sum_{\lambda_k>0} \lambda_k |M_k(u_b) - M_k(c)| + \sum_{\lambda_k<0} \lambda_k |f_k - M_k(c)| \\
&\leq \operatorname{sgn}(u_b - c)[F(u_b) - F(c)] + \textstyle\sum_{\lambda_k<0} \lambda_k \operatorname{sgn}(M_k(u_b) - M_k(c))(f_k - M_k(u_b)).
\end{aligned}$$

The functions M_k are all nondecreasing, so that

$$\begin{aligned}
&\textstyle\sum_{\lambda_k>0} \lambda_k |M_k(u_b) - M_k(c)| + \sum_{\lambda_k<0} \lambda_k |f_k - M_k(c)| \\
&\leq \operatorname{sgn}(u_b - c)[F(u_b) - F(c) + \textstyle\sum_{\lambda_k<0} \lambda_k (f_k - M_k(u_b))].
\end{aligned}$$

Thus

$$\sum_{\lambda_k>0} \lambda_k M_k(u_b) = F(u_b) - \sum_{\lambda_k<0} \lambda_k M_k(u_b)$$

gives the desired result. □

Finally we present the proof of the consistency of the entropy inequalities.

Theorem 4.1. *The sequence of solutions to* (1.3) *and* (1.4) *converges to the (unique) entropic solution that satisfies the following entropy condition:*

$$\begin{aligned}\int_{\mathbb{R}_+\times(0,T)} |u-c|\partial_t\phi \ &+ \operatorname{sgn}(u-c)(f(u)-f(c))\partial_x\phi dxdt\\ &+ \int_{\mathbb{R}_+} |u(0,x)-c|\phi(0,x)dx\\ &+ \int_0^T \operatorname{sgn}(u_b(t)-c)(f(u(t,0))-f(c))dt \geq 0.\end{aligned}$$

Proof. By Lemma 4.1 we have

$$\begin{aligned}&-\int_0^T\int_{\mathbb{R}^+}\sum_k |f_k^\epsilon - M_k(c)|(\partial_t+\lambda_k\partial_x)\phi(t,x)dxdt\\ &\quad -\int_{\mathbb{R}^+}\sum_k |M_k(u^0)-M_k(c)|\phi(0,x)dx\\ &\quad \leq \int_0^T\sum_k \lambda_k|f_k^\epsilon(t,0)-M_k(c)|\phi(t,0)dt.\end{aligned}$$

The last term can be rewritten as

$$\int_0^T \sum_{\lambda_k>0}\lambda_k|M_k(u_b(t))-M_k(c)| + \sum_{\lambda_k<0}\lambda_k|f_k^\epsilon(t,0)-M_k(c)|\phi(t,0)dt.$$

Then, we use our new inequality (4.2) to obtain

$$\begin{aligned}&-\int_0^T\int_{\mathbb{R}^+}\sum_k |f_k^\epsilon - M_k(c)|(\partial_t+\lambda_k\partial_x)\phi(t,x)dxdt\\ &-\int_{\mathbb{R}^+}\sum_k |M_k(u^0)-M_k(c)|\phi(0,x)dx\\ &\leq \int_0^T \operatorname{sgn}(u_b(t)-c)[\textstyle\sum_{\lambda_k>0}\lambda_k M_k(u_b(t))+\sum_{\lambda_k<0}\lambda_k f_k^\epsilon - F(c)]\phi(t,0)dt.\end{aligned}\tag{4.3}$$

So because of stability results already established, the previous expression tends to its limit:

$$\begin{aligned}&-\int_{\mathbb{R}_+\times(0,T)} |u-c|\partial_t\phi - \operatorname{sgn}(u-c)(F(u)-F(c))\partial_x\phi = dxdt\\ &-\int_{\mathbb{R}_+}|u(0,x)-c|\phi(0,x)dx\\ &\leq \int_0^T \operatorname{sgn}(u_b(t)-c)[\textstyle\sum_{\lambda_k>0}\lambda_k M_k(u_b(t))+\xi(t)-F(c)]\phi(t,0)dt.\end{aligned}\tag{4.4}$$

Note that the terms inside the domain and at $t=0$ are limits due to convergence in $C([0,T],L^1_{loc}(\mathbb{R}^+))$ while the boundary one is obtained only in $L^\infty(0,T)$ weak-* sense. If we take

$$\phi(t,x)=\rho(t)\max\{0,1-\frac{x}{\eta}\},\qquad \eta\in\mathbb{R}^+\ ,$$

with $\rho(0)=0$, and let η tend to zero. It holds that

$$\begin{aligned}&\int_0^T \operatorname{sgn}(u(t,0)-c)(F(u(t,0))-F(c))\rho(t)dt\\ &\leq \int_0^T \operatorname{sgn}(u_b(t)-c)[\textstyle\sum_{\lambda_k>0}\lambda_k M_k(u_b)+\xi(t)-F(c)]\rho(t)dt,\end{aligned}$$

that is,

$$\operatorname{sgn}(u(t,0)-c)(F(u(t,0))-F(c)) \leq \operatorname{sgn}(u_b(t)-c)[\sum_{\lambda_k>0}\lambda_k M_k(u_b)+\xi(t)-F(c)].$$

Taking $c > \sup(u(t,0), u_b(t))$ and $c < \inf(u(t,0), u_b(t))$ yields

$$\xi(t) = F(u(t,0)) - \sum_{\lambda_k>0}\lambda_k M_k(u_b(t)),$$

we replace $\xi(t)$ by its value in (4.4) and we get the desired result. □

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