On the asymptotic regime of a model for friction mediated by transient elastic linkages.

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Abstract

In this work we study a system of an integral equation of Volterra type coupled to an original renewal equation. This model arises in the context of cell motility [6]: the integral equation describes the trajectory of a binding site which is connected via transiently remodelling linkages to the substrate and which evolves driven by a given force. The renewal model accounts for the remodelling process of linkages which attach and break with given probabilities.

In the present paper we analyze existence and uniqueness issues for the coupled system of interest and provide a rigorous justification of the asymptotic limit of infinitesimally rapid turnover of linkages.

The renewal model for the age distribution of linkages differs from more classical ones in that it describes competition between population size and birth and because it admits a new and specific Lyapunov functional. On the other side, using a comparison principle which applies to non-convolution linear Volterra kernels and the peculiar transport properties of the linkages, one establishes a convergence result when the turnover parameter ε tends to zero.

Abstract in french. Dans cet article, on étudie un système d'une équation intégrale de Volterra couplée avec une équation de renouveau d'un type particulier. Ce modèle apparaît dans le contexte de la motilité cellulaire [6]: l'équation intégrale décrit la trajectoire d'un site d'adhésion connecté au substrat par des liaisons protéiques éphémères et soumis à une force extérieure. Le processus de remodelage des liaisons qui se déchirent ou se créent sur ce site avec une certaine probabilité est décrit par un l'équation de renouveau.

Ici, on analyse les questions d'existence et d'unicité de ce système couplé et on donne une justification rigoureuse de la limite instantanée du taux de renouvellement des liaisons (noté ε).

Le modèle de renouveau pour la distribution de l'age des liaisons diffère des modèles classiques en ce qu'il décrit la compétition entre la taille totale de la population et le taux de naissance. Pour tenir compte de cette dernière difficulté, on a exhibé une nouvelle fonctionnelle de Liapounov. Par ailleurs, en utilisant un principe de comparaison propre aux équations de Volterra à noyau non-convolutif, on établit un résultat de convergence losrque le paramètre ε tend vers zéro.

Keywords: friction coefficient, protein linkages, cell adhesion, renewal equation, effect of chemical bonds, lyapunov functional, comparison principle, integral equation, Volterra kernel,
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1. Introduction

We consider the integral equation

$$\begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_0^\infty (z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a))\,\rho_\varepsilon(a,t)\,da = f(t)\,, & t \geq 0\,, \\ z_\varepsilon(t) = z_p(t)\,, & t < 0\,, \end{cases} \tag{1}$$

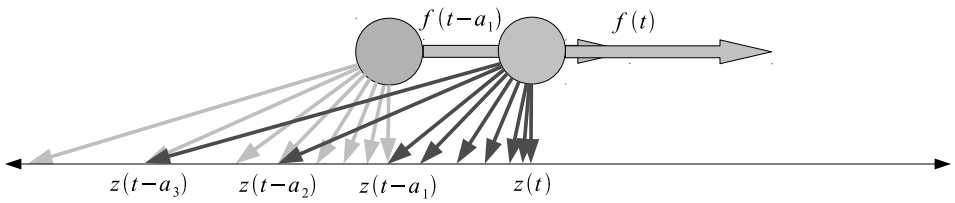

Figure 1: The position of the moving binding site at time t and time $t - a_1$ with some of the respective linkages. The scaling parameter is set to $\varepsilon = 1$.

where $z_\varepsilon = z_\varepsilon(t) \in \mathbb{R}$ represents the time dependent position of a linkage binding site and the function $f(t) \in \mathrm{Lip}(\mathbb{R}_+, \mathbb{R})$ represents a given exterior force. The kernel $\rho_\varepsilon = \rho_\varepsilon(a,t)$ is interpreted as the density of existing linkages to the substrate with respect to the age $a \geq 0$ and is defined by the renewal model

$$\begin{cases} \varepsilon \partial_t \rho_\varepsilon + \partial_a \rho_\varepsilon + \zeta_\varepsilon(a,t)\rho_\varepsilon = 0\,, & t > 0\,,\ a > 0\,, \\ \rho_\varepsilon(a=0,t) = \beta_\varepsilon(t)\left(1 - \displaystyle\int_0^\infty \rho_\varepsilon(\tilde{a},t)\,d\tilde{a}\right)\,, & t > 0\,, \\ \rho_\varepsilon(a,t=0) = \rho_{I,\varepsilon}(a)\,, & a \geq 0\,, \end{cases} \tag{2}$$

with the kinetic rate functions $\beta_\varepsilon = \beta_\varepsilon(t) \in \mathbb{R}_+$ and $\zeta_\varepsilon = \zeta_\varepsilon(a,t) \in \mathbb{R}_+$, both possibly depending on the dimensionless parameter $\varepsilon > 0$ which represents the speed of linkage turnover. The two submodels are finally complemented by their respective past and initial data $z_p \in \mathrm{Lip}((-\infty, 0])$ and $\rho_I \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$.

The system (1-2) is a model describing the mechanical effect of a set of chemical linkages dynamically remodelled in time. For instance the cross-linking proteins attaching to actin filaments in the lamellipodia of living cells can be modelled in this way. The complete model was introduced and developed in [6]. A reverse coupling between both submodels was established through the possible dependence of β_ε, the on-rate and ζ_ε, the off-rates on the geometrical configuration of the mechanical structures where the binding sites are located. In the present study, however, we do not take into account a functional dependence of these rates on the function z_ε.

The integrin equation (1) models a force balance between the time-dependent exterior force $f(t)$ and elastic forces exerted by a population of linkages which connect the moving binding site to binding sites on the substrate. The competing force contributions are visualized in figure 1 by arrows.

Linkages are originally established between the moving binding site positioned at $z_\varepsilon(t)$ and the substrate at the very same position. As a consequence linkages with a given age a connect the moving binding site to the substrate at position $z_\varepsilon(t - \varepsilon a)$ where the dimensionless scaling parameter ε represents the ratio of the age scale in the ρ_ε-model and the time scale in the z_ε-model, i.e. small ε reflects rapid lifecycle of the linkage proteins.

The model (2) for the age distribution of linkages states that chemical bonds break, respectively detach with a given rate $\zeta_\varepsilon = \zeta_\varepsilon(a,t) \geq 0$. Moreover, creation of new chemical bonds with a given rate $\beta_\varepsilon = \beta_\varepsilon(t) \geq 0$ is proportional to the abundance of empty binding sites which itself is given by the difference of the constant total number of binding sites, in this study scaled to 1, and the number of occupied ones.

The renewal is visualized in figure 1. The grey arrows connecting the ball-shaped binding site at position $z(t - a_1)$ to some of its past positions represent the set of existing linkages in the past. When going from time $t - a_1$ to time t, some of the connections break, some of them still exist like the one connecting the point $z(t - a_2)$ on the substrate to the present position of the moving binding site, and some linkages have been established in the meantime like the one connecting the moving binding site to its actual position $z(t)$.

In this sense we consider the above model to be a renewal equation, using intentionally the same nomenclature as for similar and more classical renewal models (see for instance [7] and

numerous references therein). In those models the generation of offspring is positively coupled to the abundance of existing individuals and therefore one might call them self-renewal models. However in (2) this dependence is inverse, i.e. the more chemical bonds exist, the smaller is the pool of empty binding sites to generate new linkages. Below we detail what this implies for the mathematical analysis.

In [5] the asymptotic scaling, which induces rapid turnover of the linkage proteins, was introduced and the formal limit as $\varepsilon \to 0$ was computed. In the framework of the present study it is given by

$$\begin{cases} \mu_{1,0}\, \partial_t z_0 = f \quad \text{with} \quad \mu_{1,0}(t) := \displaystyle\int_0^\infty a\rho_0(a,t)\, da\ , \quad t>0\ , \\ z_0(t=0) = z_I := z_p(0)\ , \end{cases} \tag{3}$$

where the limit distribution ρ_0 is explicitly given by

$$\rho_0(a,t) = \frac{1}{\frac{1}{\beta_0(t)} + \int_0^\infty \exp\left(-\int_0^a \zeta_0(\tilde a,t)\, d\tilde a\right) da} \exp\left(-\int_0^a \zeta_0(\tilde a,t)\, d\tilde a\right), \tag{4}$$

being the solution of

$$\begin{cases} \partial_a \rho_0 + \zeta_0(a,t)\rho_0 = 0\ , & t>0\ , \quad a>0\ , \\ \rho_0(t,a=0) = \beta_0(t)\left(1 - \displaystyle\int_0^\infty \rho_0(\tilde a,t)\, d\tilde a\right)\ , & t>0\ . \end{cases} \tag{5}$$

Combining (3) and (4) we are able to give an explicit expression for the viscosity constant $\mu_{1,0}$, which represents the macroscopic friction effect, in terms of the microscopic rate constants. In the special case where the limit off-rate does not depend on age, $\zeta_0 = \zeta_0(t)$, the viscosity constant is given by

$$\mu_{1,0}(t) = \frac{1}{\zeta_0(t)(1+\zeta_0(t)/\beta(t))}\ . \tag{6}$$

The macroscopic friction law (3) is similar to the Stokes Law. The biological setting we refer to, the relative movement of actin-filaments with respect to crossing filaments and with respect to the substrate, has conceptual parallels with the movement of solids on lubricated surfaces. In the theory of lubrication as well, there exist friction laws depending on the speed of the motion [2].

The existence and uniqueness of continuous solutions to Volterra type integral equations like (1) is a well known fact [1, 3] and even an explicit representation formula for the solution in terms of a resolvent function can be given [8, 1]. In our analysis, however, we are confronted with the difficulty that these classical results do not imply *a priori* estimates on the solution and do not provide a control which is uniform with respect to ε, our scaling parameter. The renewal model (2) on the other hand is different in nature from those treated in the existing theory. The inverse relation between the population size and the birth term does not allow, again, to apply techniques presented in [4, 7] as for instance the Generalized Relative Entropy Method. In this work we therefore develop specific tools to tackle all these peculiarities.

The program of this study is then as follows. First, for fixed ε, we prove existence and uniqueness results for the linkage age distribution model (2) in $C(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap \mathrm{L}^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$. In a second step we also give existence and uniqueness results for the integral equation (1). Then we focus on the rigorous study of the asymptotic limit of the system as ε tends to zero and we show in a two step manner that $(\rho_\varepsilon, z_\varepsilon)$ tends in a sense defined below to the solution (ρ_0, z_0) of the formal limit system (3), (5).

Concerning the age distribution model (2) we establish that its homogeneous version admits the Lyapunov functional

$$\mathcal{H}[u] := \left|\int_0^\infty u(a)\, da\right| + \int_0^\infty |u(a)|\, da\ , \tag{7}$$

which satisfies for any non negative time t

$$\frac{d}{dt}\mathcal{H}[\rho_\varepsilon(\cdot,t)-\rho_0(\cdot,t)] \leq -\frac{1}{\varepsilon}\zeta_{\min}\,\mathcal{H}[\rho_\varepsilon(\cdot,t)-\rho_0(\cdot,t)]\,. \tag{8}$$

The Lyapunov functional does not only yield a result on the convergence in time but also on the convergence as the scaling parameter ε tends to zero. The convergence result $z_\varepsilon \to z_0$ is then established via a comparison principle satisfied by certain Volterra integral equations.

The framework of our analysis relies on the following hypotheses on the on- and off-rates.

Assumption 1.1. *The dimensionless parameter $\varepsilon > 0$ is assumed to induce two families of chemical rate functions that satisfy:*

(i) For any $T > 0$ the function $\beta_\varepsilon(t)$ is a uniform Lipschitz function in $[0,T]$ and $\zeta_\varepsilon(a,t)$ is in $\mathrm{Lip}_t([0,T];L^\infty_a(\mathbb{R}_+))$, i.e.

$$\zeta_\varepsilon \in L^\infty((0,T)\times\mathbb{R}_+) \quad \textit{and} \sup_{a\in\mathbb{R}_+,\,t\in[0,T]} |\partial_t\zeta_\varepsilon(a,t)| \leq C\,.$$

for a constant $C > 0$. Moreover we suppose that for a fixed positive age $a_0 \geq 0$ the off-rate $\zeta_\varepsilon(a + t/\varepsilon, t)$ is monotonically increasing on $[a_0,\infty)$.

(ii) For limit functions $\beta_0 \in L^\infty_t$ and $\zeta_0 \in L^\infty_t L^\infty_a$ it holds that

$$\|\zeta_\varepsilon - \zeta_0\|_{L^\infty_t L^\infty_a} \to 0 \quad \textit{and} \quad \|\beta_\varepsilon - \beta_0\|_{L^\infty_t} \to 0$$

as $\varepsilon \to 0$.

(iii) We also assume that there are upper and lower bounds such that

$$0 < \zeta_{\min} \leq \zeta_\varepsilon(a,t) \leq \zeta_{\max} \quad \textit{and} \quad 0 < \beta_{\min} \leq \beta_\varepsilon(t) \leq \beta_{\max}$$

for all $\varepsilon > 0$, $a \geq 0$ and $t > 0$.

The initial data for the density model (2) satisfies some hypotheses that we sum up here:

Assumption 1.2. *The initial condition $\rho_{I,\varepsilon} \in L^\infty_a(\mathbb{R}_+)$ satisfies*

- *positivity*

$$\rho_{I,\varepsilon}(a) \geq 0\,, \qquad \textit{a.e. in } \mathbb{R}_+\,,$$

moreover, one has also that the total initial population satisfies

$$0 < \int_{\mathbb{R}_+} \rho_{I,\varepsilon}(a)da < 1\,.$$

- *boundedness of higher moments,*

$$0 < \int_{\mathbb{R}_+} a^p \rho_{I,\varepsilon}(a)\,da \leq c_p\,, \quad \textit{for } p = 1,2\,,$$

where c_p are positive constants depending only on p.

- *there exists a constant denoted $A_{\max} > a_0$ s.t.*

$$\int_{a_0}^\infty a\rho_{I,\varepsilon}(a)\,da \leq A_{\max}\int_{a_0}^\infty \rho_{I,\varepsilon}(a)\,da$$

uniformly in ε.

Concerning the integral equation (1) we assume

Assumption 1.3. *The time dependent rhs $f = f(t)$ in (1) is a uniform Lipschitz function on $[0,T]$ for any $T > 0$. The past condition z_p belongs to $\mathrm{Lip}((-\infty, 0])$, the set of uniform Lipschitz functions on $\mathbb{R}_-$.*

We are then able to claim our main result:

Theorem 1.1. *Let assumptions 1.1, 1.2 and 1.3 hold. For every fixed ε there exists a unique solution of the coupled system (1-2), $(z_\varepsilon, \rho_\varepsilon) \in C^0(\mathbb{R}_+) \times (C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2))$. Let (z_0, ρ_0) be the unique solution to the formal limit system (3-5), then for every $T > 0$ it holds that*

$$\|z_\varepsilon - z_0\|_{C^0([0,T])} + \|\rho_\varepsilon - \rho_0\|_{C^0(]0,T];L^1(\mathbb{R}_+))} \to 0$$

as $\varepsilon \to 0$.

2. Existence and uniqueness

Theorem 2.1. *Let assumptions 1.1 and 1.2 hold, then for every fixed ε there exists a unique solution $\rho_\varepsilon \in C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$ of the problem (2). It satisfies (2) in the sense of characteristics, namely*

$$\rho_\varepsilon(a,t) = \begin{cases} \beta_\varepsilon(t-\varepsilon a)\left(1 - \int_{\mathbb{R}_+} \rho_\varepsilon(\tilde{a}, t-\varepsilon a)\, d\tilde{a}\right) \exp\left(-\int_0^a \zeta_\varepsilon(\tilde{a}, t-\varepsilon(a-\tilde{a}))\, d\tilde{a}\right) , & a < t/\varepsilon , \\ \rho_{I,\varepsilon}(a - t/\varepsilon) \exp\left(-\frac{1}{\varepsilon}\int_0^t \zeta_\varepsilon((\tilde{t}-t)/\varepsilon + a, \tilde{t})\, d\tilde{t}\right) , & a \geq t/\varepsilon . \end{cases} \tag{9}$$

Proof. The existence proof relies on the Banach-Picard fixed point theorem in $C^0([0,T]; L^1(\mathbb{R}_+))$. Indeed for a given function $m \in C^0([0,T]; L^1(\mathbb{R}_+))$ we define $n := \mathcal{T}(m)$ as

$$n(a,t) := \begin{cases} \beta_\varepsilon(t-\varepsilon a)\left(1 - \int_{\mathbb{R}_+} m(\tilde{a}, t-\varepsilon a)\, d\tilde{a}\right) \exp\left(-\int_0^a \zeta_\varepsilon(\tilde{a}, t-\varepsilon(a-\tilde{a}))\, d\tilde{a}\right) , & a < t/\varepsilon , \\ \rho_{I,\varepsilon}(a - t/\varepsilon) \exp\left(-\frac{1}{\varepsilon}\int_0^t \zeta_\varepsilon((\tilde{t}-t)/\varepsilon + a, \tilde{t})\, d\tilde{t}\right) , & a \geq t/\varepsilon . \end{cases}$$

For regular data n would solve

$$\begin{cases} \varepsilon \partial_t n + \partial_a n + \zeta_\varepsilon n = 0 , & a > 0,\ t \in (0,T] , \\ n(a=0,t) = \beta_\varepsilon(t)\left(1 - \displaystyle\int_{\mathbb{R}_+} m(\tilde{a}, t)\, d\tilde{a}\right), & t > 0 , \\ n(a, t=0) = \rho_{I,\varepsilon}(a) , & a \geq 0 . \end{cases}$$

hypotheses on $\rho_{I,\varepsilon}$, β_ε and ζ_ε imply that $\mathcal{T}$ is indeed an endomorphism of $C^0([0,T]; L^1(\mathbb{R}_+))$. It is also a contraction for a time T small enough since it holds that

$$\|n_2 - n_1\|_{C^0([0,T];L^1(\mathbb{R}_+))} \leq \frac{\beta_{\max} T}{\varepsilon} \|m_2 - m_1\|_{C^0([0,T];L^1(\mathbb{R}_+))} ,$$

where $n_i := \mathcal{T}(m_i)$ for $i = 1, 2$. Thus there exists a unique fixed point in $C^0([0,T]; L^1(\mathbb{R}_+))$ by the Banach-Picard fixed point theorem if $T_0 < \varepsilon/\beta_{\max}$. As this timespan is fixed the result can be extended to $[T_0, 2T_0], [2T_0, 3T_0]$ etc., giving existence and uniqueness in $C^0(\mathbb{R}_+, L^1(\mathbb{R}_+))$ of ρ_ε such that $\rho_\varepsilon = \mathcal{T}(\rho_\varepsilon)$, which is exactly (9).

□

Lemma 2.1. *Let ρ_ε be the unique solution of problem (2) according to Theorem 2.1, then it satisfies a weak formulation of this problem, namely*

$$\int_{\mathbb{R}_+}\int_0^T \rho_\varepsilon(a,t)\left(\varepsilon\partial_t\varphi + \partial_a\varphi + \zeta_\varepsilon\varphi\right)\, dt\, da - \varepsilon\int_{\mathbb{R}_+} \rho_\varepsilon(a,t)\varphi(a,t=T)\, da + \\ + \int_0^T \rho_\varepsilon(a=0,t)\,\varphi(0,t)\, dt + \varepsilon\int_{\mathbb{R}_+} \rho_{I,\varepsilon}(a)\varphi(a,t=0)\, da = 0 , \tag{10}$$

for every $T > 0$ and every test function $\varphi \in C^\infty(\mathbb{R}_+^2) \cap L^\infty(\mathbb{R}_+^2)$.

Proof. Suppose that ρ_ε satisfies (9). We set

$$J := \int_{\mathbb{R}_+} \int_0^T \rho_\varepsilon(a,t)\,(\varepsilon\partial_t\varphi + \partial_a\varphi)\ dt\ da\ .$$

Performing the change of variables $x = (a - t/\varepsilon)/2, y = (\varepsilon a + t)/2$, one transforms $\mathbb{R}_+ \times (0,T)$ into $\Omega = \{(x,y)\} = \Omega_1 \cup \Omega_2$ where $\Omega_1 :=]-T/(2\varepsilon), 0[\times]-\varepsilon x, \varepsilon x + T/2[$ and $\Omega_2 :=]0;\infty[\times]\varepsilon x, \varepsilon x + T/2[$. Setting $\tilde\varphi(x,y) := \varphi(a,t)$ one has then that

$$\varepsilon\partial_t\varphi + \partial_a\varphi = \varepsilon\partial_y\tilde\varphi\ ,$$

and

$$J = \int_{\Omega_1} \rho_\varepsilon \varepsilon\partial_y\tilde\varphi\ dy\ dx + \int_{\Omega_2} \rho_\varepsilon\varepsilon\partial_y\tilde\varphi\ dy\ dx =: I_1 + I_2\ .$$

We treat each term separately because they correspond to the two cases of Duhamel's formula.

$$I_1 = \int_{-\frac{T}{2\varepsilon}}^0 \int_{-\varepsilon x}^{\varepsilon x + T/2} \rho_\varepsilon(0,-\varepsilon x)\ g(x,y)\ \varepsilon\partial_y\tilde\varphi(x,y)\ dy\ dx\ .$$

The function $g(x,y) := \exp\left(-\int_0^{x+\frac{y}{\varepsilon}} \zeta_\varepsilon(\tilde a, \varepsilon(\tilde a - 2x))\ d\tilde a\right)$ is in $H^1_y(]-\varepsilon x, \varepsilon x + T/2[)$ since $\zeta_\varepsilon \in L^\infty_{a,t}$ and it holds that $\tilde\varphi$ is $C^\infty \subset H^1_y$. Hence the integration by parts is well defined,

$$\begin{aligned} I_1 &= \int_{-\frac{T}{2\varepsilon}}^0 \rho_\varepsilon(0,-\varepsilon x)\left\{\varepsilon\,[g(x,y)\tilde\varphi]_{y=-\varepsilon x}^{y=\varepsilon x+T/2} - \int_0^{\varepsilon x+T/2} \zeta_\varepsilon\left(x+\frac{y}{\varepsilon}, y-\varepsilon x\right) g(x,y)\tilde\varphi(x,y)\ dy\right\} dx \\ &= \varepsilon\int_{-\frac{T}{2\varepsilon}}^0 \rho_\varepsilon(0,-\varepsilon x)\,\{\tilde\varphi(x,\varepsilon x + T/2)g(x,\varepsilon x + T/2) - \tilde\varphi(x,-\varepsilon x)\}\,dx - \int_{\Omega_1} \zeta_\varepsilon\rho_\varepsilon\tilde\varphi\ dy\ dx \\ &= \int_0^{\frac{T}{\varepsilon}} \rho_\varepsilon(a,t)\varphi(a,t)da - \int_0^T \rho_\varepsilon(0,t)\varphi(0,t)dt - \int_0^T\int_0^{\frac{T}{\varepsilon}} \zeta_\varepsilon(a,t)\rho_\varepsilon(a,t)\varphi(a,t)\ da\ dt\ , \end{aligned}$$

and similarly one gets the complementary result for I_2, which ends the proof. □

In the following two Lemmas we prove bounds on the moments of ρ_ε which we denote by

$$\mu_{p,\varepsilon}(t) := \int_{\mathbb{R}_+} a^p \rho_\varepsilon(a,t)\ da\ , \quad \text{where } p = 1, 2\ .$$

Lemma 2.2. *Let assumptions 1.1 and 1.2 hold, then the unique solution $\rho_\varepsilon \in C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}^2_+)$ of the problem (2) from Theorem 2.1 satisfies*

$$\rho_\varepsilon(a,t) \geq 0 \quad \textit{a.e. in} \quad \mathbb{R}^2_+ \quad \textit{and}$$

$$\mu_{0,\min} \leq \mu_{0,\varepsilon}(t) < 1\ , \quad \forall t \in \mathbb{R}_+ \quad \textit{where} \quad \mu_{0,\min} := \min\left(\mu_{0,\varepsilon}(0), \frac{\beta_{\min}}{\beta_{\min} + \zeta_{\max}}\right)\ . \tag{11}$$

Proof. First, we show that $\mu_{0,\varepsilon}(t) < 1$ and $\rho_\varepsilon \geq 0$ for all times. We start with initial data which satisfies both properties, hence $\mu_{0,\varepsilon}(0) = \|\rho_\varepsilon(.,t=0)\|_{L^1_a} < 1$. Due to the continuity of $\|\rho_\varepsilon\|_{L^1_a}$ it holds that $\mu_{0,\varepsilon}(t) \leq \|\rho_\varepsilon(.,t)\|_{L^1_a} < 1$ at least on a time intervall $[0,T]$ small enough. On that time interval it also holds that $\rho_\varepsilon \geq 0$ for all $a \geq 0$, since due to (9) its value is obtained by transport either from the the nonnegative initial data $\rho_{I,\varepsilon}$ or from the positive boundary.

Assume that $\mu_{0,\varepsilon}(T) = 1$. We use that ρ_ε satisfies the weak formulation (10). Choose $\varphi(a,t) = \varphi(t) \geq 0$ to obtain

$$\int_0^T \left[-\mu_{0,\varepsilon}\varepsilon\partial_t\varphi + \varphi(t)\int_0^\infty \zeta_\varepsilon\rho_\varepsilon\ da - \varphi(t)\rho_\varepsilon(0,t)\right]\ dt + \varepsilon\,(\mu_{0,\varepsilon}(T)\varphi(T) - \mu_{0,\varepsilon}(0)\varphi(0)) = 0\ . \tag{12}$$

This implies

$$\begin{aligned}\varepsilon\left\{-\varphi(T)(1-\mu_{0,\varepsilon}(T))+\varphi(0)(1-\mu_{0,\varepsilon}(0))\right\} &\leq \int_0^T \left(-\varepsilon\partial_t\varphi+\varphi(t)\beta_\varepsilon(t)\right)(1-\mu_{0,\varepsilon}(t))\,dt \\ &\leq \int_0^T \left(-\varepsilon\partial_t\varphi+\varphi(t)\beta_{\max}\right)(1-\mu_{0,\varepsilon}(t))\,dt\,.\end{aligned}$$

Set $\varphi=\exp(t\beta_{\max}/\varepsilon)$ to obtain that

$$(1-\mu_{0,\varepsilon}(T))\geq\exp(-T\beta_{\max}/\varepsilon)(1-\mu_{0,\varepsilon}(0))>0$$

contradicting the assumption $\mu_{0,\varepsilon}(T)=1$. Duhamel's principle formulated in (9) then directly implies

$$0\leq\rho_\varepsilon(a,t)\leq\max(\beta_{\max},\|\rho_{I,\varepsilon}\|_\infty)\,,\qquad\text{a.e. }(a,t)\in(\mathbb{R}_+)^2\,.$$

In order to obtain a lower bound we set $\tilde{\mu}(t):=\mu_{0,\varepsilon}(t)-\mu_{0,\min}$ with $\mu_{0,\min}$ as defined in (11). According to the same definition, we start with an initial datum which satisfies $\mu_{0,\varepsilon}(0):=\mu_{I,\varepsilon}\geq\mu_{0,\min}$. The formal computation yields $\varepsilon\partial_t\tilde{\mu}\geq-\frac{\beta_\varepsilon}{\mu_{0,\min}}\tilde{\mu}$ which we can confirm in the same way as the upper bound: observe that $\tilde{\mu}\geq0$ on a small interval $[0,T]$ due to the continuity of $\mu_{0,\varepsilon}$. As above we assume that $\tilde{\mu}(T)=0$ and obtain

$$\begin{aligned}&\varepsilon\left((\mu_{0,\varepsilon}(T)-\mu_{0,\min})\varphi(T)-(\mu_{0,\varepsilon}(0)-\mu_{0,\min})\varphi(0)\right)= \\ &\qquad=\int_0^T\left[(\mu_{0,\varepsilon}-\mu_{0,\min})\varepsilon\partial_t\varphi-\varphi(t)\int_0^\infty\zeta_\varepsilon\rho_\varepsilon\,da+\varphi(t)(\beta_\varepsilon(1-\mu_{0,\varepsilon}))\right]\,dt\geq \\ &\qquad\geq\int_0^T\left[(\mu_{0,\varepsilon}-\mu_{0,\min})\varepsilon\partial_t\varphi-\varphi(t)\zeta_{\max}\mu_{0,\varepsilon}+\varphi(t)(\beta_{\min}(1-\mu_{0,\varepsilon}))\right]\,dt= \\ &=\int_0^T\left[(\mu_{0,\varepsilon}-\mu_{0,\min})(\varepsilon\partial_t\varphi-\varphi(t)(\zeta_{\max}+\beta_{\min}))+\varphi(t)(\beta_{\min}-\mu_{0,\min}(\beta_{\min}+\zeta_{\max}))\right]\,dt\geq0\,.\end{aligned}$$

By choosing $\varphi=\exp(t(\zeta_{\max}+\beta_{\min})/\varepsilon)$ and using the definition of $\mu_{0,\min}$, we conclude that

$$(\mu_{0,\varepsilon}(T)-\mu_{0,\min})\geq\exp(-T(\zeta_{\max}+\beta_{\min})/\varepsilon)(\mu_{0,\varepsilon}(0)-\mu_{0,\min})>0\,,$$

which contradicts the assumption $\mu_{0,\varepsilon}(T)=\mu_{0,\min}$ and thus finishes the proof of the lower bound in (11).

□

In a more straightforward manner one gets for higher moments as well

Lemma 2.3. *Let assumption 1.2 hold, then*

$$\mu_{p,\min}<\mu_{p,\varepsilon}(t)\leq k\quad\textit{for}\quad p=1,2\,,\quad\textit{where}\quad\mu_{p,\min}:=\min\left(\mu_{p,\varepsilon}(0),\frac{\mu_{p-1,\min}}{\zeta_{\max}}\right),$$

and the generic constant k is independent of both time and ε.

Proof. The proof is made by induction. The case of the zeroth order moment is already treated as $\mu_{0,\varepsilon}$ which is uniformely bounded by 1. We set $q_{\varepsilon,k}(a,t)=a^k\rho_\varepsilon(a,t)$ for $k=1,2$ and assume that the property is true for $k-1$. It holds that

$$\begin{cases}\varepsilon\partial_t q_{\varepsilon,k}+\partial_a q_{\varepsilon,k}+\zeta_\varepsilon q_{\varepsilon,k}-pq_{\varepsilon,k-1}=0\,, & a>0\,,\ t>0\,,\\ q_{\varepsilon,k}(a=0,t)=0\,, & t>0\,,\\ q_{\varepsilon,k}(a,t=0)=a^k\rho_{I,\varepsilon}(a)\,, & a\geq0\,.\end{cases}$$

After integration in age one obtains

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}q_{\varepsilon,k}(a,t)\,da\leq-\int_{\mathbb{R}_+}\zeta_{\min}q_{\varepsilon,k}(a,t)\,da+p\int_{\mathbb{R}_+}q_{\varepsilon,k-1}\,da\,,$$

which by Gronwall's inequality implies

$$\int_{\mathbb{R}_+} q_{\varepsilon,k}(a,t)\,da \le e^{-\frac{\zeta_{\min} t}{\varepsilon}} \int_{\mathbb{R}_+} a^k \rho_{I,\varepsilon}(a)\,da + k \sup_{s\in[0,t]} \int_{\mathbb{R}_+} q_{\varepsilon,k-1}(a,s)\,da\ .$$

Now take the supremum with respect to T on both sides. The fact that the property is true for $k-1$ ends the proof.

For the lower bound we proceed as in the case of Lemma 2.2, so we just give the formal sketch of the proof: for any constant c one has

$$\varepsilon\partial_t(\mu_{p,\varepsilon}-c) \ge -\zeta_{\max}(\mu_{p,\varepsilon}-c) - \zeta_{\max}c + \mu_{p-1,\varepsilon}(t) \ge -\zeta_{\max}(\mu_{p,\varepsilon}-c) - \zeta_{\max}c + \mu_{p-1,\min}\ .$$

Two situations occur:

- either $\mu_{p,\varepsilon}(0) > \mu_{p-1,\min}/\zeta_{\max}$. We set $c := \mu_{p-1,\min}/\zeta_{\max}$. One gets after integration in time
$$\mu_{p,\varepsilon}(t) - \frac{\mu_{p-1,\min}}{\zeta_{\max}} \ge e^{-\frac{\zeta_{\max} t}{\varepsilon}} \left(\mu_{p,\varepsilon}(0) - \frac{\mu_{p-1,\min}}{\zeta_{\max}}\right) > 0\ .$$
- or $\mu_{p,\varepsilon}(0) \le \mu_{p-1,\min}/\zeta_{\max}$. In this case setting $c = \mu_{p,\varepsilon}(0)$ gives, after integration in time,
$$\mu_{p,\varepsilon}(t) - \mu_{p,\varepsilon}(0) \ge \frac{1}{\varepsilon}\int_0^t e^{-\frac{(t-s)\zeta_{\max}}{\varepsilon}} ds(-\zeta_{\max}\mu_{p,\varepsilon}(0) + \mu_{p-1,\min}) \ge 0\ ,$$

which ends the proof. □

Lemma 2.4. *Consider the expectation value of a given density ρ_ε with respect to the tail $a > t/\varepsilon$,*

$$A_\varepsilon[\rho_\varepsilon](t) := \frac{\int_0^\infty a\ \rho_\varepsilon(\frac{t}{\varepsilon}+a,t)\,da}{\int_0^\infty \rho_\varepsilon(\frac{t}{\varepsilon}+a,t)\,da}\ , \tag{13}$$

then under assumptions 1.1 and 1.2, one has

$$A_\varepsilon[\rho_\varepsilon](t) \le A_{\max} \quad a.e.\ t>0.$$

uniformly wrt ε.

Proof. Observe that $\frac{d}{dt}\rho_\varepsilon(\frac{t}{\varepsilon}+a,t) = -\frac{1}{\varepsilon}\zeta_\varepsilon(t/\varepsilon+a,t)\rho_\varepsilon(\frac{t}{\varepsilon}+a,t)$ and that

$$\frac{d}{dt}A_\varepsilon[\rho_\varepsilon](t) = A_\varepsilon[\rho_\varepsilon](t)\left(-\int_0^\infty q_{1,\varepsilon,t}(a)\frac{1}{\varepsilon}\zeta_\varepsilon(t,t/\varepsilon+a)\,da + \int_0^\infty q_{0,\varepsilon,t}(a)\frac{1}{\varepsilon}\zeta_\varepsilon(t,t/\varepsilon+a)\,da\right)\ ,$$

where

$$q_{1,\varepsilon,t}(a) := \frac{a\ \rho_\varepsilon(\frac{t}{\varepsilon}+a,t)}{\int_0^\infty a\rho_\varepsilon(\frac{t}{\varepsilon}+a,t)\,da} \quad \text{and} \quad q_{0,\varepsilon,t}(a) := \frac{\rho_\varepsilon(\frac{t}{\varepsilon}+a,t)}{\int_0^\infty \rho_\varepsilon(\frac{t}{\varepsilon}+a,t)\,da}\ .$$

Let $Q_{i,\varepsilon,t} := \int_0^a q_{i,\varepsilon,t}(\tilde a)\,d\tilde a$ and define the transformation $T_{\varepsilon,t}(a) := Q^{-1}_{1,\varepsilon,t}(Q_{0,\varepsilon,t}(a))$ which allows to rewrite the above identity as

$$\frac{d}{dt}A_\varepsilon[\rho_\varepsilon](t) = A_\varepsilon[\rho_\varepsilon](t)\left(-\int_0^\infty q_{0,\varepsilon,t}(a)\frac{1}{\varepsilon}(\zeta_\varepsilon(t/\varepsilon+T_{\varepsilon,t}(a),t) - \zeta_\varepsilon(t/\varepsilon+a,t))\,da\right)\ . \tag{14}$$

Finally observe that $T_{\varepsilon,t}(a) \ge a$ since the inequality $Q_{0,\varepsilon,t}(a) \ge Q_{1,\varepsilon,t}(a)$ is equivalent to

$$\frac{\int_0^1 a\ \rho_\varepsilon(\frac{t}{\varepsilon}+a,t)\,da}{\int_1^\infty a\ \rho_\varepsilon(\frac{t}{\varepsilon}+a,t)\,da} \le \frac{\int_0^1 \rho_\varepsilon(\frac{t}{\varepsilon}+a,t)\,da}{\int_1^\infty \rho_\varepsilon(\frac{t}{\varepsilon}+a,t)\,da}\ , \tag{15}$$

which can be easily verified. If ζ_ε were monotonically increasing with respect to a, then the right hand side of (14) would be negative. In the weaker case defined in the assumptions of the present Lemma, where ζ_ε is only monotone on $[a_0, \infty)$, define

$$\bar{\rho}_\varepsilon(a,t) = \begin{cases} \rho_\varepsilon(a,t) & a > \frac{t}{\varepsilon} + a_0 \\ 0 & \text{otherwise,} \end{cases}$$

to exclude the area where the decay rate is not monotonically increasing. For fixed $t > 0$ either it holds that $\bar{\rho} \equiv 0$, which directly implies that $A_\varepsilon[\rho_\varepsilon](t) \leq a_0 \leq A_{\max}$, or in the opposite case we use that $\int_{a_0}^{\infty} \rho_{I,\varepsilon}(a,t)\, da > 0$ and obtain

$$A_\varepsilon[\rho_\varepsilon](t) \leq A_\varepsilon[\bar{\rho}_\varepsilon](t) \leq A_\varepsilon[\bar{\rho}_{\varepsilon,I}] \leq A_{\max},$$

where the first inequality can be reduced to (15), while the second one is due to an analogous application of (14). The integral in the numerator is bounded because the first moment of the initial datum $\rho_{I,\varepsilon}$ is bounded. □

We give existence and uniqueness results for (1).

Theorem 2.2. *Let $\rho_\varepsilon \in C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$ be given and let assumption 1.3 hold, then there exists for every fixed $\varepsilon > 0$ a unique function $z_\varepsilon \in C^0(\mathbb{R}_+)$ solving* (1)*.*

Proof. Setting $k_\varepsilon(\tilde{a}, t) := \frac{1}{\mu(t)} \frac{1}{\varepsilon} \rho_\varepsilon(\frac{t-\tilde{a}}{\varepsilon}, t)$ we write (1) as

$$z_\varepsilon(t) - \int_0^t z_\varepsilon(\tilde{a}) k_\varepsilon(\tilde{a}, t)\, d\tilde{a} = \tilde{f}_\varepsilon\,, \quad \text{with} \quad \tilde{f}_\varepsilon(t) := \varepsilon \frac{1}{\mu_{0,\varepsilon}(t)} f(t) + \int_t^\infty z_p(\tilde{a}) k_\varepsilon(\tilde{a}, t)\, d\tilde{a}$$

for all $t \geq 0$. Using the results of Theorem 2.1 we obtain that according to section 9.5 in [1] (Def. 5.2 and Thm. 5.4) the kernel k_ε of the integral equation is of bounded continuous type, which, together with the continuity of $\tilde{f}_\varepsilon(t)$, implies the existence of unique solution $z_\varepsilon \in C^0(\mathbb{R}_+)$. □

3. Convergence

Consider the difference $\hat{\rho}_\varepsilon := \rho_\varepsilon - \rho_0$. A formal computation using (2) and (5) implies that it satisfies

$$\begin{cases} \varepsilon \partial_t \hat{\rho}_\varepsilon + \partial_a \hat{\rho}_\varepsilon + \zeta_\varepsilon(a,t) \hat{\rho}_\varepsilon = \mathcal{R}_\varepsilon\,, & a > 0\,,\ t > 0\,, \\ \hat{\rho}_\varepsilon(a=0,t) = -\beta_\varepsilon(t) \displaystyle\int_0^\infty \hat{\rho}_\varepsilon(\tilde{a}, t)\, d\tilde{a} + M_\varepsilon\,, & t > 0\,, \\ \hat{\rho}_\varepsilon(a, t=0) = \rho_{\varepsilon,I}(a) - \rho_0(a,0)\,, & a \geq 0\,, \end{cases} \tag{16}$$

with $\mathcal{R}_\varepsilon := -\varepsilon \partial_t \rho_0 - \rho_0(\zeta_\varepsilon - \zeta_0)$ and $M_\varepsilon := (\beta_\varepsilon - \beta_0)\left(1 - \int_0^\infty \rho_0\, da\right)$. Like for its counterpart ρ_ε, we find that $\hat{\rho}_\varepsilon$ satisfies the above system (16) in the sense of integration along characteristics. Namely combining the system (9) with (4) we obtain:

Corollary 3.1. *The function $\hat{\rho}_\varepsilon$ satisfies the following integrated version of* (16)*,*

$$\hat{\rho}_\varepsilon(a,t) = \begin{cases} \left(-\beta_\varepsilon(t-\varepsilon a) \int_0^\infty \hat{\rho}_\varepsilon(\tilde{a}, (t-\varepsilon a))\, d\tilde{a} + M_\varepsilon(t-\varepsilon a)\right) \exp\left(-\int_0^a \zeta_\varepsilon(\tilde{a}, t-\varepsilon(a-\tilde{a}))\, d\tilde{a}\right) \\ \qquad + \int_0^a R_\varepsilon(t-\varepsilon(a-\bar{a})) \exp\left(-\int_{\bar{a}}^a \zeta_\varepsilon(\tilde{a}, t-\varepsilon(a-\tilde{a}))\, d\tilde{a}\right) d\bar{a} \quad a < t/\varepsilon\,, \\ (\rho_{\varepsilon,I}((a-t/\varepsilon)) - \rho_0((a-t/\varepsilon),0)) \exp\left(-\frac{1}{\varepsilon}\int_0^t \zeta_\varepsilon((\tilde{t}-t)/\varepsilon + a, \tilde{t})\, d\tilde{t}\right) \\ \qquad + \frac{1}{\varepsilon}\int_0^t R_\varepsilon(\bar{t}) \exp\left(-\frac{1}{\varepsilon}\int_{\bar{t}}^t \zeta_\varepsilon((\tilde{t}-t)/\varepsilon + a, \tilde{t})\, d\tilde{t}\right) d\bar{t}\,, \qquad a \geq t/\varepsilon\,. \end{cases} \tag{17}$$

Finally we formally multiply (16) by $\mathrm{sign}(\hat\rho_\varepsilon)$ to obtain

$$\begin{cases} \varepsilon\partial_t|\hat\rho_\varepsilon| + \partial_a|\hat\rho_\varepsilon| + \zeta_\varepsilon(a,t)|\hat\rho_\varepsilon| = \mathcal{R}_\varepsilon\,\mathrm{sign}(\hat\rho_\varepsilon)\,, & a>0\,,\ t>0\,,\\ |\hat\rho_\varepsilon(a=0,t)| = \left|-\beta_\varepsilon(t)\displaystyle\int_0^\infty \hat\rho_\varepsilon(\tilde a,t)\,d\tilde a + M_\varepsilon\right|\,, & t>0\,,\\ |\hat\rho_\varepsilon(a,t=0)| = |\rho_{\varepsilon,I}(a)-\rho_0(a,0)|\,, & a\geq 0\,, \end{cases} \tag{18}$$

which we also re-interpret using the method of characteristics:

Lemma 3.1. *$|\hat\rho_\varepsilon|$ satisfies the system* (18) *in the same way as $\hat\rho_\varepsilon$ fulfils* (16) *in the sense of* (17)

Proof. We reparametrise (17) like in the proof of Lemma 2.1 by $\tilde\rho(x,y)=\hat\rho(a,t)$ and obtain $\varepsilon\partial_y\tilde\rho_\varepsilon+\zeta_\varepsilon\tilde\rho_\varepsilon = R_\varepsilon$ in the domain $\Omega_1\cup\Omega_2$ parametrized by the variables (x,y). Solving this equation in the y variable and thanks to the assumptions it is easy to show that $\tilde\rho_\varepsilon$ is indeed continuous with respect to y for every fixed x. Thus one can write in the weak sense that $\partial_y|\tilde\rho_\varepsilon| = \mathrm{sign}(\tilde\rho_\varepsilon)\partial_y\tilde\rho_\varepsilon$ for every fixed x. Thus $\varepsilon\partial_y|\tilde\rho_\varepsilon| + \zeta_\varepsilon|\tilde\rho_\varepsilon| = \mathrm{sign}(\tilde\rho_\varepsilon)R_\varepsilon$ holds a.e. with respect to y for every fixed x. We then integrate and transform back to obtain the system which is the analogon to (17). Using Lemma 2.1 one concludes then that $|\hat\rho_\varepsilon|$ solves (18) in the weak sense. □

Taking advantage of both systems (16) and (18) we find that

Lemma 3.2. *Let $\zeta_{\min}>0$ the lower bound to $\zeta_\varepsilon(a,t)$ according to assumptions 1.1 and let $\hat\rho_\varepsilon$ be the solution to* (16)*, then it holds that*

$$\frac{d}{dt}\mathcal{H}[\hat\rho_\varepsilon] \leq -\frac{1}{\varepsilon}\zeta_{\min}\mathcal{H}[\hat\rho_\varepsilon] + \frac{2}{\varepsilon}\left(\|\mathcal{R}_\varepsilon\|_{L^1_a(\mathbb{R})} + |M_\varepsilon|\right)\,. \tag{19}$$

in a weak sense analogous to equation (12)*.*

Proof. Observe that the integrations in this proof are expressed in a formal way but can be made rigorous in a weak sense like in the step leading to (12).

On one hand the system (18) implies that

$$\frac{d}{dt}\int_0^\infty |\hat\rho_\varepsilon|\,da \leq \frac{1}{\varepsilon}\left(\beta_\varepsilon\left|\int_0^\infty \hat\rho_\varepsilon\,da\right| - \int_0^\infty \zeta_\varepsilon\,|\hat\rho_\varepsilon|\,da\right) + \int_0^\infty \frac{1}{\varepsilon}\mathcal{R}_\varepsilon\,\mathrm{sign}(\hat\rho_\varepsilon)\,da + \frac{1}{\varepsilon}|M_\varepsilon|\,. \tag{20}$$

On the other hand using (16) we write

$$\frac{d}{dt}\int_0^\infty \hat\rho_\varepsilon\,da = \frac{1}{\varepsilon}\left(-\beta_\varepsilon\int_0^\infty \hat\rho_\varepsilon\,da - \int_0^\infty \zeta_\varepsilon\hat\rho_\varepsilon\,da\right) + \frac{1}{\varepsilon}\int_0^\infty \mathcal{R}_\varepsilon\,da + \frac{1}{\varepsilon}M_\varepsilon\,,$$

which implies

$$\begin{aligned}\frac{d}{dt}\left|\int_0^\infty \hat\rho_\varepsilon\,da\right| = \frac{1}{\varepsilon}\left(-\beta_\varepsilon\left|\int_0^\infty \hat\rho_\varepsilon\,da\right| - \mathrm{sign}\left(\int_0^\infty \hat\rho_\varepsilon\right)\int_0^\infty \zeta_\varepsilon\hat\rho_\varepsilon\,da\right) +\\ + \mathrm{sign}\left(\int_0^\infty \hat\rho_\varepsilon\,da\right)\frac{1}{\varepsilon}\left(\int_0^\infty \mathcal{R}_\varepsilon\,da + M_\varepsilon\right)\,.\end{aligned} \tag{21}$$

The sum of (20) and (21) controls the evolution of the functional (7),

$$\begin{aligned}\frac{d}{dt}\mathcal{H}[\hat\rho_\varepsilon] \leq -\frac{1}{\varepsilon}\int_0^\infty \zeta_\varepsilon \underbrace{\left(|\hat\rho_\varepsilon| + \mathrm{sign}\left(\int_0^\infty \hat\rho_\varepsilon\,da\right)\hat\rho_\varepsilon\right)}_{=:A}\,da +\\ + \frac{1}{\varepsilon}\int_0^\infty \left(\mathrm{sign}\,(\hat\rho_\varepsilon) + \mathrm{sign}\left(\int_0^\infty \hat\rho_\varepsilon\,da\right)\right)\mathcal{R}_\varepsilon\,da + \frac{1}{\varepsilon}\left(|M_\varepsilon| + M_\varepsilon\,\mathrm{sign}\left(\int_0^\infty \hat\rho_\varepsilon\,da\right)\right)\,,\end{aligned} \tag{22}$$

where it is easy to check that $A\geq 0$ for almost any age a and any time t. We therefore conclude

$$\frac{d}{dt}\mathcal{H}[\hat\rho_\varepsilon] \leq -\frac{\zeta_{\min}}{\varepsilon}\left(\left|\int_0^\infty \hat\rho_\varepsilon\,da\right| + \int_0^\infty |\hat\rho_\varepsilon|\,da\right) + \frac{2}{\varepsilon}\left(\int_0^\infty |\mathcal{R}_\varepsilon|\,da + |M_\varepsilon|\right)\,,$$

which implies the result. □

We add three remarks which explain and illustrate the consequences of the above crucial Lemma:

Remark 3.1. *Under more general conditions then in the present study, namely without a positive lower bound on ζ_ε as assumed in assumption 1.1, the functional* (7) *is still a Lyapunov functional. If $R_\varepsilon = M_\varepsilon = 0$ it satisfies*

$$\frac{d}{dt}\mathcal{H}[\hat{\rho}_\varepsilon] = -\frac{1}{\varepsilon}\bar{\zeta}_\varepsilon \mathcal{H}[\hat{\rho}_\varepsilon] \leq 0$$

in a weak sense analogous to equation (12)*. Hence, up to a scaling factor, it decreases at an exponential rate which is a certain mean value of the decay rate, $\bar{\zeta}_\varepsilon := \int_0^\infty \zeta_\varepsilon(t,a)\pi(a,t)\,da$ where $\pi(a,t)$ stands for the probability density $\pi(a,t) := \left(|\hat{\rho}_\varepsilon| + \operatorname{sign}\left(\int_0^\infty \hat{\rho}_\varepsilon\right)\hat{\rho}_\varepsilon\right)/\mathcal{H}[\hat{\rho}_\varepsilon]$ (cf.* (22)*).*

Remark 3.2. *Under assumption 1.1, the Lyapunov functional does not only control the solution $\hat{\rho}_\varepsilon$ in the L^1_a norm but it also controls $\hat{\mu}_\varepsilon := \mu_{0,\varepsilon} - \mu_0$ which is related to the boundary value at $a = 0$, for any time.*

Remark 3.3. *Let the data be such that $R_\varepsilon = M_\varepsilon = 0$ and let assumption 1.1 hold, then* (8) *implies time asymptotic exponential convergence of ρ_ε towards ρ_0 wrt the L^1_a norm as well as of the averages $\mu_{0,\varepsilon}$ towards μ_0.*

Lemma 3.2 implies the result on ρ_ε as $\varepsilon \to 0$,

Lemma 3.3. *Let $\zeta_{\min} > 0$ be the lower bound to $\zeta_\varepsilon(a,t)$ according to assumption 1.1, then it holds that*

$$\mathcal{H}[\hat{\rho}_\varepsilon(.,t)] \leq \mathcal{H}[\rho_{\varepsilon,I} - \rho_0(.,0)]e^{\frac{-\zeta_{\min}t}{\varepsilon}} + \frac{2}{\zeta_{\min}}\left\| \|\mathcal{R}_\varepsilon\|_{L^1_a(\mathbb{R}_+)} + |M_\varepsilon| \right\|_{L^\infty_t(\mathbb{R}_+)}$$

for all $t \geq 0$.

Proof. We intend to apply Gronwall's inequality to the inequality (19) given in the weak sense. Hence we choose the testfunction $\varphi = \exp(\zeta_0/\varepsilon t)$ as it was done in the proof of Lemma 2.2 and obtain

$$\begin{aligned}
\mathcal{H}[\hat{\rho}_\varepsilon(.,t)] &\leq \mathcal{H}[\hat{\rho}_\varepsilon(.,0)]e^{\frac{-\zeta_{\min}t}{\varepsilon}} + \int_0^t e^{\frac{-\zeta_{\min}(t-\tilde{t})}{\varepsilon}}\frac{2}{\varepsilon}\left(\int_0^\infty |\mathcal{R}_\varepsilon(a,\tilde{t})|\,da + |M_\varepsilon|\right)d\tilde{t} \\
&\leq \mathcal{H}[\hat{\rho}_\varepsilon(.,0)]e^{\frac{-\zeta_{\min}t}{\varepsilon}} + \frac{2}{\zeta_{\min}}\left(1 - e^{\frac{-\zeta_{\min}t}{\varepsilon}}\right)\left\| \|\mathcal{R}_\varepsilon\|_{L^1_a(\mathbb{R}_+)} + |M_\varepsilon| \right\|_{L^\infty_t(\mathbb{R}_+)} ,
\end{aligned}$$

which implies the result. □

Theorem 3.1. *Let ρ_ε be the solution to the system* (2) *according to Theorem 2.1 and let the ρ_0 be as defined in* (4)*, then it holds that*

$$\rho_\varepsilon \to \rho_0 \quad in \quad C^0(]0,\infty); L^1(\mathbb{R}_+)) \quad as \quad \varepsilon \to 0\,,$$

where the convergence with respect to time is in the sense of uniform convergence on compact subintervals.

Proof. This is an immediate consequence of Lemma 3.3, because it holds that $|\mathcal{H}[\rho_{\varepsilon,I} - \rho_0(.,0)]| \leq 4$ due to (4) and assumption 1.2 and because the residual terms tend to zero in the respective norms as $\varepsilon \to 0$ by assumption 1.1. □

Remark 3.4. *Note that in general $\rho_{\varepsilon,I}$ does not converge to $\rho_0(.,0)$ in L^1_a as $\varepsilon \to 0$. A boundary layer will be observable if their difference does not oscillate and its profile will be shaped like a multiple of $e^{\frac{-\zeta_{\min}t}{\varepsilon}}$, which is again a consequence of Lemma 3.3.*

In the opposite case we obtain

Corollary 3.2. *Considering the asymptotic behaviour as $\varepsilon \to 0$: Under the additional assumption that $\rho_{\varepsilon,I} \to \rho_0(.,0))$ in $L^1(\mathbb{R}_+)$ it holds by coercivity that $\mathcal{H}[\rho_{\varepsilon,I} - \rho_0(.,0)] \to 0$ and therefore the convergence $\rho_\varepsilon \to \rho_0$ in L^1_a is uniform with respect to $t \in \mathbb{R}_+$. In fact it holds that*

$$\|\rho_\varepsilon - \rho_0\|_{L^\infty_t L^1_a} \leq \sup_{t\geq 0} \mathcal{H}[\hat{\rho}_\varepsilon] \leq \mathcal{H}[\rho_{\varepsilon,I} - \rho_0(.,0)] + \frac{2}{\zeta_{\min}} \left\| \|\mathcal{R}_\varepsilon\|_{L^1_a(\mathbb{R}_+)} + |M_\varepsilon| \right\|_{L^\infty_t(\mathbb{R}_+)} .$$

We need to estimate the convergence of the first moment as well:

Lemma 3.4. *Let ρ_ε be the solution to the system (2) according to Theorem 2.1 and let ρ_0 be as defined in (4), then it holds for $t > 0$ that*

$$\int_0^\infty a|\rho_\varepsilon - \rho_0| \, da \leq e^{\frac{-\zeta_{\min} t}{\varepsilon}} \int_0^\infty a \, |\rho_{\varepsilon,I}(a) - \rho_0(a,0)| \, da + \frac{1}{\zeta_{\min}} C_\varepsilon ,$$

where the family of constants $C_\varepsilon \in \mathbb{R}$ is such that $C_\varepsilon \to 0$ as $\varepsilon \to 0$.

Proof. The proof follows the same lines as above, but is simpler because the presence of the factor a cancels boundary terms. Indeed, integrating (18) against a by setting $\phi(t,a) = a\tilde{\phi}(t)$ in its weak formulation we obtain the weak formulation of

$$\begin{aligned} \varepsilon \partial_t \int_0^\infty a|\hat{\rho}_\varepsilon| \, da &= -\int_0^\infty \zeta_\varepsilon a|\hat{\rho}_\varepsilon| \, da + \int_0^\infty |\hat{\rho}_\varepsilon| \, da + \int_0^\infty a\mathcal{R}_\varepsilon \operatorname{sign}(\hat{\rho}_\varepsilon) \, da \\ &\leq -\zeta_{\min} \int_0^\infty a|\hat{\rho}_\varepsilon| \, da + K_\varepsilon , \end{aligned}$$

where $K_\varepsilon := \int_0^\infty |\hat{\rho}_\varepsilon| \, da + \varepsilon \int_0^\infty a \, |\partial_t \rho_0| \, da + \|\zeta_\varepsilon - \zeta_0\|_{L^\infty_{a,t}(\mathbb{R}^2_+)} \int_0^\infty |a\rho_0| \, da$. An argumentation which is analogous to the one in the proof of Lemma 3.3 implies that

$$\int_0^\infty a|\hat{\rho}_\varepsilon| \, da \leq e^{\frac{-\zeta_{\min} t}{\varepsilon}} \int_0^\infty a|\hat{\rho}_\varepsilon(a,0)| \, da + \frac{1}{\zeta_{\min}} \left(1 - e^{\frac{-\zeta_{\min} t}{\varepsilon}}\right) C_\varepsilon \tag{23}$$

for all $t \geq 0$, where $C_\varepsilon := \|K_\varepsilon\|_{L^\infty_t(\mathbb{R}_+)}$ satisfies $C_\varepsilon \to 0$ as $\varepsilon \to 0$ due to Lemma 3.3 and assumption 1.1. Indeed one has

$$\begin{aligned} \int_{\mathbb{R}_+} a\rho_0(a,t) \, da &\leq \frac{\beta_{\max}}{\zeta^2_{\max}} \quad \text{and} \\ \int_{\mathbb{R}_+} a|\partial_t \rho_0|(a,t) \, da &\leq k(\zeta_{\min}, \zeta_{\max}, \|\beta_\varepsilon\|_{W^{1,\infty}_t}, \|\partial_t \zeta_\varepsilon\|_{L^\infty_{t,a}}) . \end{aligned} \tag{24}$$

Since the first moments of ρ_0 and $\rho_{\varepsilon,I}$ are bounded by (24) and assumption 1.2 respectively, the expression $\int_0^\infty a|\hat{\rho}_\varepsilon(a,0)| \, da$ in (23) is uniformly bounded, which finishes the proof. □

Having defined properly, for any fixed ε, the solutions of the coupled system (1-2), we are finally able to prove the main theorem: as ε goes to 0, $(\rho_\varepsilon, z_\varepsilon)$ tends to (ρ_0, z_0), which solves the limit system (5).

Setting $\tilde{z}_\varepsilon := z_\varepsilon - z_0$, where z_0 solves exactly (3), one has:

$$\begin{aligned} &\frac{1}{\varepsilon} \int_0^\infty (\tilde{z}_\varepsilon(t) - \tilde{z}_\varepsilon(t - \varepsilon a)) \, \rho_\varepsilon(a,t) \, da = h_\varepsilon(t) \\ &\qquad \text{with} \quad h_\varepsilon(t) := f(t) - \frac{1}{\varepsilon} \int_0^\infty (z_0(t) - z_0(t - \varepsilon a)) \rho_\varepsilon \, da . \end{aligned} \tag{25}$$

To prepare the proof of the main theorem we state

Lemma 3.5. *For $0 < \tilde{t} < T$ it holds that*

$$\|h_\varepsilon\|_{L^\infty(\tilde{t},T)} \le C_1 \exp\left(-\frac{\tilde{t}\zeta_{\min}}{\varepsilon}\right) + C_2\varepsilon + \tilde{C}_\varepsilon \ , \tag{26}$$

for constants $C_1 > 0$ and $C_2 > 0$ and a family of constants $\tilde{C}_\varepsilon > 0$ with $C_\varepsilon \to 0$ as $\varepsilon \to 0$.

Proof. We concentrate on the second part of the rhs,

$$\begin{aligned}
\varepsilon h_\varepsilon(t) &= \varepsilon f(t) - \int_0^{t/\varepsilon}\int_{t-\varepsilon a}^{t} \partial_t z_0(s)\, ds\, \rho_\varepsilon(a,t)\, da - \int_{t/\varepsilon}^{\infty} (z_0(t) - z_0(t-\varepsilon a))\rho_\varepsilon(a,t)\, da \\
&= \varepsilon f(t) - \int_0^{t/\varepsilon}\int_{t-\varepsilon a}^{t} \frac{f(s)}{\mu_{1,0}} ds\, \rho_\varepsilon(a,t)\, da - \int_{t/\varepsilon}^{\infty}\int_0^{t} \frac{f(s)}{\mu_{1,0}} ds\, \rho_\varepsilon(a,t)\, da \\
&= \underbrace{\int_0^{t/\varepsilon}\int_{t-\varepsilon a}^{t} \left\{\frac{f(t)}{\mu_{1,0}(t)} - \frac{f(s)}{\mu_{1,0}(s)}\right\} ds\, \rho_\varepsilon(a,t)\, da + \int_{t/\varepsilon}^{\infty}\int_0^{t} \left\{\frac{f(t)}{\mu_{1,0}(t)} - \frac{f(s)}{\mu_{1,0}}\right\} ds\, \rho_\varepsilon(a,t)\, da}_{=:I_1} \\
&\quad + \underbrace{\varepsilon f(t) - \int_0^{t/\varepsilon}\int_{t-\varepsilon a}^{t} \frac{f(t)}{\mu_{1,0}(t)} ds\, \rho_\varepsilon(a,t)\, da - \int_{t/\varepsilon}^{\infty}\int_0^{t} \frac{f(t)}{\mu_{1,0}(t)} ds\, \rho_\varepsilon(a,t)\, da}_{=:I_2} \ .
\end{aligned}$$

Due to the regularity assumptions 1.1 on β_ε and ζ_ε and the assumptions 1.3 on $f(t)$, it is easy to prove that the function $g := f/\mu_{1,0}$ is uniformly Lipschitz with respect to time with a Lipschitz constant L_g. This implies

$$\begin{aligned}
|I_1| &\le L_g \left\{\int_0^{t/\varepsilon}\int_{t-\varepsilon a}^{t} (t-s)\, ds\, \rho_\varepsilon(a,t)\, da + \int_{t/\varepsilon}^{\infty}\int_0^{t} (t-s)\, ds\, \rho_\varepsilon(a,t)\, da\right\} \\
&= \frac{L_g}{2}\left\{\int_0^{t/\varepsilon} (\varepsilon a)^2 \rho_\varepsilon(a,t)\, da + \int_{t/\varepsilon}^{\infty} t^2 \rho_\varepsilon(a,t)\, da\right\} \\
&\le \frac{L_g}{2}\left\{\int_0^{t/\varepsilon} (\varepsilon a)^2 \rho_\varepsilon(a,t)\, da + \int_{t/\varepsilon}^{\infty} (\varepsilon a)^2 \rho_\varepsilon(a,t)\, da\right\} \le \frac{\varepsilon^2 L_g}{2}\mu_{2,\varepsilon} \le C_2\varepsilon^2 \ ,
\end{aligned}$$

where $\mu_{2,\varepsilon} := \int a^2 \rho_\varepsilon(a,t)\, da$. The upper bound of Lemma 2.3 allows to state that the constant C_2 does not depend on ε. On the other hand

$$\begin{aligned}
|I_2| &= \left|\varepsilon f(t) - \int_0^{t/\varepsilon} \varepsilon a \frac{f(t)}{\mu_{1,0}(t)} \rho_\varepsilon(a,t)\, da - \int_{t/\varepsilon}^{\infty} t \frac{f(t)}{\mu_{1,0}(t)} \rho_\varepsilon(a,t)\, da\right| \\
&= \left|\varepsilon f(t)\left\{1 - \int_0^{\infty} \frac{a\rho_\varepsilon(a,t)}{\mu_{1,0}} da\right\} + \frac{f(t)}{\mu_{1,0}(t)} \int_{t/\varepsilon}^{\infty} (\varepsilon a - t)\, \rho_\varepsilon(a,t)\, da\right| \\
&\le \varepsilon \left\|\frac{f}{\mu_{1,0}}\right\|_{L^\infty(0,T)} \left(|\mu_{1,0}(t) - \mu_{1,\varepsilon}(t)| + \int_{t/\varepsilon}^{\infty} \left(a - \frac{t}{\varepsilon}\right) \rho_\varepsilon(a,t)\, da\right) \ ,
\end{aligned}$$

where, using the second case in (9), it holds that

$$\int_{t/\varepsilon}^{\infty} \left(a - \frac{t}{\varepsilon}\right) \rho_\varepsilon(a,t)\, da \le \mu_{1,\varepsilon}(0) \exp\left(-\frac{t\zeta_{\min}}{\varepsilon}\right) \quad \text{for} \quad t > 0 \ .$$

This, together with Lemma 3.4, defines the constants C_1 and $\tilde{C}_\varepsilon$ in the result. □

Now we are ready to prove the main theorem.

proof of Theorem 1.1. The idea of the proof is to use a comparison principle to construct a majorizing function $U_\varepsilon \geq |\tilde{z}_\varepsilon|$ such that $U_\varepsilon \to 0$ as $\varepsilon \to 0$.

The comparison principle applies to the integral equation (25) in a rewritten form, namely by setting $k_\varepsilon(\tilde{a},t) := \frac{1}{\mu_\varepsilon(t)}\frac{1}{\varepsilon}\rho_\varepsilon(\frac{t-\tilde{a}}{\varepsilon},t)$ it becomes

$$\tilde{z}_\varepsilon(t) = \int_0^t \tilde{z}_\varepsilon(\tilde{a})k_\varepsilon(\tilde{a},t)\,d\tilde{a} + \tilde{h}_\varepsilon \quad \text{with} \quad \tilde{h}_\varepsilon(t) := \varepsilon\frac{1}{\mu_\varepsilon(t)}h_\varepsilon(t) + \int_{-\infty}^0 \tilde{z}_\varepsilon(\tilde{a})k_\varepsilon(\tilde{a},t)\,d\tilde{a} \tag{27}$$

for all $t \geq 0$. For the kernel of this integral operator we find that

$$0 \leq \int_0^t k_\varepsilon(\tilde{a},t)\,d\tilde{a} = \int_0^{t/\varepsilon} \frac{\rho_\varepsilon(a,t)}{\mu_\varepsilon(t)}da \leq 1 - \mu_{0,\varepsilon}(0)\exp\left(-\frac{t\,\zeta_{\max}}{\varepsilon}\right) < 1\,,$$

which implies that the Volterra kernel k_ε is of modulus

$$|||k_\varepsilon|||_{B^\infty(0,T)} := \sup_{0\leq t\leq T}\int_0^t |k_\varepsilon(\tilde{a},t)|d\tilde{a} \leq 1 - \mu_{0,\varepsilon}(0)\exp\left(-\frac{T\,\zeta_{\max}}{\varepsilon}\right) < 1$$

according to the Definition 5.1 in chapter 9 of [1]. Hence, by Proposition 8.1 and the generalized Gronwall Lemma 8.2 (p. 257) in chapter 9 of [1] a comparison principle holds: the control of the right hand side of of the equation implies the control of the solution. First observe that

$$|\tilde{z}_\varepsilon(t)| - \int_0^t |\tilde{z}_\varepsilon(\tilde{a})|k_\varepsilon(\tilde{a},t)\,d\tilde{a} \leq |\tilde{h}_\varepsilon|\,.$$

We will construct a function U_ε which satisfies

$$|\tilde{h}_\varepsilon(t)| \leq \underbrace{\varepsilon\frac{1}{\mu_\varepsilon(t)}\,|h_\varepsilon(t)|}_{=:\tilde{h}_{1,\varepsilon}} + \underbrace{\int_{-\infty}^0 |\tilde{z}_\varepsilon(\tilde{a})|\,k_\varepsilon(\tilde{a},t)\,d\tilde{a}}_{=:\tilde{h}_{2,\varepsilon}} \leq U_\varepsilon(t) - \int_0^t U_\varepsilon(\tilde{a})k(\tilde{a},t)\,d\tilde{a} \tag{28}$$

and hence is a majorizing function such that $U_\varepsilon(t) \geq |\tilde{z}_\varepsilon(t)|$ for all $t \geq 0$ due to the comparison principle.

To find such a function U_ε we also split up the integral operator applied to U_ε,

$$\begin{aligned}
U_\varepsilon(t) - \int_0^t U_\varepsilon(\tilde{a})k(\tilde{a},t)\,d\tilde{a} &= \\
&= \int_{-\infty}^t \left(U_\varepsilon(t) - U_\varepsilon(\tilde{a})\right)\,k_\varepsilon(\tilde{a},t)\,d\tilde{a} + \int_{-\infty}^0 U_\varepsilon(\tilde{a})\,k_\varepsilon(\tilde{a},t)\,d\tilde{a} \\
&= \int_0^\infty \left(\int_{t-\varepsilon a}^t \partial_t U_\varepsilon(\tilde{t})\,d\tilde{t}\right)\,\frac{\rho_\varepsilon(a,t)}{\mu_\varepsilon(t)}\,da + \int_{t/\varepsilon}^\infty U_\varepsilon(t-\varepsilon a)\,\frac{\rho_\varepsilon(a,t)}{\mu_\varepsilon(t)}\,da =: H_{1,\varepsilon} + H_{2,\varepsilon}\,,
\end{aligned}$$

and intend to specify U_ε such that $H_{1,\varepsilon} \geq \tilde{h}_{1,\varepsilon}$ and $H_{2,\varepsilon} \geq \tilde{h}_{2,\varepsilon}$. To this end we make the ansatz

$$U_\varepsilon = \varepsilon C + \frac{1}{\mu_{1,\min}}\begin{cases}\int_0^t \|h_\varepsilon\|_{L^\infty(\tilde{t},T)}\,d\tilde{t} & t > 0\,,\\ t\,\|h_\varepsilon\|_{L^\infty(0,T)} & t \leq 0\,,\end{cases} \tag{29}$$

with a constant $C > 0$ which we will choose appropriately. The motivation for this ansatz is the following. Both, the integral equation (27) and the formal limit equation (3) represent a growth dynamic with the growth given by the inhomogenity. To construct the majorizing function we hence take a suitable norm of the inhomogentiy and combine it with the structure of the formal limit function, since this can be given explicitly. This explains the integral part in (29). Furthermore, setting $t = 0$ in (27), one observes that $\tilde{z}_\varepsilon(t) = O(\varepsilon)$ due to the Lipschitz-continuity of the past data z_p according to assumption 1.3. This motivates the additional εC term in (29), where $C > 0$ will be chosen large enough.

Since U_ε is differentiable one can rewrite $H_{1,\varepsilon}$ and verify that it controls $\tilde{h}_{1,\varepsilon}$,

$$\begin{aligned} H_{1,\varepsilon}(t) &= \int_0^\infty \left(\int_{t-\varepsilon a}^t \partial_t U(\tilde{t})\, d\tilde{t} \right) \frac{\rho_\varepsilon(a,t)}{\mu_\varepsilon(t)}\, da \geq \\ &\geq \frac{1}{\mu_{1,\min}} \int_0^\infty \left(\int_{t-\varepsilon a}^t \|h_\varepsilon\|_{L^\infty(t,T)}\, d\tilde{t} \right) \frac{\rho_\varepsilon(a,t)}{\mu_\varepsilon(t)} da = \\ &= \frac{\varepsilon \mu_{1,\varepsilon}(t)}{\mu_\varepsilon(t)\, \mu_{1,\min}} \|h_\varepsilon\|_{L^\infty(t,T)} \geq \varepsilon \frac{1}{\mu_\varepsilon(t)} |h_\varepsilon(t)| = \tilde{h}_{1,\varepsilon}(t) \end{aligned}$$

a.e. on $\mathbb{R}_+$. For the difference of the second components we find that

$$\begin{aligned} H_{2,\varepsilon} - \tilde{h}_{2,\varepsilon} &= \int_{t/\varepsilon}^\infty U(t-\varepsilon a)\, \frac{\rho_\varepsilon(a,t)}{\mu_\varepsilon(t)}\, da - \int_{-\infty}^0 |\tilde{z}_\varepsilon(\tilde{a})| k_\varepsilon(\tilde{a},t)\, d\tilde{a} \\ &= \int_{t/\varepsilon}^\infty \left(\varepsilon C + (t-\varepsilon a) \frac{\|h_\varepsilon\|_{L^\infty(0,T)}}{\mu_{1,\min}} - |\tilde{z}_\varepsilon(t-\varepsilon a)| \right) \frac{\rho_\varepsilon(a,t)}{\mu_{0,\varepsilon}(t)}\, da \\ &= \int_0^\infty \left(\varepsilon C + (-\varepsilon a) \frac{\|h_\varepsilon\|_{L^\infty(0,T)}}{\mu_{1,\min}} - |\tilde{z}_\varepsilon(-\varepsilon a)| \right) \frac{\rho_\varepsilon(\frac{t}{\varepsilon}+a,t)}{\mu_\varepsilon(t)}\, da \\ &\geq \int_0^\infty \left(\varepsilon C + (-\varepsilon a) \frac{\|h_\varepsilon\|_{L^\infty(0,T)}}{\mu_{1,\min}} - L\varepsilon a \right) \frac{\rho_\varepsilon(\frac{t}{\varepsilon}+a,t)}{\mu_\varepsilon(t)}\, da \\ &= \varepsilon \int_0^\infty \left(C - a \left(\frac{\|h_\varepsilon\|_{L^\infty(0,T)}}{\mu_{1,\min}} + L \right) \right) \frac{\rho_\varepsilon(\frac{t}{\varepsilon}+a,t)}{\mu_\varepsilon(t)}\, da \geq 0\,, \end{aligned}$$

where $L > 0$ is a Lipschitz constant for $\tilde{z} = z_p - z_0$ on $\mathbb{R}_-$ according to assumption 1.3 and C has to be chosen such that

$$C \geq \left(\frac{\|h_\varepsilon\|_{L^\infty(0,T)}}{\mu_{1,\min}} + L \right) \frac{\int_0^\infty a\, \rho_\varepsilon\left(\frac{t}{\varepsilon}+a,t\right)\, da}{\int_0^\infty \rho_\varepsilon\left(\frac{t}{\varepsilon}+a,t\right)\, da}$$

using the uniform in ε bound on the expectation value of the tail established in Lemma 2.4. The comparison principle which we discussed above applied to (28) finally implies for all $0 \leq t \leq T$ that

$$0 \leq |\tilde{z}_\varepsilon(t)| \leq U_\varepsilon(t) = \varepsilon C + \frac{1}{\mu_{1,\min}} \int_0^t \|h_\varepsilon\|_{L^\infty(\tilde{t},T)}\, d\tilde{t} \ \to 0 \quad \text{as} \quad \varepsilon \to 0$$

due to Lemma 3.5, hence $z_\varepsilon \to z_0$ in $C^0((0,T))$. □

4. A simple example

We give here a simple example illustrating the approximation performed when using system (2-1) in order to approximate system (5-3).

Lemma 4.1. *We set both ζ_ε and β_ε to fixed values independent on ε, i.e.*

$$\zeta_\varepsilon = \zeta_0 = \zeta, \quad \beta_\varepsilon = \beta_0 = \beta.$$

Moreover defining the initial condition at equilibrium:

$$\rho_{I,\varepsilon} = \rho_0 = \frac{\beta\, \zeta}{\beta + \zeta} e^{-\zeta a}.$$

We obtain $\mu_{0,\varepsilon} = \mu_{0,0} = \beta/(\beta+\zeta)$, $\mu_{1,\varepsilon} = \mu_{1,0} = \beta/(\zeta(\beta+\zeta))$ and $\rho_0(a) = \mu_{0,0}\zeta e^{-\zeta a}$ and then one solves directly equation (1)*:*

$$z_\varepsilon(t) = \int_0^t \frac{f}{\mu_{1,0}} ds + \varepsilon \frac{f(t)}{\mu_{0,0}} + \frac{1}{\mu_{0,0}} \int_0^\infty z_p(-\varepsilon a)\, \rho_0\, da\,,$$

and hence

$$z_\varepsilon(t) - z_0(t) = \varepsilon \frac{f(t)}{\mu_{0,0}} - \int_{-\infty}^{0} z_p'(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds$$

with $z_0(t) = z_p(0) + \int_0^t f(s)ds/\mu_{1,0}$. *Note that the last term is an* ε *order term according to hypotheses 1.3, indeed it holds that*

$$\left| \int_{-\infty}^{0} z_p'(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds \right| \leq \frac{\varepsilon}{\zeta} \|z_p\|_{W^{1,\infty}(\mathbb{R}_-)} .$$

Proof. In this case one can rephrase the equation for $t \geq 0$ as

$$z_\varepsilon(t) - \frac{\zeta}{\varepsilon} \int_0^t z_\varepsilon(s) \exp\left(-\frac{\zeta(t-s)}{\varepsilon}\right) ds = \varepsilon \frac{f(t)}{\mu_{0,0}} + \frac{\zeta}{\varepsilon} \int_{-\infty}^{0} z_p(s) \exp\left(-\frac{\zeta(t-s)}{\varepsilon}\right) ds .$$

Due to the separation of variable made possible by this specific form of the kernel, one can rewrite this equation for all $t \geq 0$ as

$$q_\varepsilon(t) - \frac{\zeta}{\varepsilon} \int_0^t q_\varepsilon(s) ds = \varepsilon \exp\left(\frac{\zeta t}{\varepsilon}\right) \frac{f(t)}{\mu_{0,0}} + \frac{\zeta}{\varepsilon} \int_{-\infty}^{0} z_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds , \tag{30}$$

where

$$q_\varepsilon(t) = z_\varepsilon(t) \exp\left(\frac{\zeta t}{\varepsilon}\right) , \quad t \geq 0 .$$

Note that for $t = 0^+$ the integral equation provides the initial data

$$q_\varepsilon(0^+) = \varepsilon f(0)/\mu_{0,0} + \frac{\zeta}{\varepsilon} \int_{-\infty}^{0} z_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds .$$

Differentiating (30) for strictly positive times, one gets

$$\dot{q}_\varepsilon(t) - \frac{\zeta}{\varepsilon} q_\varepsilon(t) = \frac{\exp\left(\frac{\zeta t}{\varepsilon}\right)}{\mu_{0,0}} \left(\zeta f(t) + \varepsilon f'(t)\right) , \quad t > 0 .$$

Solving this differential equation in $]0, T[$ and using the initial data given above, one gets

$$q_\varepsilon(t) = \exp\left(\frac{\zeta}{\varepsilon} t\right) \left(\varepsilon \frac{f(t)}{\mu_{0,0}} + \frac{\zeta}{\varepsilon} \int_{-\infty}^{0} z_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds + \int_0^t \frac{f}{\mu_{1,0}}(s)\, ds \right) ,$$

where we used that $\mu_{1,0} = \mu_{0,0}/\zeta$. □

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