ON A STRUCTURED MODEL FOR LOAD DEPENDENT REACTION KINETICS OF TRANSIENT ELASTIC LINKAGES MEDIATING NONLINEAR FRICTION. *

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Abstract. We consider a microscopic model for friction mediated by transient elastic linkages introduced in [8, 10]. In this study we extend results and the general approach employed in [5]. We introduce a new unknown and reformulate the model. Based on this framework, we derive new a-priori estimates. In a first step this approach allows us to reproduce results of [5] concerning the convergence of the system to a macroscopic friction law in the semi-coupled case, but under weaker assumptions. Furthermore we consider the fully coupled case and prove existence and uniqueness of the solution.

Key words. friction coefficient, protein linkages, effect of chemical bonds, cell adhesion, renewal equation, integro-differential equations, Volterra kernel, fixed point, non-linear strain-stress

AMS subject classifications. 35Q92, 35B40, 45D05.

1. Introduction.

In this study we consider a mathematical model for the load-dependent turnover of chemical bonds modeled as elastic linkages. We are especially interested in the asymptotic limit of fast linkage turnover which allows to derive non-linear friction laws relevant in cell mechanics.

This model and its asymptotic counterpart have found many applications in biomathematical studies, specifically in cell mechanics and tumor biology. In [9] it has been used to describe the effect of cross-linking proteins in a lamellipodial F-actin meshwork. In this study the simplicity of the limit friction model allowed to achieve numerically feasible simulations of the deformation of keratocytes in response to external stimuli which reproduce characteristic cell shapes. In [6] the same sub-model for actin associated cross-linker proteins has been used in the derivation of a mathematical model which explains and quantifies the viscous properties of a one dimensional bundle of short actin filaments and its ability to propagate contractile force. Furthermore in [12], the authors use the asymptotic limit model in order to relate adhesion forces with the eulerian velocities of cells and embed these terms in a visco-elasto-plastic model for tumor growth. They also use experimental data found in [1] and [15] to reconstruct the load dependence of the off-rate of protein linkages (s. below) from the forces measured at tear-off.

We focus on the following system of equations which describes the evolution of the time-dependent position of a single binding site denoted by $z_\varepsilon(t) \in \mathbb{R}$,

$$
\begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_0^\infty \left(z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a) \right) \rho_\varepsilon(t,a) \, da = f(t) \,, & t \geq 0 \,, \\ z_\varepsilon(t) = z_p(t) \,, & t < 0 \,, \end{cases} \tag{1.1}
$$

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where the function $f = f(t) \in \mathbb{R}$ represents a given exterior force. The known past positions are given by the Lipschitz function $z_p(t) \in \mathbb{R}$ for $t < 0$. The time-dependent density function $\varrho_\varepsilon(t,a)$ represents the age-distribution of the linkages and solves itself the aged structured problem (1.2) with a non-local boundary term,

$$\begin{cases} \varepsilon \partial_t \varrho_\varepsilon + \partial_a \varrho_\varepsilon + \zeta_\varepsilon \, \varrho_\varepsilon = 0 \,, & t > 0 \,, \; a > 0 \,, \\ \varrho_\varepsilon(a=0,t) = \beta_\varepsilon(t) \left(1 - \int_0^\infty \varrho_\varepsilon(t,\tilde{a}) \, d\tilde{a} \right) \,, & t > 0 \,, \\ \varrho_\varepsilon(a,t=0) = \varrho_{I,\varepsilon}(a) \,, & a \geq 0 \,, \end{cases} \tag{1.2}$$

with the kinetic rate functions $\beta_\varepsilon = \beta_\varepsilon(t) \in \mathbb{R}_+$ and $\zeta_\varepsilon \in \mathbb{R}_+$, both possibly depending on the dimensionless parameter $\varepsilon > 0$.

Bonds are created as the binding site at $z_\varepsilon(t)$ moves on the substrate. We denote by $a \geq 0$ the age of a chemical bond which connects the massless single binding site at $z_\varepsilon(t)$ to the point $z_\varepsilon(t - \varepsilon a)$ on the substrate where that bond has been created. The system (1.2) models ageing of linkages and the boundary term describes the creation of new bonds which is proportional to the availability of free binding sites. The more linkages already exist, the less new bonds with age $a = 0$ are created, and *vice-versa*.

Finally, at any point in time, equation (1.1) claims the equilibrium of the sum of the external force and all elastic spring forces (see fig. 1.1) caused by these linkages.

The dimensionless parameter ε denotes the typical age of linkages as compared to the timescale of the problem and hence $1/\varepsilon$ represents the rate of linkage turnover. What we have in mind is to pass to a limit where this turnover is fast (ε small). Simultaneously, we make the assumption that linkages are very stiff, their elasticiy scaling like $1/\varepsilon$, which allows to obtain meaningful limit equations (see below). The key transformations which allows to obtain the scaled system from an unscaled one, are

$$z = \frac{1}{\varepsilon} z_\varepsilon, \quad \rho(t,a) = \frac{1}{\varepsilon} \varrho_\varepsilon \left(t, \frac{a}{\varepsilon} \right), \quad \beta = \frac{1}{\varepsilon} \beta_\varepsilon, \quad \zeta = \frac{1}{\varepsilon} \zeta_\varepsilon$$

and to take a/ε as the new independent variable for the age. For details of the scaling see [8, 5, 7].

In this work and in many others (see [12] and references therein), one of the crucial points is the precise behavior of ζ_ε. If the off-rate

$$\zeta_\varepsilon := \zeta_\varepsilon(a,t) \tag{1.3}$$

is a given function, then we call the system semi-coupled : in a first step, one can solve (1.2) to obtain ϱ_ε and then use it in order to solve for z_ε. From the modeling point of view, the more interesting case consists in defining ζ_ε such that it depends on z_ε, after scaling, as

$$\zeta_\varepsilon(t,a) := \zeta \left(\frac{|z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)|}{\varepsilon} \right) \,, \quad t \geq 0 \,, \quad a \geq 0 \,, \tag{1.4}$$

for a given monotonically increasing function $\zeta = \zeta(s) > 0$. In this case, we call the system fully coupled since ϱ_ε and z_ε are now interdependent. In fact by (1.4) the linkages' off-rate depends on their extension *i.e* on their mechanical load. A typical situation is, for example, an exponential increase of the off-rate as the elastic linker is extended, $\zeta_\varepsilon(t,a) = \zeta_0 \exp\left(|z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)|/\varepsilon\right)$ (cf. [14, 2, 12, 13]).

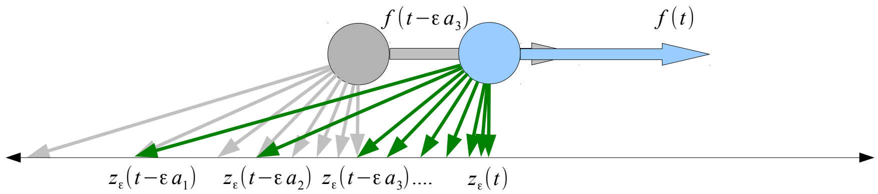

FIG. 1.1. *Positions of the moving binding site at time t and time $t-\varepsilon a_3$ with the respective linkages, (see [5]).*

We especially focus on the behavior of solutions as ε tends to zero. The tuple (ϱ_0, z_0) satisfies the formal limit of the system above

$$\begin{cases} \mu_{1,0}\,\partial_t z_0 = f \quad \text{with} \quad \mu_{1,0}(t) := \displaystyle\int_0^\infty a\varrho_0(t,a)\,da\;, \quad t>0\;, \\ z_0(t=0) = z_p(0)\;, \end{cases} \tag{1.5}$$

where ϱ_0 is explicitly given by

$$\varrho_0(t,a) = \frac{1}{\frac{1}{\beta_0(t)} + \int_0^\infty \exp\left(-\int_0^b \zeta_0\,d\tilde a\right) db} \exp\left(-\int_0^a \zeta_0\,d\tilde a\right), \tag{1.6}$$

being the solution of

$$\begin{cases} \partial_a \varrho_0 + \zeta_0\,\varrho_0 = 0\;, & t>0\;, \quad a>0\;, \\ \varrho_0(t,a=0) = \beta_0(t)\left(1 - \displaystyle\int_0^\infty \varrho_0(t,\tilde a)\,d\tilde a\right)\;, & t>0\;. \end{cases} \tag{1.7}$$

In the fully coupled case (1.4), the limit off-rate is given by $\zeta_0 = \zeta(a\,|\partial_t z_0|)$, whereas in the semi-coupled case, ζ_0 is defined as the limit of the family ζ_ε as ε tends to zero.

Combining (1.5) and (1.6), we are able to give an explicit expression for the viscosity constant $\mu_{1,0}$, which represents the macroscopic friction effect, in terms of the microscopic rate constants. In the general fully coupled case (1.4), the viscosity $\mu_{1,0}$ depends on the velocity of the binding site $\partial_t z_0$ and it is given by

$$\mu_{1,0}(t) = \frac{\int_0^\infty a \exp\left(-\int_0^a \zeta_0(\tilde a\,|\partial_t z_0|)\,d\tilde a\right) da}{\frac{1}{\beta_0(t)} + \int_0^\infty \exp\left(-\int_0^a \zeta_0(\tilde a\,|\partial_t z_0|)\,d\tilde a\right) da}\,. \tag{1.8}$$

For the semi-coupled system (1.1), (1.2), (1.3), the viscosity $\mu_{1,0}$, however, does not depend on the velocity of the binding site, and, in the special case where the limit off-rate does also not depend on age, $\zeta_0 = \zeta_0(t)$, the viscosity constant is given by $\mu_{1,0}(t) = (\zeta_0(t)(1+\zeta_0(t)/\beta(t)))^{-1}$. For instance, in the fully coupled case, following example 2, section 4, [12], one can define ζ as

$$\zeta(w) := \begin{cases} \zeta_0 & \text{if } w < w_0, \\ \zeta_1 & \text{otherwise.} \end{cases}$$

where $0 < \zeta_0 < \zeta_1 < \infty$. We denote $v := \partial_t z_0$, which is a velocity. One is able to compute explicitly $\mu_{1,0}$ in (1.8) :

$$\mu_{1,0}(v) := \frac{v\,e^{\frac{w_0\,\zeta_0}{v}}\,\zeta_1^2 - w_0\,\zeta_0\,\zeta_1^2 - v\,\zeta_1^2 + w_0\,\zeta_0^2\,\zeta_1 + v\,\zeta_0^2}{v\,\zeta_0\,\zeta_1\left(\zeta_0\,e^{\frac{w_0\,\zeta_0}{v}}\,\zeta_1 + e^{\frac{w_0\,\zeta_0}{v}}\,\zeta_1 - \zeta_1 + \zeta_0\right)}\,.$$

We plot (1.5) as a force-velocity relation, $\mu_{1,0}(v)v = f$, displayed in fig. 1.2, where we chose $w_0 = 5$, $\beta = 1$, $\zeta_0 = 1$ and various values of ζ_1 in order to emphasize whether the relation is bijective or not. These results are to be compared with curves plotted on p.1914 of [12]. One can observe that depending on the values of ζ_1, there can be one or more values of v for a given force, which is also observed in [12]. In [12] different regions of the velocity-force plan were related to various cellular regimes (see fig. 5 p. 1918).

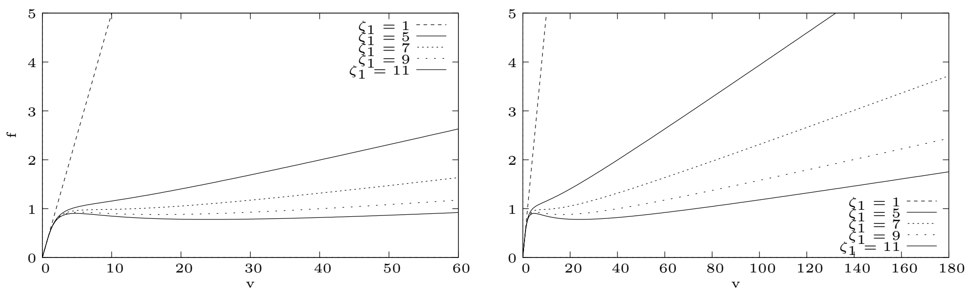

FIG. 1.2. *Velocity force diagram for a specific piecewise definition of ζ and various values of ζ_1. On the left we zoom on smaller values of v.*

In the semi-coupled case (1.3), the authors gave a first series of results concerning existence and uniqueness for fixed ε in [5]. They proved as well the convergence of $(\varrho_\varepsilon, z_\varepsilon)$ to (ϱ_0, z_0) when ε goes to 0. Although this limit leads to a linear Darcy type friction law, as stated in example 1, p. 1913, [12], the mathematical justification is far from being simple. In particular, in [5] p. 487, specific hypotheses concerning ζ_ε were made : ζ_ε was a definite positive bounded Lipschitz function wrt age, it was also supposed monotone non increasing on $[a_0, \infty)$, a_0 being some arbitrary chosen positive age. These assumptions guaranteed that

a) if $0 < \zeta_{\min} < \zeta_\varepsilon(t, a)$, one obtains convergence of ϱ_ε without hypotheses on the initial data wrt to $\varrho_0(0, a)$, (which means that we were able to control the boundary layer in time for the ϱ_ε model), and mild convergence rates of the on- and off-rates (typically $o(1)$ wrt ε).
b) if $\zeta_\varepsilon(t, a) < \zeta_{\max} < \infty$, the total mass $\mu_{0,\varepsilon}(t) := \int_{\mathbb{R}_+} \varrho_\varepsilon(t, a)da$ is positive definite for all times, which was necessary for the convergence of z_ε,
c) if ζ_ε is monotone for $a > a_0$ for some $a_0 > 0$, comparison results allowed to prove convergence of z_ε.

As stated above and in [7, 12], realistic cases involve less restrictive hypotheses on ζ_ε.

In this article, we introduce a new variable u_ε (see (2.2)) that shall replace the unknown z_ε and that can be seen as a discrete difference involving z_ε (see next sections for further details). This transforms equation (1.1) into a scalar integro-differential equation that can be compared to the renewal equation presented in [3, 11] *i.e.* it is an age structured equation with a non-local integral source term depending on the unknown itself, see (2.1).

In the semi-coupled case, the new formalism allows to obtain the global existence and uniqueness of the tuple $(\varrho_\varepsilon, u_\varepsilon)$ in suitable functional spaces, for any fixed ε. Thanks to this new framework, we are also able to prove convergence when ε goes to zero, supposing that $0 < \zeta_{\min} \le \zeta_\varepsilon(t, a) \le \zeta_{\max} < \infty$ but without the monotonicity assumption c) above.

We use then the new framework in order to construct a solution pair, for a fixed ε, in the case of a strong coupling: we suppose that ζ is regular but depends on $|u_\varepsilon|$ and s.t. $0 < \zeta_{\min} \leq \zeta_\varepsilon(w) \leq \zeta_{\max} < \infty$ for all $w \in \mathbb{R}$.

For what concerns further relaxing the assumption in item a) above, when ζ_ε is only non-negative, one needs a well prepared initial condition $\varrho_{I,\varepsilon}$ and β_ε and ζ_ε shall converge as $o(\varepsilon)$ to their respective limits in order to guarantee convergence as ε goes to zero in the semi-coupled case. For what concerns item b), a work is in preparation that fully treats the case when ζ is an unbounded function ([4]) for both the semi- and fully coupled cases.

The outline of the paper is as follows. In the next section, we detail the precise framework and the main tools of our analysis and we state our main results. Then, in Section 3, we recall results concerning the ϱ_ε model (1.2) already established in [5]. In Section 4, we introduce the variable u_ε and eliminate z_ε from the system so to express the problem in terms of $(\varrho_\varepsilon, u_\varepsilon)$ exclusively. Furthermore, in Section 5, we prove *a priori* estimates for the reformulated system. These results apply to the semi-coupled case as well as to the fully coupled case. Hence in Section 6, we treat the semi-coupled case and prove the existence of a unique solution $(u_\varepsilon, \varrho_\varepsilon)$ as well as the weak convergence of u_ε towards the solution u_0 of the limit problem. Furthermore we show that this implies the strong convergence of z_ε as formulated in Theorem 2.1. In the last section, we prove, based on results of previous sections, and extending them, the existence and uniqueness of a solution of the fully coupled system (1.1), (1.2), (1.4) for fixed ε by fixed point techniques.

2. Analytical framework and main results. In this study, it is our aim to introduce a new analytic method to deal with the system (1.1), (1.2) and either (1.3) or (1.4) and to obtain new results based on it. To this end, we introduce the new variable u_ε as the solution of the system

$$\begin{cases} \varepsilon\partial_t u_\varepsilon + \partial_a u_\varepsilon = \dfrac{1}{\mu_{0,\varepsilon}}\left(\varepsilon\partial_t f + \displaystyle\int_0^\infty \zeta_\varepsilon u_\varepsilon \varrho_\varepsilon \, da\right), & t > 0,\ a > 0, \\ u_\varepsilon(t,0) = 0, & t > 0, \\ u_\varepsilon(0,a) = u_{I,\varepsilon}(a), & a \geq 0, \end{cases} \tag{2.1}$$

where $\mu_{0,\varepsilon}(t) := \int_0^\infty \varrho_\varepsilon(\tilde{a}, t)\, d\tilde{a}$ and

$$u_{I,\varepsilon}(a) := \frac{z_\varepsilon(0) - z_p(-\varepsilon a)}{\varepsilon},$$

and where according to (1.1), it holds that

$$z_\varepsilon(0) = \frac{1}{\mu_{0,\varepsilon}(0)}\left(\int_0^\infty z_p(-\varepsilon a)\ \varrho_{I,\varepsilon}(a)\, da + \varepsilon f(0)\right).$$

In fact, z_ε, being the solution of (1.1) and u_ε, being the solution of (2.1), contain the same information. We state, on the one hand, that, given z_ε, one actually obtains the function u_ε according to

$$u_\varepsilon(t,a) = \begin{cases} \frac{z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a)}{\varepsilon}, & t > \varepsilon a, \\ \frac{z_\varepsilon(t) - z_p(t-\varepsilon a)}{\varepsilon}, & t \leq \varepsilon a. \end{cases} \tag{2.2}$$

On the other hand, given u_ε one obtains z_ε evaluating

$$z_\varepsilon(t) = z_\varepsilon(0) + \int_0^t \frac{1}{\mu_{0,\varepsilon}(\tilde{t})} \left(\varepsilon \partial_t f(\tilde{t}) + \int_0^\infty \zeta_\varepsilon(\tilde{t}, a)\, u_\varepsilon(\tilde{t}, a)\, \varrho_\varepsilon(\tilde{t}, a)\, da \right) d\tilde{t}\ . \tag{2.3}$$

Finally the original integral equation (1.1) may be recasted as

$$\int_0^\infty \varrho_\varepsilon(t, a)\, u_\varepsilon(t, a)\, da = f(t)\ , \quad t \geq 0\ , \tag{2.4}$$

and in the fully coupled case (1.4) is replaced by

$$\zeta_\varepsilon = \zeta(|u_\varepsilon|)\ , \tag{2.5}$$

which defines the coupling with (1.2) in a straightforward way.

In our analysis, system (2.1) replaces the original integral equation (1.1) and allows to derive *a priori* bounds for u_ε. First, we obtain the previously unknown *a priori* estimate

$$\int_0^\infty \varrho_\varepsilon(t, a)\, |u_\varepsilon(t, a)|\, da \leq \int_0^\infty \varrho_{I,\varepsilon}(a)\, |u_{I,\varepsilon}|(a)\, da + \int_0^t |\partial_t f|\, dt\ , \tag{2.6}$$

which holds provided $\zeta_\varepsilon \geq 0$ only (see Lemma 5.1). Observe that the estimate formulated in (2.6) includes an ε-dependent weight-function. Then, under the supplementary hypothesis that ζ_ε is bounded from above, we get, in Lemma 5.2, a pointwise bound on u_ε

$$|u_\varepsilon(t, a)| \leq \alpha_0 + \alpha_1 a, \quad \forall (t, a) \in (0, T) \times \mathbb{R}_+, \tag{2.7}$$

where the coefficients α_0 and α_1 depend on the data, on T but not on ε.

In the semi-coupled case (1.3), we obtain weak convergence results for u_ε converging towards the formal limit of the model (2.1). Finally, this argument ensures also the strong convergence of z_ε to z_0, thus reproducing the results obtained in [5]. The main advantage of this approach is that no hypotheses are required on the monotonicity of ζ_ε.

In the fully coupled case (1.4), we prove the existence and uniqueness of a solution to (1.2), (2.1), (2.5) for a fixed ε.

The analysis in this paper relies on the following set of assumptions. The initial data for the density model (1.2) satisfies the following hypotheses.

Assumption 2.1. *The initial condition $\varrho_{I,\varepsilon} \in L^\infty_a(\mathbb{R}_+)$ is*

(i) nonnegative, i.e. $\varrho_{I,\varepsilon}(a) \geq 0\ , \quad$ a.e. in $\mathbb{R}_+$.

(ii) Moreover, the total initial population satisfies

$$0 < \int_0^\infty \varrho_{I,\varepsilon}(a) da < 1\ ,$$

(iii) and higher moments are bounded,

$$0 < \int_0^\infty a^p \varrho_{I,\varepsilon}(a)\, da \leq c_p\ , \quad \textit{for } p = 1, 2\ ,$$

where c_p are positive constants depending only on p.

Concerning the integral equation (1.1) and its new analogue (2.1) we assume

ASSUMPTION 2.2. *The time dependent exterior force $f = f(t)$ in (1.1) is locally a Lipschitz function on $\mathbb{R}_+$,* i.e. $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+)$. *The past condition z_p belongs to* $\mathrm{Lip}(\mathbb{R}_-)$.

For the chemical reaction rates we assume:

ASSUMPTION 2.3. *The dimensionless parameter $\varepsilon > 0$ is assumed to induce a family of on-rates for the protein linkages that satisfy*

(i) For the limit function $\beta_0 \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+)$ it holds that $\|\beta_\varepsilon - \beta_0\|_{L^\infty_t} \to 0$ as $\varepsilon \to 0$.

(ii) We also assume that there are upper and lower bounds such that

$$0 < \beta_{\min} \le \beta_\varepsilon(t) \le \beta_{\max},$$

for all $\varepsilon > 0$ and $t > 0$.

For the off-rates we distinguish between the fully coupled and the semi-coupled cases. In the semi-coupled case we make assumptions for the off-rates which are analogues of Assumptions 2.3.

ASSUMPTION 2.4. *There is a family ζ_ε of functions that satisfy*

(i) the limit function $\zeta_0 \in W^{1,\infty}(\mathbb{R}_+; L^\infty(\mathbb{R}_+))$, and $\|\zeta_\varepsilon - \zeta_0\|_{L^\infty_t L^\infty_a} \to 0$ as $\varepsilon \to 0$.

(ii) We also assume that there are upper and lower bounds such that

$$0 < \zeta_{\min} \le \zeta_\varepsilon(t, a) \le \zeta_{\max},$$

for all $\varepsilon > 0$, and a.e. $a \ge 0$ and a.e. $t > 0$.

In the fully coupled case we assume instead

ASSUMPTION 2.5. *The function $\zeta = \zeta(s)$, $s \in \mathbb{R}$ is Lipschitz-continuous with Lipschitz-constant $\zeta_{lip} := \|\zeta'(\cdot)\|_{L^\infty(\mathbb{R})}$ and there are upper and lower bounds such that*

$$0 < \zeta_{\min} \le \zeta(s) \le \zeta_{\max} < \infty,$$

for all $s \in \mathbb{R}$.

In order to set up the analytic framework to deal with the function u_ε, we introduce the weight function

$$\omega(a) := \frac{1}{1+a} \tag{2.8}$$

and we define the functional space

$$X_T := \left\{ g \in L^\infty_{\mathrm{loc}}((0,T) \times \mathbb{R}_+) \text{ s.t. } \sup_{t \in (0,T)} \|g(t,a)\,\omega(a)\|_{L^\infty_a} < \infty \right\} \tag{2.9}$$

and the corresponding norm is denoted $\|\cdot\|_{X_T}$. Given two real times $T_{n+1} > T_n$, one denotes as well $X_{(T_n, T_{n+1})}$ the space where the time interval in the (2.9) is replaced by (T_n, T_{n+1}).

If Y is a Banach space, then we denote by $C([0,T]; Y)$ the set of continuous functions with values in Y equipped with the norm $\|u\|_{C([0,T];Y)} := \sup_{t \in [0,T]} \|u(t,\cdot)\|_Y$. When the time interval is infinite, and if not specified, the norm is to be understood in the local sense.

Our first result applies to the semi-coupled system of equations (1.1), (1.2) and (1.3) and relaxes the technical assumptions used in [5].

Theorem 2.1. *Let assumptions 2.1, 2.2, 2.3 and 2.4 hold, then for every fixed ε there exists a unique solution of the coupled system* (1.1), (1.2), (1.3) *denoted by* $(z_\varepsilon, \rho_\varepsilon) \in \mathrm{Lip}([0,T]) \times \left(C([0,T];\, L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}^2_+)\right)$. *Moreover if the same assumptions hold for $T = \infty$ then the time of existence and uniqueness extends to $T = \infty$ as well. Let (z_0, ϱ_0) be the unique solution to the formal limit system (1.5-1.7), then for every finite $T > 0$ it holds that*

$$\|z_\varepsilon - z_0\|_{C([0,T])} + \|\varrho_\varepsilon - \varrho_0\|_{C(]0,T];L^1(\mathbb{R}_+))} \to 0$$

as $\varepsilon \to 0$.

We consider in a second step the fully coupled system described by (1.2) coupled to (2.1), (2.5) or, alternatively, (1.2), (1.1), (1.4). We prove existence and uniqueness of the solution $(u_\varepsilon, \varrho_\varepsilon)$ for a fixed $\varepsilon > 0$.

Theorem 2.2. *Let assumptions 2.1, 2.2, 2.3 and 2.5 hold, then there exists a unique solution* $(\varrho_\varepsilon, u_\varepsilon) \in C([0,T]; L^1(\mathbb{R}_+)) \times X_T$ *solving the coupled system* (1.2)-(2.1)-(2.5) *for any positive time T. The maximal time of existence is infinite and the stability results Lemma 5.1 (i.e.*(2.6)*) and Lemma 5.2 (i.e.* (2.7)*) hold. This results provide existence and uniqueness of the couple* $(\varrho_\varepsilon, z_\varepsilon) \in C([0,T]; L^1(\mathbb{R}_+)) \times \mathrm{Lip}([0,T])$ *solving* (1.1)-(1.2)-(1.4).

We underline that such an existence and uniqueness result is completely new. An open problem is still the convergence of such a non-linear model when ε goes to zero.

3. Preliminary results. The following preliminary results on the solution ϱ_ε of (1.2) have been obtained in [5].

Theorem 3.1. *Let assumptions 2.1, 2.3 and 2.4 hold, then for every fixed ε there exists a unique solution* $\varrho_\varepsilon \in C(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}^2_+)$ *of the problem* (1.2). *We say that ϱ_ε is a mild solution since it satisfies* (1.2) *in the sense of characteristics, namely*

$$\varrho_\varepsilon(t,a) = \begin{cases} \beta_\varepsilon(t-\varepsilon a)\left(1 - \int_0^\infty \varrho_\varepsilon(\tilde a, t-\varepsilon a)\, d\tilde a\right) \times \\ \qquad\qquad \times \exp\left(-\int_0^a \zeta_\varepsilon(\tilde a, t - \varepsilon(a-\tilde a))\, d\tilde a\right) \;, & \text{when } a < t/\varepsilon \;, \\ \varrho_{I,\varepsilon}(a - t/\varepsilon)\exp\left(-\frac{1}{\varepsilon}\int_0^t \zeta_\varepsilon((\tilde t - t)/\varepsilon + a, \tilde t)\, d\tilde t\right) \;, & \text{if } a \geq t/\varepsilon \;. \end{cases} \tag{3.1}$$

Moreover, it is a weak solution as well since it satisfies

$$\int_0^\infty \int_0^T \varrho_\varepsilon(t,a)\left(\varepsilon\partial_t\varphi + \partial_a\varphi + \zeta_\varepsilon\varphi\right)\, dt\, da - \varepsilon\int_0^\infty \varrho_\varepsilon(t,a)\varphi(a, t=T)\, da + \\ + \int_0^T \varrho_\varepsilon(a=0,t)\,\varphi(0,t)\, dt + \varepsilon\int_0^\infty \varrho_{I,\varepsilon}(a)\varphi(a,t=0)\, da = 0 \;, \tag{3.2}$$

for every $T > 0$ and every test function $\varphi \in C^\infty(\mathbb{R}^2_+) \cap L^\infty(\mathbb{R}^2_+)$.

Obviously ϱ_ε can be more regular provided that the data is regular and that a compatibility condition is satisfied at the origin :

Corollary 3.1. *We fix $T > 0$ possibly equal to infinity, if* $\beta_\varepsilon \in C([0,T])$, $\zeta_\varepsilon \in C_{\mathrm{loc}}([0,T] \times \mathbb{R}_+)$, $\varrho_{I,\varepsilon} \in C_{\mathrm{loc}}(\mathbb{R}_+)$ *and the compatibility condition :* $\varrho_{I,\varepsilon}(0) = \beta_\varepsilon(0)(1 - \mu_{0,\varepsilon}(0))$ *is satisfied, then* $\varrho_\varepsilon \in C_{\mathrm{loc}}([0,T] \times \mathbb{R}_+)$.

The following two Lemmas formulate bounds on the moments of ϱ_ε which we denote by

$$\mu_{p,\varepsilon}(t) := \int_0^\infty a^p \varrho_\varepsilon(t,a)\, da \;, \quad \text{where } p = 1, 2 \;.$$

Lemma 3.1. *Let assumptions 2.1, 2.3 and 2.4 hold, then the unique solution $\varrho_\varepsilon \in C(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$ of the problem (1.2) from Theorem 3.1 satisfies*

$$\varrho_\varepsilon(t,a) \geq 0 \quad a.e.\ in \quad \mathbb{R}_+^2 \quad and$$

$$\mu_{0,\min} \leq \mu_{0,\varepsilon}(t) < 1\ , \quad \forall t \in \mathbb{R}_+ \quad where \quad \mu_{0,\min} := \min\left(\mu_{0,\varepsilon}(0), \frac{\beta_{\min}}{\beta_{\min} + \zeta_{\max}}\right). \tag{3.3}$$

In a more straightforward manner one gets for higher moments as well

Lemma 3.2. *Let assumption 2.1 hold, then*

$$\mu_{p,\min} < \mu_{p,\varepsilon}(t) \leq k \quad for \quad p = 1,2\ , \quad where \quad \mu_{p,\min} := \min\left(\mu_{p,\varepsilon}(0), \frac{\mu_{p-1,\min}}{\zeta_{\max}}\right),$$

and the generic constant k is independent of both time and ε.

Furthermore the following results on the convergence of ϱ_ε as ε tends to 0 have been obtained. We define the functional

$$\mathcal{H}[u] := \left|\int_0^\infty u(a)\ da\right| + \int_0^\infty |u(a)|\ da\ , \tag{3.4}$$

and we obtain

Lemma 3.3. *Let $\zeta_{\min} > 0$ be the lower bound to $\zeta_\varepsilon(t,a)$ according to assumption 2.4, then it holds for all $t \geq 0$ that*

$$\mathcal{H}[\varrho_\varepsilon(t,.) - \varrho_0(t,.)] \leq \mathcal{H}[\rho_{\varepsilon,I} - \rho_0(0,.)]e^{\frac{-\zeta_{\min} t}{\varepsilon}} + \frac{2}{\zeta_{\min}}\left\|\|\mathcal{R}_\varepsilon\|_{L_a^1(\mathbb{R}_+)} + |M_\varepsilon|\right\|_{L_t^\infty(\mathbb{R}_+)}$$

with $\mathcal{R}_\varepsilon := -\varepsilon\partial_t\varrho_0 - \varrho_0(\zeta_\varepsilon - \zeta_0)$ and $M_\varepsilon := (\beta_\varepsilon - \beta_0)\left(1 - \int_0^\infty \varrho_0\ da\right)$. As a consequence we conclude

Theorem 3.2. *Let ϱ_ε be the solution to the system (1.2) according to Theorem 3.1 and let the ϱ_0 be as defined in (1.6), then it holds that*

$$\varrho_\varepsilon \to \varrho_0 \quad in \quad C_{\mathrm{loc}}(]0,\infty); L^1(\mathbb{R}_+)) \quad as \quad \varepsilon \to 0\ .$$

Remark 3.1. *Note that in general $\rho_{\varepsilon,I}$ does not converge to $\rho_0(.,0)$ in L_a^1 as $\varepsilon \to 0$. A boundary layer will be observable and its profile will be shaped like a multiple of $e^{\frac{-\zeta_{\min} t}{\varepsilon}}$, which is again a consequence of Lemma 3.3.* In the opposite case we obtain

Corollary 3.2. *Considering the asymptotic behavior as $\varepsilon \to 0$: Under the additional assumption that $\rho_{\varepsilon,I} \to \rho_0(.,0))$ in $L^1(\mathbb{R}_+)$ it holds by coercivity that $\mathcal{H}[\rho_{\varepsilon,I} - \rho_0(.,0)] \to 0$ and therefore the convergence $\varrho_\varepsilon \to \varrho_0$ in L_a^1 is uniform with respect to $t \in \mathbb{R}_+$. In fact it holds that*

$$\begin{aligned}\|\varrho_\varepsilon - \varrho_0\|_{L_t^\infty L_a^1} &\leq \sup_{t\geq 0}\mathcal{H}[\varrho_\varepsilon(t,.) - \varrho_0(t,.)] \leq \\ &\leq \mathcal{H}[\rho_{\varepsilon,I} - \rho_0(.,0)] + \frac{2}{\zeta_{\min}}\left\|\|\mathcal{R}_\varepsilon\|_{L_a^1(\mathbb{R}_+)} + |M_\varepsilon|\right\|_{L_t^\infty(\mathbb{R}_+)}\ .\end{aligned}$$

We provide estimates on the convergence of the first moment as well.

Lemma 3.4. *Let ϱ_ε be the solution to the system (1.2) according to Theorem 3.1 and let ϱ_0 be defined as in (1.6), then it holds for $t > 0$ that*

$$\int_0^\infty a|\varrho_\varepsilon - \varrho_0|\, da \leq e^{\frac{-\zeta_{\min} t}{\varepsilon}} \int_0^\infty a\,|\rho_{\varepsilon,I}(a) - \rho_0(a,0)|\, da + \frac{1}{\zeta_{\min}} C_\varepsilon\ ,$$

where the family of constants $C_\varepsilon \in \mathbb{R}$ is such that $C_\varepsilon \to 0$ as $\varepsilon \to 0$.

Remark 3.2. *Integrating in time estimates from Lemma 3.3 and above, one concludes that $(1+a)\varrho_\varepsilon$ actually converges to $(1+a)\varrho_0$ strongly in $L^1((0,T)\times\mathbb{R}_+)$ for any finite time T.*

4. Reformulating the Volterra integral equation. Given the functions ϱ_ε and ζ_ε, our goal in this section is to replace $z_\varepsilon = z_\varepsilon(t)$ satisfying (1.1) by a new quantity which we denote by $u_\varepsilon = u_\varepsilon(t,a)$. It is defined as the solution of (2.1). Observe that in Section 6 and Section 7 we will actually prove the existence of a unique solution to (2.1).

In the following two results we state, on the one hand, that, given z_ε, one actually obtains the function u_ε according to (2.2) and that on the other hand, given u_ε, one recovers z_ε evaluating (2.3). Finally we also show that as a consequence of (2.1) the original equation (1.1) transforms into (2.4).

In this paper we consider "mild" solutions to (2.1) which satisfy (2.1) after integration along characteristics, namely

$$u_\varepsilon(t,a) := \begin{cases} \int_0^a g(t-\varepsilon\tilde a)\, d\tilde a\ , & t > \varepsilon a, \\ \int_0^{t/\varepsilon} g(t-\varepsilon\tilde a)\, d\tilde a + u_{I,\varepsilon}(a - t/\varepsilon)\ , & t \leq \varepsilon a, \end{cases} \tag{4.1}$$

where

$$g(t) := \frac{1}{\mu_{0,\varepsilon}(t)} \left(\varepsilon \partial_t f + \int_0^\infty \varrho_\varepsilon(t,a)\, \zeta_\varepsilon\, u_\varepsilon(t,a)\, da \right)\ .$$

Being a mild solution u_ε is as well a weak solution of (2.1) in the sense that it satisfies

$$\begin{aligned} -\int_0^T \int_0^\infty u_\varepsilon(\varepsilon\partial_t\varphi + \partial_a\varphi)\, da\, dt + \varepsilon \left[\int_0^\infty u_\varepsilon(s,a)\,\varphi(s,a)\, da \right]_{s=0}^{s=T} = \\ = \int_0^T \frac{1}{\mu_{0,\varepsilon}} \left(\varepsilon\partial_t f + \int_0^\infty \zeta_\varepsilon\, \varrho_\varepsilon\, u_\varepsilon\, da \right) \left(\int_0^\infty \varphi(t,\tilde a)\, d\tilde a \right) dt \end{aligned} \tag{4.2}$$

for any function $\varphi \in C_c^\infty([0,T]; C_c^\infty(\mathbb{R}_+))$.

Lemma 4.1. *Let z_ε be a Lipschitz-continuous solution to (1.1), then the function u_ε recovered according to (2.2) satisfies the system (2.1) in the sense of integration along characteristics (4.1).*

Proof. In fact, the function u_ε if defined by (2.2) solves the following system

$$u(t,a) := \begin{cases} \int_0^a z'_\varepsilon(t-\varepsilon\tilde a)\, d\tilde a\ , & t > \varepsilon a, \\ \int_0^{t/\varepsilon} z'_\varepsilon(t-\varepsilon\tilde a)\, d\tilde a + u_{I,\varepsilon}(a - t/\varepsilon)\ , & t \leq \varepsilon a. \end{cases} \tag{4.3}$$

Hence the weak formulation of the following system holds

$$\begin{cases} \varepsilon\partial_t u_\varepsilon + \partial_a u_\varepsilon = z'_\varepsilon(t)\ , & a > 0\,,\ t > 0\ , \\ u_\varepsilon(t,0) = 0\ , & t > 0\ , \\ u_\varepsilon(0,a) = u_{I,\varepsilon}\ , & a \geq 0\ . \end{cases}$$

Testing against ϱ_ε and integrating in age, one gets that

$$\varepsilon\partial_t \int_0^\infty \varrho_\varepsilon u_\varepsilon da + \int_0^\infty \zeta_\varepsilon \varrho_\varepsilon u_\varepsilon da = z_\varepsilon'(t)\mu_{0,\varepsilon}(t) \ , \tag{4.4}$$

but then one uses again the fact z_ε solves (1.1), which in term of u_ε means (2.4). According to Lemma 3.1 it holds that $\mu_{0,\varepsilon}(t) > \mu_{0,\min}$, hence we might isolate $z_\varepsilon'(t)$ in (4.4) and combine it with (2.4) and (4.3) to obtain (4.1). □

LEMMA 4.2. *Let u_ε satisfy (2.1) in the sense of (4.1), then the reformulated version (2.4) of the original equation (1.1) holds and z_ε recovered from u_ε according to (2.3) satisfies (1.1).*

Proof. Testing (2.1) against ϱ_ε and integrating in age gives

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty u_\varepsilon(t,a)\varrho_\varepsilon(t,a)\, da = \varepsilon \partial_t f \ ,$$

which after integration in time gives

$$\int_0^\infty u_\varepsilon(t,a)\varrho_\varepsilon(t,a)\, da - f(t) = \int_0^\infty u_{I,\varepsilon}(a)\varrho_\varepsilon(0,a)\, da - f(0) = 0 \ , \quad \forall t > 0 \ ,$$

implying (2.4).

Now we would like to confirm that z_ε given by (2.3) satisfies (1.1). Using (4.1) and (2.3), one checks that, if $t \geq \varepsilon a$,

$$\frac{z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a)}{\varepsilon} = \frac{1}{\varepsilon}\int_{t-\varepsilon a}^t g(\tilde{t})d\tilde{t} = \int_0^a g(t-\varepsilon\tilde{a})d\tilde{a} = u_\varepsilon(t,a),$$

and if $t \leq \varepsilon a$,

$$\begin{aligned}\frac{z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a)}{\varepsilon} &= \frac{1}{\varepsilon}\left(\int_0^t g(\tilde{t})d\tilde{t} + z_\varepsilon(0) - z_p(t-\varepsilon a)\right) \\ &= \int_0^{t/\varepsilon} g(t-\varepsilon\tilde{a})d\tilde{a} + u_{I,\varepsilon}(a - t/\varepsilon) = u_\varepsilon(t,a).\end{aligned}$$

This gives when evaluating the left hand side of (1.1) that

$$\int_{\mathbb{R}_+} \left(\frac{z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a)}{\varepsilon}\right)\varrho_\varepsilon(t,a)da = \int_{\mathbb{R}_+} u_\varepsilon(t,a)\varrho_\varepsilon(t,a)da,$$

which thanks to (2.4) proves that z_ε actually solves (1.1). □

In the rest of the paper we solve the coupled model for the tuple $(\varrho_\varepsilon, u_\varepsilon)$.

5. *A priori* estimates.

The uniform *a priori* estimates below will be very useful for the further analysis. They apply both to the semi and the fully coupled cases.

LEMMA 5.1. *If $\zeta_\varepsilon \geq 0$, then the solution of system (2.1) satisfies the ε-uniform estimate (2.6).*

Proof. Multiplying formally the equation by $\operatorname{sign}(u_\varepsilon)$, testing against ϱ_ε and integrating with respect to a gives

$$\varepsilon\partial_t \int_0^\infty \varrho_\varepsilon(t,a)|u_\varepsilon(t,a)|\, da + \int_0^\infty \varrho_\varepsilon\zeta_\varepsilon|u_\varepsilon|da \leq \varepsilon|\partial_t f| + \int_0^\infty \varrho_\varepsilon\zeta_\varepsilon|u_\varepsilon|da \ .$$

The rigorous proof follows the same steps as in the proof of Lemma 3.1 p. 493, [5] and is left to the reader. On both sides the same term $\int_0^\infty \zeta_\varepsilon \varrho_\varepsilon |u_\varepsilon| \, da$ appears and thus cancels. One concludes then directly the claim after integration in time. □

The following result refers to the typical profile in age of the function u_ε.

LEMMA 5.2. *Let ζ_ε be such that $0 \le \zeta_{\min} \le \zeta_\varepsilon \le \zeta_{\max}$, then on any fixed time interval $(0,T)$ the profile $a \mapsto u_\varepsilon(t,a)$ is at most linear in age. Indeed, if we suppose that there are two constants α_0 and α_1 such that*

– *the growth factor of the rhs is controlled,*

$$\alpha_1 \ge \frac{1}{\mu_{0,\min}} \left\{ \|\partial_t f\|_{L^\infty(0,T)} + \zeta_{\max} \left(\int_0^\infty \varrho_{I,\varepsilon} |u_{I,\varepsilon}(a)| da + \int_0^T |\partial_t f| ds \right) \right\} ,$$

– *as well as the Lipschitz constant of the past data,*

$$\alpha_1 \ge \left\| z_p' \right\|_{L^\infty(\mathbb{R}_-)} , \quad \alpha_0 \ge \frac{1}{\mu_{0,\min}} \left(\left\| z_p' \right\|_{L^\infty(\mathbb{R}_-)} \mu_{1,\varepsilon}(0) + f(0) \right) .$$

Then one has

$$|u_\varepsilon(t,a)| \le \alpha_0 + \alpha_1 a, \quad (t,a) \in (0,T) \times \mathbb{R}_+ .$$

Proof. Observe that

$$\begin{aligned} \varepsilon \partial_t(\alpha_0 + \alpha_1 a - |u_\varepsilon|) + \partial_a(\alpha_0 + \alpha_1 a - |u_\varepsilon|) \ge \\ \ge \alpha_1 - \frac{1}{\mu_{0,\min}} \left\{ |\partial_t f| + \int_0^\infty \zeta_\varepsilon \varrho_\varepsilon |u_\varepsilon| da \right\} \\ \ge \alpha_1 - \frac{1}{\mu_{0,\min}} \left\{ \|\partial_t f\|_{L^\infty(0,T)} + \zeta_{\max} \int_0^\infty \varrho_\varepsilon |u_\varepsilon| da \right\} . \end{aligned}$$

Then, using Lemma 5.1 one recovers that under the first assumption on α_1 one has

$$\varepsilon \partial_t(\alpha_0 + \alpha_1 a - |u_\varepsilon|) + \partial_a(\alpha_0 + \alpha_1 a - |u_\varepsilon|) \ge 0, \quad (t,a) \in (0,T) \times \mathbb{R}_+ .$$

Hence it holds that $\alpha_0 + \alpha_1 a - |u_\varepsilon| \ge 0$ along the characteristics of the transport operator $\varepsilon \partial_t + \partial_a$ provided this quantity is nonnegative at the boundaries where $a = 0$ or $t = 0$ respectively. When $a = 0$, it is straightforward to observe that $\alpha_0 - |u_\varepsilon| = \alpha_0 \ge 0$. On the other hand, when $t = 0$, the boundary term is $\alpha_0 + \alpha_1 a - |u_{I,\varepsilon}|$ and we need to estimate the initial datum,

$$\begin{aligned} |u_{I,\varepsilon}(a)| = \left| \frac{z_\varepsilon(0) - z_p(-\varepsilon a)}{\varepsilon} \right| \le \left| \frac{z_\varepsilon(0) - z_p(0)}{\varepsilon} \right| + \left| \frac{z_p(0) - z_p(-\varepsilon a)}{\varepsilon} \right| = \\ = \frac{1}{\mu_{0,\varepsilon}(0)} \left| f(0) + \frac{1}{\varepsilon} \int_0^\infty (z_p(-\varepsilon a) - z_p(0)) \, \varrho_\varepsilon(a,0) \, da \right| + \left| \frac{z_p(0) - z_p(-\varepsilon a)}{\varepsilon} \right| \le \\ \le \frac{1}{\mu_{0,\min}} \left(\left\| z_p' \right\|_{L^\infty(\mathbb{R}_-)} \mu_{1,\varepsilon}(0) + f(0) \right) + \left\| z_p' \right\|_{L^\infty(\mathbb{R}_-)} a \le \alpha_0 + \alpha_1 a . \end{aligned}$$

Due to the assumptions on α_0 and α_1 this ends the proof. □

LEMMA 5.3. *Under assumptions 2.4 or 2.5 , let $(\varrho_\varepsilon, u_\varepsilon)$ be the solution of problem (1.2)-(2.1), then z_ε given by formula (2.3) is a Lipschitz continuous function on any finite time interval $(0,T)$.*

Proof. The proof is a straightforward consequence of Lemmas 5.1 and 5.2. They provide an L^∞ bound uniform in ε on the rhs of (2.1). Thanks to formula (2.3) one gets directly the Lipschitz continuity of z_ε. □

REMARK 5.1. *Lemma 5.3 completes Lemma 4.2 and proves the equivalence between $(\varrho_\varepsilon, u_\varepsilon)$ solutions of system (1.2) and (2.1) belonging to $C([0,T]; L^1(\mathbb{R}_+)) \times C([0,T], L^\infty(\mathbb{R}_+,\omega))$ and $(\varrho_\varepsilon, z_\varepsilon)$, solving system (1.2)-(1.1) which are in $C([0,T]; L^1(\mathbb{R}_+)) \times \mathrm{Lip}([0,T])$. This results holds in both semi and fully coupled cases.*

6. Existence of solutions and convergence in the semi-coupled case. In this section we consider the semi-coupled case which consists of the equations (1.2) and (1.3) coupled to either (2.1) or (1.1) and prove Theorem 2.1. The framework for the analysis is the function space defined in (2.9) which relies on the weight function ω defined in (2.8). The following result is a straightforward consequence of the definition.

THEOREM 6.1. *Let the Assumptions 2.1, 2.2, 2.3 and 2.4 hold and let ϱ_ε be the unique solution of (1.2) according to Theorem 3.1, then for any fixed ε and any $T > 0$ there exists a unique $u_\varepsilon \in X_T$ solving problem (2.1). Moreover the maximal time of existence is infinite and stability results stated in Lemmas 5.1 and 5.2 hold.*

Proof. The proof follows by a fixed point argument. We define the mapping $\Phi(w) = u$ such that

$$u(t,a) := \begin{cases} \int_0^a h(t-\varepsilon\tilde{a})\, d\tilde{a}\ , & t > \varepsilon a, \\ \int_0^{t/\varepsilon} h(t-\varepsilon\tilde{a})\, d\tilde{a} + u_{I,\varepsilon}(a - t/\varepsilon)\ , & t \le \varepsilon a, \end{cases} \tag{6.1}$$

where $h(t) := (\varepsilon\partial_t f + \int_0^\infty \varrho_\varepsilon(t,a)\zeta_\varepsilon(t,a) w(t,a) da)/\mu_{0,\varepsilon}(t)$. A simple computation shows that

$$\|u\|_{X_T} \le \|h\|_{L^\infty(0,T)} \frac{T}{\varepsilon + T} + \|u_{I,\varepsilon}\|_{L^\infty(\mathbb{R}_+,\omega)}$$

which allows then with the specific definition of h to write:

$$\|u\|_{X_T} \le \frac{1}{\mu_{0,\min}} \varepsilon \|\partial_t f\|_{L^\infty(0,T)} + \zeta_{\max}\left(1 + \frac{k}{\mu_{0,\min}}\right) \|w\|_{X_{T_1}} + \|u_{I,\varepsilon}\|_{L^\infty(\mathbb{R}_+,\omega)}\ ,$$

where k is the constant from Lemma 3.2. This proves that Φ is an endomorphism for any given time T. By similar arguments one shows as well that if we set $u_i = \Phi(w_i)$ for $i \in \{1,2\}$, then it holds that

$$\|u_1 - u_2\|_{X_T} \le C_2 \frac{T}{\varepsilon} \|w_1 - w_2\|_{X_T}$$

for a constant $C_2 > 0$. Choosing then $T < \varepsilon/C_2$ proves local existence in time in the interval $[0,T]$ by the Banach-Picard fixed point theorem. As the contraction time does not depend on the initial data, we can extend the same result by continuation and existence and uniqueness in X_T follow for any $T > 0$. □

Next we obtain a weak convergence result for u_ε which in a second step implies the strong convergence of z_ε.

THEOREM 6.2. *Under the assumptions of Theorem 6.1 one has*

$$u_\varepsilon \rightharpoonup u_0 \ \text{weakly-* in } X_T \tag{6.2}$$

as $\varepsilon \to 0$, where u_0 satisfies

$$\begin{cases} \partial_a u_0 = \int_0^\infty \zeta_0 u_0 \varrho_0 \, da\ , & t \ge 0, \quad a > 0, \\ u_0(t,0) = 0 & t \ge 0, \quad a = 0. \end{cases} \tag{6.3}$$

and

$$\int_0^\infty u_0(t,a)\varrho_0(t,a)da = f(t)\ , \qquad a.e\ t\in(0,T). \tag{6.4}$$

Furthermore it also holds that

$$z_\varepsilon \to z_0 \text{ strongly in } L^\infty(0,T) \text{ as } \varepsilon\to 0\ ,$$

where $z_0 = z_p(0) + \int_0^t \frac{f(\tilde t)}{\mu_{1,0}(\tilde t)}\, d\tilde t$ *is the unique solution of* (1.5).

Proof. As already noticed in Remark 3.2, Theorem 3.2 and Lemma 3.4 imply that

$$(1+a)\,\varrho_\varepsilon \to (1+a)\,\varrho_0$$

in $L^1((0,T)\times\mathbb{R}_+)$ strongly. On the other hand, one has, by Lemma 5.2, that $\|u_\varepsilon\|_{X_T}\leq \max\{\alpha_0,\alpha_1\}$, which implies (6.2), i.e.

$$\frac{u_\varepsilon}{1+a} \rightharpoonup \frac{u_0}{1+a}$$

in $L^\infty((0,T)\times\mathbb{R}_+)$ in the weak-* sense for a limit function $u_0\in X_T$. As $\zeta_\varepsilon\to\zeta_0$ in $L^\infty((0,T)\times\mathbb{R}_+)$ by Assumption 2.4 one concludes that for every $\psi\in L^\infty((0,T)\times\mathbb{R}_+)$ one has

$$\int_0^T\int_0^\infty (\zeta_\varepsilon\, u_\varepsilon\, \varrho_\varepsilon\, \psi)\, da\, dt \to \int_0^T\int_0^\infty (\zeta_0\, u_0\, \varrho_0\, \psi)\, da\, dt\ .$$

Observe that this implies the weak convergence of $\int_0^\infty (\zeta_\varepsilon\, u_\varepsilon\, \varrho_\varepsilon\,)\, da$ in $L^1(0,T)$ since we might choose $\psi=\psi(t)$. Passing hence to the limit $\varepsilon\to 0$ in (4.2) we obtain that $u_0 = w(t)\, a$ for a function $w=w(t)$. Due to Lemma 4.2 it holds that $\int_0^\infty u_\varepsilon(t,a)\varrho_\varepsilon(t,a)da = f(t)$ for every $t\geq 0$. By analogous arguments one obtains the weak convergence of $\int_0^\infty u_\varepsilon\varrho_\varepsilon da$ to $\int_0^\infty u_0\varrho_0 da$ in $L^1(0,T)$. Hence one concludes that the limit satisfies the identity (6.4) which proves that $u_0 = w(t)a$ where $w(t) = f(t)/\mu_{1,0}(t)$.

A triangular inequality gives that

$$\begin{aligned}
&\left|\int_0^t \frac{1}{\mu_{0,\varepsilon}(\tilde t)}\int_0^\infty \zeta_\varepsilon\, \varrho_\varepsilon\, u_\varepsilon da\, d\tilde t - \int_0^t \frac{1}{\mu_{0,0}(\tilde t)}\int_0^\infty \zeta_0\, \varrho_0\, u_0\, da\, d\tilde t\right| \leq \\
&\qquad \leq \left(\int_0^t \left|\frac{1}{\mu_{0,\varepsilon}(\tilde t)} - \frac{1}{\mu_{0,0}(\tilde t)}\right| d\tilde t\right) \zeta_{\max} \left\|\int_0^\infty \varrho_\varepsilon\, |u_\varepsilon|\, da\right\|_{L^\infty(0,T)} + \\
&\qquad\qquad + \left|\int_0^t \frac{1}{\mu_{0,0}(\tilde t)}\int_0^\infty (\zeta_\varepsilon\, \varrho_\varepsilon\, u_\varepsilon - \zeta_0\varrho_0\, u_0)\, da\, d\tilde t\right|
\end{aligned}$$

where both terms on the right hand side tend to zero as $\varepsilon\to 0$ thanks to the weak convergence of $\int_0^\infty \zeta_\varepsilon\, u_\varepsilon\, \varrho_\varepsilon\, da$ in $L^1(0,T)$ combined with the strong convergence of $1/\mu_{0,\varepsilon}$ in $L^1(0,T)$ due to Theorem 3.2 and Lemma 3.1. This allows to pass to the limit in the third term in the right hand side of (2.3). Moreover, as z_p' is uniformly bounded an easy check gives that $z_\varepsilon(0)\to z_p(0)$ when ε goes to zero. These two facts prove that $z_\varepsilon\to z_0$ strongly in $L^\infty(0,T)$, z_0 solving:

$$\begin{aligned}
z_0(t) = z_p(0) + \int_0^t \frac{1}{\mu_{0,0}}\left(\int_0^\infty \zeta_0 u_0 \varrho_0\, da\right) d\tilde t = \\
= z_p(0) + \int_0^t \partial_a u_0\, d\tilde t = z_p(0) + \int_0^t \frac{f(\tilde t)}{\mu_{1,0}(\tilde t)}\, d\tilde t\ ,
\end{aligned}$$

which concludes the proof. □

Finally **Theorem 2.1** summarizes the results of the Theorems 3.2 and 6.2.

7. Existence of a unique solution in the fully coupled case. We prove the result stated in **Theorem 2.2** using the Banach Fixed Point theorem.

Proof. For an arbitrary time $T_0 > 0$, which we will determine in the end of the proof, let $A_0 := B_{X_{T_0}}(0, C_0)$ be the ball centered at the origin in X_{T_0} with radius C_0. We construct a mapping that given $w \in A_0$ defines the function $\varrho_\varepsilon \in C([0,T_0]; L^1(\mathbb{R}_+))$ as the solution of

$$\begin{cases} \varepsilon\partial_t\varrho_\varepsilon + \partial_a\varrho_\varepsilon + \zeta(w)\,\varrho_\varepsilon = 0\,, & t>0\,,\ a>0\,, \\ \varrho_\varepsilon(a=0,t) = \beta_\varepsilon(t)\left(1 - \int_0^\infty \varrho_\varepsilon(t,\tilde a)\,d\tilde a\right)\,, & t>0\,, \\ \varrho_\varepsilon(a,t=0) = \varrho_{I,\varepsilon}(a)\,, & a\geq 0\,. \end{cases} \tag{7.1}$$

Results from Theorem 3.1 and Lemmas 3.1 and 3.2 imply existence and uniqueness of a solution as well as uniform bounds for moments of order up to 2. Then w and ϱ_ε are used as the input in order to compute the function $u \in X_{T_0}$ solving the problem :

$$\begin{cases} \varepsilon\partial_t u + \partial_a u = \dfrac{1}{\mu_{0,\varepsilon}}\left(\varepsilon\partial_t f + \displaystyle\int_0^\infty \zeta(w)\,w\,\varrho_\varepsilon\,da\right)\,, & t>0\,,\ a>0\,, \\ u(t,0) = 0\,, & t>0\,, \\ u(0,a) = u_{I,\varepsilon}(a)\,, & a\geq 0\,. \end{cases} \tag{7.2}$$

By the same arguments as in the proof of Theorem 6.1, the solution $u \in X_{T_0}$ exists for any given function $w \in X_{T_0}$ and u can be controlled by the norm in X_{T_0},

$$\begin{aligned} \|u\|_{X_{T_0}} &\leq \left(\varepsilon\frac{\|\partial_t f\|_\infty}{\mu_{0,\min}} + \zeta_{\max}\left(1 + \frac{k}{\mu_{0,\min}}\right)\|w\|_{X_{T_0}}\right)\frac{T_0}{\varepsilon} + \|u_{I,\varepsilon}\|_{L^\infty_\omega} \\ &\leq (c_1 + c_2 C_0)\frac{T_0}{\varepsilon} + \|u_{I,\varepsilon}\|_{L^\infty_\omega}\,, \end{aligned} \tag{7.3}$$

where $c_1, c_2 > 0$ are constants defined by the preceding computation. Choosing the time T_0 such that $T_0 \leq t_0$ where $t_0 := \varepsilon(C_0 - \|u_{I,\varepsilon}\|_{L^\infty_\omega})/(c_1 + c_2 C_0)$, one then ensures that $u \in A_0$.

Next, we shall prove that this mapping is contractive in A_0. To this end, given two elements (w_1, w_2) and their respective images (u_1, u_2), we define $\hat u := u_2 - u_1$ and $\hat w = w_2 - w_1$. We define also the corresponding densities (ρ_1, ρ_2) and the respective zeroth and first order moments $(\mu_{0,1}, \mu_{0,2})$, ($\mu_{1,1}$ and $\mu_{1,2}$) as well as their differences $\hat\mu_i := \mu_{i,2} - \mu_{i,1}$, $i \in \{0,1\}$. It holds that

$$\hat u(t,a) = \begin{cases} \int_0^a \hat g(t - \varepsilon\tilde a)\,d\tilde a\,, & t > \varepsilon a\,, \\ \int_0^{t/\varepsilon} \hat g(t - \varepsilon\tilde a)\,d\tilde a\,, & t \leq \varepsilon a\,, \end{cases} \tag{7.4}$$

where

$$\hat g(t) = \varepsilon\partial_t f\left(\frac{1}{\mu_{0,2}} - \frac{1}{\mu_{0,1}}\right) + \int_0^\infty \left(\zeta(w_2)\frac{\rho_2}{\mu_{0,2}}w_2 - \zeta(w_1)\frac{\rho_1}{\mu_{0,1}}w_1\right)da\,.$$

We estimate

$$|\hat g(t)| \leq \frac{\varepsilon|\partial_t f|}{\mu_{0,1}\mu_{0,2}}\,|\hat\mu_0| + \sum_{i=1}^4 I_i\,,$$

and we detail the right hand side as follows,

$$
\begin{aligned}
I_1 &:= \int_0^\infty \zeta_{lip} |\hat{w}| \left| \frac{\rho_2}{\mu_{0,2}} w_2 \right| da \leq \zeta_{lip} \left(1 + \frac{3k}{\mu_{0,\min}}\right) \|\hat{w}\|_{X_{T_0}} \|w_2\|_{X_{T_0}} \ , \\
I_2 &:= \int_0^\infty |\hat{\rho}| \left| \zeta(w_1) \frac{1}{\mu_{0,2}} w_2 \right| da \leq \frac{\zeta_{\max}}{\mu_{0,\min}} \|w_2\|_{X_{T_0}} \int_0^\infty (1+a) |\hat{\rho}| \ da \ , \\
I_3 &:= \frac{1}{\mu_{0,1}\mu_{0,2}} |\hat{\mu}_0| \int_0^\infty |\zeta(w_1)\rho_1 w_2| \, da \\
&\leq \frac{\zeta_{\max}}{\mu_{0,\min}^2} \left(1 + \frac{k}{\mu_{0,\min}}\right) \|w_2\|_{X_{T_0}} \|\hat{\mu}_0\|_{L^\infty(0,T_0)} \ , \\
I_4 &:= \int_0^\infty |\hat{w}| \left| \zeta(w_1) \frac{\rho_1}{\mu_{0,1}} \right| da \leq \zeta_{\max} \left(1 + \frac{k}{\mu_{0,\min}}\right) \|\hat{w}\|_{X_{T_0}} \ .
\end{aligned}
$$

Using the same arguments as in the proof of Lemma 3.3 (see the proof of Lemma 3.3. p. 495 in [5]) we show that

$$
\begin{aligned}
\|\hat{\mu}_0\|_{L^\infty(0,T_0)} \leq \|\hat{\rho}\|_{L^\infty(0,T_0;L^1(\mathbb{R}))} &\leq \frac{2}{\zeta_{\min}} \left\| \int_0^\infty |\zeta(w_2) - \zeta(w_1)| \rho_2 \, da \right\|_{L^\infty(0,T_0)} \\
&\leq \frac{2\zeta_{lip}}{\zeta_{\min}} \left(1 + \frac{k}{\mu_{0,\min}}\right) \|\hat{w}\|_{X_{T_0}}
\end{aligned}
$$

and an analogous result to Lemma 3.2,

$$
\|\hat{\mu}_1\|_{L^\infty(0,T_0)} \leq \|a|\hat{\rho}|\|_{L^\infty(0,T_0;L^1(\mathbb{R}))} \leq C \|\hat{w}\|_{X_{T_0}}
$$

for a constant $C > 0$. Using these results we obtain

$$
\|u_1 - u_2\|_{X_{T_0}} \leq \frac{T_0}{\varepsilon} (c_3 + c_4 C_0) \|w_1 - w_2\|_{X_{T_0}} \ ,
$$

which is contractive provided that $T_0 < t_1$ where $t_1 := \varepsilon/(c_3 + c_4 C_0)$. Choosing for example $T_0 < \min(t_0, t_1)$ proves local existence of in time in the interval $[0, T_0]$ by the Banach-Picard fixed point theorem.

We extend that result to longer times by induction. We suppose that the solutions $(\varrho_\varepsilon, u_\varepsilon)$ solving (1.2)-(2.1)-(1.4) exist until the time T_n *i.e.* $(\varrho_\varepsilon, u_\varepsilon) \in C([0, T_n]; L^1(\mathbb{R}_+, (1+a)^2)) \times X_{T_n}$ and that one has the bound :

$$
\left\| (1+a)^2 \varrho_\varepsilon \right\|_{L^\infty((0,T_n);L^1(\mathbb{R}_+))} \leq k_1 (1 + \mu_{1,\varepsilon}(0) + \mu_{2,\varepsilon}(0)). \tag{7.5}
$$

where $k_1 := 2(1 + 1/\zeta_{\min} + 1/\zeta_{\min}^2)$. Then on the next interval $[T_n, T_{n+1}]$, one uses again a fixed point strategy. We set the mapping defined above by solving (7.1)-(7.2) on (T_n, T_{n+1}) with initial datum $u_\varepsilon(T_n, \cdot)$ for u_ε, and $\varrho_\varepsilon(T_n, \cdot)$ for ϱ_ε. We denote by $(\rho(w), u(w))$ the solutions of (7.1)-(7.2) on (T_n, T_{n+1}) for a given function $w \in X_{(T_n, T_{n+1})}$. Firstly, we prove using similar arguments as in Lemma 3.1 that

$$
\left\| (1+a)^2 \rho(w) \right\|_{L^\infty((T_n,T_{n+1});L^1(\mathbb{R}_+))} \leq k_1 (1 + \mu_{1,\varepsilon}(T_n) + \mu_{2,\varepsilon}(T_n))
$$

and similarly to Lemma 3.3 one has as well :

$$
\|(1+a)\hat{\rho}\|_{L^\infty((T_n,T_{n+1});L^1(\mathbb{R}_+))} \leq k_2 k_1 (1 + \mu_{1,\varepsilon}(T_n) + \mu_{2,\varepsilon}(T_n)) \|\hat{w}\|_{X_{(T_n,T_{n+1})}}
$$

where we denote $k_2 := (1/\zeta_{\min} + 1/\zeta_{\min}^2)$, $\hat{\rho} := \rho_2(w_2) - \rho_1(w_1)$, and $\hat{w} := w_2 - w_1$. Thanks to the induction hypothesis (7.5), these bounds can be estimated as

$$\left\|(1+a)^2 \rho(w)\right\|_{L^\infty((T_n,T_{n+1});L^1(\mathbb{R}_+))} \leq k_2'(1 + \mu_{1,\varepsilon}(0) + \mu_{2,\varepsilon}(0)) =: k_2'', \tag{7.6}$$

where $k_2' := k_1^2$ and

$$\begin{aligned} \|(1+a)\hat{\rho}\|_{L^\infty((T_n,T_{n+1});L^1(\mathbb{R}_+))} \leq & k_3'(1 + \mu_{1,\varepsilon}(0) + \mu_{2,\varepsilon}(0))\|\hat{w}\|_{X_{(T_n,T_{n+1})}} \\ =: & \; k_3'' \, \|\hat{w}\|_{X_{(T_n,T_{n+1})}}. \end{aligned}$$

where $k_3' := k_2 k_1^2$. The norm of the initial condition for $u(w)$ might indeed be larger then $\|u_{I,\varepsilon}\|_{L^\infty(\mathbb{R}_+,\omega)}$, at worst we may have attained the bound C_n during the previous periods and we might have to choose $C_{n+1} > C_n$. Rewriting (7.3) in $[T_n, T_{n+1})$ and denoting $\Delta T_n := T_{n+1} - T_n$, one has indeed

$$\begin{aligned} &\|u\|_{X_{(T_n,T_{n+1})}} \leq \\ &\leq \frac{1}{\mu_{0,\min}} \left(\varepsilon \|\partial_t f\|_\infty + \zeta_{\max} \left\| \int_{\mathbb{R}_+} (1+a)\varrho_\varepsilon(t,a) da \right\|_{L^\infty(T_n,T_{n+1})} \|w\|_{X_{(T_n,T_{n+1})}} \right) \frac{\Delta T_n}{\varepsilon} \\ &+ \|u_\varepsilon(T_n,\cdot)\|_{L^\infty_\omega} \\ &\leq \frac{1}{\mu_{0,\min}} \left(\varepsilon \|\partial_t f\|_\infty + \zeta_{\max} k_2'' \|w\|_{X_{(T_n,T_{n+1})}} \right) \frac{\Delta T_n}{\varepsilon} + C_n \\ &\leq (c_1 + c_2' C_{n+1}) \frac{\Delta T_n}{\varepsilon} + C_n \;, \end{aligned}$$

where we used (7.6). A similar computation gives for the contraction part that :

$$\|\hat{u}\|_{X_{(T_n,T_{n+1})}} \leq \left(\sum_{i=1}^5 J_i \right) \frac{\Delta T_n}{\varepsilon},$$

where

$$\begin{aligned} J_1 :=& \frac{\varepsilon}{\mu_{0,\min}^2} \|\partial_t f\|_{L^\infty(T_n,T_{n+1})} \|\hat{\mu}_0\|_{L^\infty(T_n,T_{n+1})} \leq \frac{\varepsilon k_3''}{\mu_{0,\min}^2} \|\partial_t f\|_{L^\infty(T_n,T_{n+1})} \|\hat{w}\|_{X_{(T_n,T_{n+1})}}, \\ J_2 :=& \frac{1}{\mu_{0,\min}^2} \sup_{(T_n,T_{n+1})} \left(\int_{\mathbb{R}_+} \zeta(w)|w|\varrho_\varepsilon da \; |\hat{\mu}_0| \right) \leq \frac{\zeta_{\max} k_2'' k_3''}{\mu_{0,\min}^2} \|w_2\|_{X_{(T_n,T_{n+1})}} \|\hat{w}\|_{X_{(T_n,T_{n+1})}}, \\ J_3 :=& \frac{1}{\mu_{0,\min}} \sup_{(T_n,T_{n+1})} \left(\int_{\mathbb{R}_+} |\hat{\zeta} w_2| \rho_2 da \right) \leq \frac{\zeta_{lip} k_2''}{\mu_{0,\min}} \|w_2\|_{X_{(T_n,T_{n+1})}} \|\hat{w}\|_{X_{(T_n,T_{n+1})}}, \\ J_4 :=& \frac{1}{\mu_{0,\min}} \sup_{(T_n,T_{n+1})} \left(\int_{\mathbb{R}_+} |\zeta_1 \hat{w}| \rho_2 da \right) \leq \frac{\zeta_{\max} k_2''}{\zeta_{\min}} \|\hat{w}\|_{X_{(T_n,T_{n+1})}}, \\ J_5 :=& \frac{1}{\mu_{0,\min}} \sup_{(T_n,T_{n+1})} \left(\int_{\mathbb{R}_+} \zeta_1 |w_1||\hat{\rho}| da \right) \leq \frac{\zeta_{\max} k_3''}{\zeta_{\min}} \|\hat{w}\|_{X_{(T_n,T_{n+1})}} \|w_1\|_{X_{(T_n,T_{n+1})}}, \end{aligned}$$

leading to

$$\|\hat{u}\|_{X_{(T_n,T_{n+1})}} \leq \left\{ (c_3' + c_4' C_{n+1}) \frac{\Delta T_n}{\varepsilon} \right\} \|\hat{w}\|_{X_{(T_n,T_{n+1})}}.$$

Existence and uniqueness of the extended solution on the interval $[0,T_{n+1}] = [0,T_n] \cup (T_n, T_{n+1}]$ hold provided that the period ΔT_n is chosen sufficiently small *i.e.*

$$\Delta T_n < \min \left\{ \frac{\varepsilon \Delta C_n}{c_1 + c_2' C_{n+1}}, \frac{\varepsilon}{c_3' + c_4' C_{n+1}} \right\} . \tag{7.7}$$

where $\Delta C_n = C_{n+1} - C_n$. The fixed point theorem provides a pair $(\varrho_\varepsilon, u_\varepsilon)$ defined on $C([0,T_{n+1}]; L^1(\mathbb{R}_+, (1+a)^2)) \times X_{T_{n+1}}$, then Lemmas 3.1 and 3.2 establish that $\mu_{0,\varepsilon}(T_{n+1}) \leq 1$ and $\mu_{1,\varepsilon}(T_{n+1}) \leq \mu_{1,\varepsilon}(0) + \zeta_{\min}^{-1}$. For $\mu_{2,\varepsilon}$, the second order moment, a similar estimate holds as well. This proves (7.5) up to T_{n+1}. The induction step is complete.

Thus, in an iterative way we are able to extend the solution up to periods $[T_n, T_{n+1})$ for any $n > 0$. We choose $C_n := 2\|u_{I,\varepsilon}\|_{L^\infty_\omega(\mathbb{R}_+)}(n+1)$. Since both series, $\frac{\varepsilon \Delta C_n}{c_1 + c_2' C_n}$ and $\frac{\varepsilon}{c_3' + c_4' C_n}$ are scaled versions of the divergent series $\sum_{n=0}^\infty \frac{1}{1+n}$, the periods ΔT_n can be chosen such that $T_n \to \infty$ as n grows large, which finishes the proof. ∎

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