FRICTION MEDIATED BY TRANSIENT ELASTIC LINKAGES : ASYMPTOTIC EXPANSIONS AND FAT TAILS*

SAMAR ALLOUCH† AND VUK MILISIC‡

Abstract. We construct an asymptotic expansion for the integral delay operator already introduced in [7, 8] and show how to improve convergence rates already obtained in the latter papers. Moreover, we weaken one of the major hypotheses made on the off-rates in our previous works : we do not assume exponential decay of the linkages' density. Instead, polynomial decrease is allowed, leading to stronger adhesions and slower motions of adhesion sites.

Key words. integral equation, renewal problem, load bounded variation, comparison principle, Volterra equation, singular perturbation problem

MSC codes. 35B40, 45D05

1. Introduction.

This work is a continuation of a series of works related to the mathematical study of adhesion forces in the context of cell motility (see [1], [7], [8], [9] and [10]). Cell adhesion and migration play a crucial role in many biological phenomena such as embryonic development, inflammatory responses, wound healing and tumor metastasis. The adhesion model discussed further has been designed at the scale of a single binding site [13, Chapter 5]. Lately, it has also been used at the mesoscopic cell scale ([3], [11]). The cell is modelled as a point particle whose position on the real line is denoted X_ε and depends on t. The position X_ε is obtained solving a force balance equation

$$
\begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_{\mathbb{R}_+} \left(X_\varepsilon(t) - X_\varepsilon(t-\varepsilon a)\right) \rho_\varepsilon(a,t)da = f(t), & t > 0, \\ X_\varepsilon(t) = X_p(t), & t \leq 0, \end{cases} \tag{1.1}
$$

where f is an external force and the left hand side a continuum of elastic spring forces with respect to past positions. The density of linkages ρ_ε is either a given function or it can also be a solution of an age-structured model [12, 14] :

$$
\begin{cases} \left(\varepsilon\partial_t + \partial_a + \zeta(a,t)\right)\rho_\varepsilon(a,t) = 0, & (a,t) \in \mathbb{R}_+ \times (0,T), \\ \rho_\varepsilon(0,t) = \beta(t)\left(1 - \displaystyle\int_{\mathbb{R}_+} \rho_\varepsilon(a,t)da\right), & (a,t) \in \{0\} \times (0,T), \\ \rho_\varepsilon(a,0) = \rho_I(a), & (a,t) \in \mathbb{R}_+ \times \{0\}. \end{cases} \tag{1.2}
$$

In the latter system, $\beta \in \mathbb{R}_+$ (resp. $\zeta \in \mathbb{R}_+$) is the kinetic on-rate (resp. off-rate) function and the speed of linkage turnover is represented by the small parameter known as $\varepsilon > 0$ ([7], [8], and [9]). Under the assumption that the death rate ζ admits a strictly positive lower bound $\zeta_{\min}$, in [7], the authors studied rigorously the

*Submitted to the editors September 29, 2022

Funding: This work was funded by CoFund H2020-MSCA-COFUND-2019 Grant agreement 945332

†Laboratoire Analyse, Géométrie & Applications (LAGA), Université Paris 13 (allouch@math.univ-paris13.fr)

‡Laboratoire de Mathématiques de Bretagne Atlantique, Université Bretagne Occidentale, Brest, France, (vuk.milisic@univ-brest.fr,https://milisic.perso.math.cnrs.fr/)

asymptotic limit of the systems (1.1) and (1.2) when ε goes to zero. They obtained the convergence results

$$\|X_\varepsilon - X_0\|_{C^0([0,T])} + \|\rho_\varepsilon - \rho_0\|_{C^0((0,T];L^1(\mathbb{R}_+))} \to 0,$$

where the limit X_0 solves

$$\begin{cases} \mu_{1,0}(t)X_0' = f(t), & t > 0, \\ X_0(t=0) = X_p(0), & t = 0, \end{cases} \tag{1.3}$$

where $\mu_{1,0}(t)$ denotes the first moment of ρ_0, namely $\mu_{1,0}(t) := \int_0^\infty a\rho_0(a,t)da$ and ρ_0 satisfies :

$$\begin{cases} \partial_a \rho_0 + \zeta(a,t)\rho_0 = 0, & t > 0, a > 0, \\ \rho_0(a=0,t) = \beta(t)\left(1 - \int_0^\infty \rho_0(t,\tilde{a})\ d\tilde{a}\right), & t > 0. \end{cases} \tag{1.4}$$

We underline that, to some extent, the ε scaling can be associated with the long time behavior of solutions of (1.1) and (1.2) [11, Theorem 4.4].

In [7],using the Lyapunov functional

$$\mathcal{H}[u] := \left| \int_0^\infty u(a)da \right| + \int_0^\infty |u(a)|da\ , \tag{1.5}$$

the authors have proved the convergence of ρ_ε towards ρ_0. In the same article they showed as well the convergence of the position X_ε thanks to a comparison principle specific to Volterra equations [4, Chapter 9, Section 8]. Here as well, the main goal is to study the asymptotic behavior of solutions of the coupled problem (1.1)-(1.2) under two major constrains :

1) increase the order of approximation with respect to ε,
2) weaken hypotheses on ζ allowing ρ_ε and ρ_0 to have *fat tails* with respect to the age variable.

More precisely, we aim at constructing the N^{th}-order asymptotic approximation of the solution X_ε satisfying (1.1) as

$$\tilde{X}_{\varepsilon,N} = X_{outer}(t) + X_{inner}(\tau) + O(\varepsilon^N), \tag{1.6}$$

where $\tau = t/\varepsilon$ is the stretched variable. The construction of this asymptotic development is done in two steps. First, we construct X_{outer} containing a series of macroscopic correctors in power of ε. These correctors are valid away from $t = 0$. Next, we construct X_{inner} containing microscopic correctors that correct the fast variation near the boundary layer at $t = 0$.

We start by studying problem (1.1) with a given non-negative density of linkages ρ such that

$$\mu_{N+1} := \int_0^{\mathbb{R}_+} (1+a)^{N+1} \varrho(a)da < +\infty$$

and such that it is not compactly supported (cf. Hypotheses 1 for a more precise definition). In this case, we construct an expansion (1.6) for any fixed integer N. The macroscopic part X_{outer} consists of correctors solving first order differential equations (see (3.7)). Initial conditions of these correctors are then defined by matching inner and outer expansion. An error estimate is obtained in Theorem 2 leading to

$$\left\| X_\varepsilon - \tilde{X}_{\varepsilon,N} \right\|_{C([0,T])} \lesssim \varepsilon^N.$$

Finally, we consider the general case where the density also depends on ε and solves (1.2). We are mainly concerned with the asymptotic behavior of X_ε as the perturbation parameter ε approaches zero. First, we study the asymptotic behavior of ρ_ε when ε goes to 0. The novelty here, compared to [7], is that we weakened the assumptions on the death rate ζ. Namely we assume that there exists a non-increasing function $m \in L^1(\mathbb{R}_+; (1+a)^3)$ such that

$$\zeta(a,t) \geq -\frac{m'(a)}{m(a)}, \quad \text{a.e. } a \in \mathbb{R}_+ \tag{1.7}$$

This hypothesis allows ζ to go to zero for large a and allows ϱ to have fat tails. In comparison, in [7], the hypothesis on ζ was stronger : $\zeta(a,t) \geq \zeta_{\min} > 0$. This allowed to use of Gronwall's Lemma and get *a priori* estimates :

$$\|\rho_\varepsilon(\cdot,t) - \rho_0(\cdot,t)\|_{L^1(\mathbb{R}_+)} \leq \mathcal{H}[\rho_I(\cdot) - \rho_0(\cdot,0)] \exp\left(-\zeta_{\min} t/\varepsilon\right) + o_\varepsilon(1).$$

Showing an exponential decay in time and age of the initial layer near to $t = 0$ [6]. In our case, if ζ satisfies the condition (1.7), we cannot use Gronwall's Lemma to establish the convergence when ε tends to 0. For this sake, we enrich the asymptotic expansion of ρ_ε with supplementary terms. We introduce ρ_1, the first order macroscopic solution of :

$$\begin{cases} \left(\partial_a + \zeta(a,t)\right) \rho_1(a,t) = -\partial_t \rho_0(a,t), & a > 0,\ t > 0, \\ \rho_1(0,t) = -\beta(t) \displaystyle\int_{\mathbb{R}_+} \rho_1(a,t) da, & a = 0,\ t > 0, \end{cases} \tag{1.8}$$

and $\mathfrak{r}_0$ the initial layer approximation solving :

$$\begin{cases} \left(\partial_t + \partial_a + \zeta(a,0)\right) \mathfrak{r}_0(a,t) = 0, & a > 0,\ t > 0, \\ \mathfrak{r}_0(0,t) = -\beta(0) \displaystyle\int_{\mathbb{R}_+} \mathfrak{r}_0(a,t) da, & a = 0,\ t > 0, \\ \mathfrak{r}_0(a,0) = \rho_I(a) - \rho_0(a,0), & a > 0,\ t = 0. \end{cases} \tag{1.9}$$

This enhances the earlier error estimates. Indeed, for any $t > 0$, one has

$$\mathcal{H}[\rho_\varepsilon(\cdot,t) - \rho_0(\cdot,t) - \varepsilon\rho_1(\cdot,t) - \mathfrak{r}_0(\cdot,t/\varepsilon)] \lesssim o_\varepsilon(1),$$

leading to :

$$\|\rho_\varepsilon - \rho_0\|_{L^1(\mathbb{R}_+ \times (0,T),(1+a)))} \leq o_\varepsilon(1).$$

In turn this result is used to establish the strong convergence of X_ε towards X_0 solving (1.3) extending the results from [8] to this more general framework.

The paper's outline is structured as follows : in section 2, we list notations which will be used throughout this paper. In section 3, we analyze the case where the kernel in (1.1) is fixed and depends on the age variable. Moreover, we construct the asymptotic expansion of X_ε and show error estimates. Finally, in section 4, we analyze the asymptotic behavior of ρ_ε and X_ε solutions of (1.2) and (1.1) respectively when ε tends to 0. Then we extend results from [8] to our setting and conclude.

2. Notations. Before presenting our main result, we list some notations and assumptions that will be used in this paper. In the rest of the paper, we'll use some notations for the functional spaces, for instance $L^p_t L^q_a := L^p((0,T); L^q(\mathbb{R}_+))$ for any real $(p,q) \in [1,\infty]^2$. We refer to [15] for a general framework for BV functions (used in Section 4), our notations being coherent with this reference.

Hereafter, in the following sections, capital letters $(X_i)_{i\in\mathbb{N}}$ denote the macroscopic correctors defined on $[0,T]$, and the microscopic correctors $(x_{i,j})_{(i,j)\in\mathbb{N}^2}$ or $(w_k)_{k\in\mathbb{N}}$ are defined on $\mathbb{R}$. They are then renamed with a tilde when rescaled with respect to ε : $\tilde{x}_{i,j}(t) := x_{i,j}(t/\varepsilon)$ for $t \in (0,T)$ and $(i,j) \in \mathbb{N}^2$.

3. The linkages' density is constant in time. In this section, we begin to study the simple model of the problem (1.1) with a kernel ρ constant in time. We assume that the data of the problem satisfies : the following assumptions :

ASSUMPTIONS 1. *Assume that :*
i) the source term is such that $f \in C^N(\mathbb{R}_+)$.
ii) the past condition $X_p \in C^{N+1}(\mathbb{R}_+)$.
iii) for all $a \in \mathbb{R}_+$, there exists $M \subset (a,\infty)$, M compact and $|M| > 0$ such that $\varrho(\tilde{a}) > 0$ for almost every $\tilde{a} \in M$.
iv) moreover

$$\mu_{N+1} := \int_{\mathbb{R}_+} (1+a)^{N+1} \varrho(a) da < \infty.$$

3.1. Construction of the expansion. First, we start with the construction of the terms forming the N^{th}-order approximation of X_ε solution of (1.1) for a fixed kernel as

$$\tilde{X}_{\varepsilon,N} := \underbrace{\sum_{i=0}^{N-1} \varepsilon^i X_i(t)}_{\text{outer expansion}} + \underbrace{Y_N(t) + Z_N(t) + W_N(t)}_{\text{inner expansion}}, \tag{3.1}$$

where these terms are set later on. Define the operator $\mathcal{L}_\varepsilon : C([0,T]) \to C([0,T])$ that maps X to

$$\mathcal{L}_\varepsilon[X](t) := \frac{1}{\varepsilon}\left\{ \mu_0 X(t) - \int_0^{\frac{t}{\varepsilon}} X(t-\varepsilon a)\varrho(a)da \right\}.$$

Then problem (1.1) can be rephrased as :

$$\mathcal{L}_\varepsilon[X_\varepsilon](t) = f(t) + \frac{1}{\varepsilon}\int_{\frac{t}{\varepsilon}}^{\infty} X_p(t-\varepsilon a)\varrho(a)da, \quad \forall t > 0 \tag{3.2}$$

and we aim at constructing $\tilde{X}_{\varepsilon,N}$ such that it satisfies

$$\mathcal{L}_\varepsilon[\tilde{X}_{\varepsilon,N}] = f(t) + \frac{1}{\varepsilon}\int_{\frac{t}{\varepsilon}}^{\infty} X_p(t-\varepsilon a)\varrho(a)da + O(\varepsilon^N). \tag{3.3}$$

PROPOSITION 1. *Assume that Hypotheses 1 hold, let the sequence of functions $(X_i)_{i\in\{0,\dots,N-1\}}$ be given and for all $i \in \{0,\dots,N-1\}$ assume that $X_i \in W^{N+1,\infty}$*

$([0,T])$, *then one has the expansion :*

$$\mathcal{L}_\varepsilon[X_i](t) = \sum_{k=1}^{N-i} \frac{\varepsilon^{k-1}(-1)^{k+1}}{k!} X_i^{(k)}(t)\left(\mu_k - \Xi_{0,k}(t)\right) + \frac{1}{\varepsilon}\int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a)da X_i(t) \\ + \varepsilon^{N-i}\mathcal{R}_i^{N+1-i},$$

where

$$\mu_k := \int_{\mathbb{R}_+} a^k \varrho(a)da, \quad \Xi_{0,k}(t) := \int_{\frac{t}{\varepsilon}}^{\infty} a^k \varrho(a)da =: \xi_{0,k}\left(\frac{t}{\varepsilon}\right),$$

and the rest can be controlled :

$$\left|\mathcal{R}_i^{N+1-i}\right| \leq \frac{1}{(N+1-i)!}\left\|X_i^{(N+1-i)}\right\|_\infty \int_{\mathbb{R}_+} a^{N+1-i}\varrho(a)da,$$

Proof. One writes :

$$\mathcal{L}_\varepsilon[X_i](t) = \frac{1}{\varepsilon}\int_0^{\frac{t}{\varepsilon}} (X_i(t) - X_i(t-\varepsilon a))\varrho(a)da + \frac{1}{\varepsilon}\int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a)da\ X_i(t)$$

then using the Taylor expansion :

$$X_i(t-\varepsilon a) = \sum_{k=0}^{N-i} \frac{\varepsilon^k a^k}{k!}(-1)^k X_i^{(k)}(t) \\ + \frac{(-\varepsilon a)^{N+1-i}}{(N-i)!}\int_0^1 X_i^{(N+1-i)}(t - s\varepsilon a)(1-s)^{N-i}ds$$

so that the first term above becomes :

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^{\frac{t}{\varepsilon}} \left(X_i(t) - X_i(t-\varepsilon a)\right)\varrho(a)da \\
&\quad= \sum_{k=1}^{N-i}\int_0^{\frac{t}{\varepsilon}} \frac{\varepsilon^{k-1}a^k}{k!}\varrho(a)da(-1)^{k+1}X_i^{(k)}(t) + \varepsilon^{N-i}\mathcal{R}_i^{N+1-i} \\
&\quad= \sum_{k=1}^{N-i} \frac{(-1)^{k+1}}{k!}X_i^{(k)}(t)\left(\varepsilon^{k-1}\mu_k - \frac{1}{\varepsilon}\int_{\frac{t}{\varepsilon}}^{\infty}(\varepsilon a)^k\varrho(a)da\right) + \varepsilon^{N-i}\mathcal{R}_i^{N+1-i}
\end{aligned}$$

which provides the result. □

PROPOSITION 2 (Outer expansions).
Under the same hypothesis as above, the zeroth-order macroscopic limit is given by

$$\mu_1 X_0^{(1)} = f, \tag{3.4}$$

and at any order $\ell \in \{1,\dots,N\}$, we have :

$$\mu_1 X'_{\ell-1} = \sum_{k=2}^{\ell}(-1)^k\frac{\mu_k}{k!}X_{\ell-k}^{(k)}. \tag{3.5}$$

Proof. The result proved in Proposition 1 leads to :

$$
\begin{aligned}
\mathcal{L}_\varepsilon\left[\sum_{i=0}^{N-1}\varepsilon^i X_i\right] &= \sum_{i=0}^{N-1}\varepsilon^i\sum_{k=1}^{N-i}\frac{\varepsilon^{k-1}(-1)^{k+1}}{k!}X_i^{(k)}(t)\left(\mu_k-\Xi_{0,k}(t)\right)\\
&\quad+\sum_{i=0}^{N-1}\varepsilon^{i-1}\int_{\frac{t}{\varepsilon}}^{\infty}\varrho(a)da\ X_i(t)+S_{N,0},
\end{aligned}
\tag{3.6}
$$

where we set $S_{N,0}:=\varepsilon^N\sum_{i=0}^{N-1}\mathcal{R}_i^{N+1-i}$ and $|S_{N,0}|\leq\max_{i\in\{0,\dots,N-1\}}\Big\{\|X_i\|_{W^{N+1,\infty}(0,T)}$ $\mu_{N+1-i}\}$. Considering the first sum gives :

$$
\begin{aligned}
\sum_{i=0}^{N-1}\varepsilon^i\sum_{k=1}^{N-i}&\frac{\varepsilon^{k-1}(-1)^{k+1}}{k!}X_i^{(k)}(t)\left(\mu_k-\Xi_{0,k}(t)\right)\\
&=\sum_{k=1}^{N}\sum_{\ell=k}^{N}\varepsilon^{\ell-1}\frac{(-1)^{k+1}}{k!}X_{\ell-k}^{(k)}(t)\left(\mu_k-\Xi_{0,k}(t)\right)\\
&=\sum_{\ell=1}^{N}\varepsilon^{\ell-1}\sum_{k=1}^{\ell}\frac{(-1)^{k+1}}{k!}X_{\ell-k}^{(k)}(t)\left(\mu_k-\Xi_{0,k}(t)\right)
\end{aligned}
$$

Separating powers of ε and considering that terms containing functions $\Xi_{0,k}$ belong to the initial layer (these depend only on the microscopic variable t/ε) provides :

$$
\sum_{k=1}^{\ell}\mu_k\frac{X_{\ell-k}^{(k)}(t)}{k!}(-1)^{k+1}=\begin{cases}0 & \text{if } \ell\neq 1,\\ f & \text{otherwise,}\end{cases}
\tag{3.7}
$$

and by relating the lowest derivative with the highest index to the rest of the correctors, we establish macroscopic nested ODEs (3.4) and (3.5). ∎

Remark 3.1. The initial conditions of the macroscopic correctors X_i are to be defined later (cf Theorem 1).

PROPOSITION 3 (Inner expansion). *It is threefold.*

- *The first part accounts for terms containing $\Xi_{0,k}$ in the first sum of (3.6) :*

$$
Y_N(t):=\sum_{m=1}^{N}\varepsilon^m\sum_{q=1}^{m}\sum_{k=1}^{q}\frac{(-1)^{k+1}}{k!(m-q)!}X_{q-k}^{(k+m-q)}(0)\tilde{x}_{m-q,k}(t)
\tag{3.8}
$$

  where $\tilde{x}_{j,k}:=x_{j,k}(t/\varepsilon)$ and the microscopic correctors solve :

$$
\mathcal{L}_1[x_{j,k}](t)=\xi_{j,k}(t):=t^j\int_t^\infty a^k\varrho(a)da,
\tag{3.9}
$$

  and $\mathcal{L}_1$ is the operator $\mathcal{L}_\varepsilon$ taken for ε set to 1.
- *The second part corrects the second sum in (3.6) and reads :*

$$
Z_N(t):=-\sum_{m=1}^{N}\varepsilon^m\sum_{q=0}^{m-1}\frac{1}{(m-q)!}X_q^{(m-q)}(0)\tilde{x}_{m-q,0}-\sum_{i=0}^{N-1}\varepsilon^i X_i(0).
\tag{3.10}
$$

- *The last part concerns the remainders related to the past source term in* (3.3)

$$W_N(t) := \sum_{i=0}^{N} \frac{\varepsilon^i}{i!} X_p^{(i)}(0)\tilde{w}_i(t),$$

where $\tilde{w}_i(t) := w_i(t/\varepsilon)$ *and* $(w_\ell)_\ell$ *solve for* $\ell \in \mathbb{N}$,

$$\begin{cases} \int_{\mathbb{R}_+} (w_\ell(t) - w_\ell(t-a))\varrho(a)da = 0, & t > 0, \\ w_\ell(t) = t^\ell, & t \leq 0. \end{cases} \tag{3.11}$$

Proof. First, we begin by constructing the first part of the initial layer Y_N. We consider the second term in (3.6) and we use Taylor's expansion :

$$X_{\ell-k}^{(k)}(t) = \sum_{j=0}^{N-k} X_{\ell-k}^{(k+j)}(0)\frac{t^j}{j!} + \varepsilon^N \mathcal{R}_{k,l}^N,$$

where

$$\mathcal{R}_{k,l}^N := \frac{\varepsilon^{-k+1}}{(N-k)!}\left(\frac{t}{\varepsilon}\right)^{N-k+1} \int_0^1 (1-s)^{N-k} X_{\ell-k}^{(N-k+1)}(st)ds,$$

which implies that

$$\begin{aligned} &\sum_{\ell=1}^{N} \varepsilon^{\ell-1} \sum_{k=1}^{\ell} \frac{(-1)^k}{k!} X_{\ell-k}^{(k)}(t)\Xi_{0,k}(t) \\ &= \sum_{\ell=1}^{N} \varepsilon^{\ell-1} \sum_{k=1}^{\ell} \frac{(-1)^k}{k!}\Xi_{0,k}(t)\left\{\sum_{j=0}^{N-k} X_{\ell-k}^{(k+j)}(0)\frac{t^j}{j!} + \mathcal{R}_{k,l}^N\right\} \\ &= \sum_{\ell=1}^{N}\sum_{k=1}^{\ell}\sum_{j=0}^{N-k} \frac{(-1)^k}{k!j!}\varepsilon^{\ell+j-1} X_{\ell-k}^{(k+j)}(0)\left(\frac{t}{\varepsilon}\right)^j \Xi_{0,k}(t) + S_{N,1} =: I + S_{N,1} \end{aligned} \tag{3.12}$$

where

$$S_{N,1} := \sum_{\ell,k} \varepsilon^{\ell+N-k}\frac{(-1)^k}{k!}\Xi_{0,k}(t)\mathcal{R}_{k,l}^N, \tag{3.13}$$

that can be estimated as :

$$|S_{N,1}| \leq C\varepsilon^N \mu_{N+1} \tag{3.14}$$

The first triple sum can be decomposed thanks to Proposition 6 as

$$\begin{aligned} I &:= \sum_{m=1}^{N} \varepsilon^{m-1}\left(\sum_{q=1}^{m}\sum_{k=1}^{q} \frac{(-1)^k}{k!(m-q)!} X_{q-k}^{(k+m-q)}(0)\Xi_{m-q,k}(t)\right) \\ &\quad + \sum_{m=N+1}^{2N-1} \varepsilon^{m-1}\left(\sum_{q=m+1-N}^{N}\sum_{k=1}^{q+N-m} \frac{(-1)^k}{k!(m-q)!} X_{q-k}^{(k+m-q)}(0)\Xi_{m-q,k}(t)\right) \\ &\quad =: I_1 + O(\varepsilon^N) \end{aligned}$$

In order to compensate I_1, we define microscopic correctors $\tilde{x}_{j,k}$ as (3.9) and set Y_N as in (3.8). Now, we need to correct the third term in (3.6), which we do with the same technique as above :

$$\begin{aligned}
&\sum_{i=0}^{N-1} \varepsilon^{i-1} X_i(t) \int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a)da \\
&= \sum_{i=0}^{N-1} \varepsilon^{i-1} \left\{ \sum_{j=0}^{N-i} \frac{t^j}{j!} X_i^{(j)}(0) + \frac{t^{N+1-i}}{(N-i)!} \int_0^1 X_i^{N+1-i}(st)(1-s)^{N-i} ds \right\} \int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a)da \\
&= \sum_{i=0}^{N-1} \sum_{j=0}^{N-i} \frac{\varepsilon^{i+j-1}}{j!} X_i^{(j)}(0) \frac{t^j}{\varepsilon^j} \Xi_{0,0}(t) + S_{N,2} = \sum_{i=0}^{N-1} \sum_{j=0}^{N-i} \frac{\varepsilon^{i+j-1}}{j!} \Xi_{j,0}(t) X_i^{(j)}(0) + S_{N,2} \\
&= \sum_{m=1}^{N} \varepsilon^{m-1} \sum_{q=0}^{m-1} \frac{1}{(m-q)!} \Xi_{m-q,0}(t) X_q^{(m-q)}(0) + \sum_{i=0}^{N-1} \varepsilon^{i-1} \Xi_{0,0}(t) X_i(0) + S_{N,2}
\end{aligned}$$

where $\Xi_{j,0}(t) := \xi_{j,0}(t/\varepsilon)$ and

$$S_{N,2}(t) := \sum_{i=0}^{N-1} \varepsilon^{i-1} \frac{t^{N+1-i}}{N-i!} \int_0^1 X_i^{N+1-i}(st)(1-s)^{N-i} ds \Xi_{0,0}(t), \tag{3.15}$$

and one has :

$$|S_{N,2}| \leq C\varepsilon^N \int_{\mathbb{R}_+} (1+a)^{N+1} \varrho(a)da \sup_{i\in\{0,\dots,N\}} \|X_i\|_{W^{N+1-i,\infty}(0,t)}. \tag{3.16}$$

It suffices then to add the correction Z_N defined as in (3.10). Lastly, it remains to correct the terms of the past. To find them, we need to develop $X_p(t-\varepsilon a)$ around 0, which is stated as :

$$\begin{aligned}
X_p(t-\varepsilon a) = &\sum_{i=0}^{N} \varepsilon^i X_p^{(i)}(0) \frac{(t/\varepsilon - a)^i}{i!} \\
&+ \varepsilon^{N+1} \frac{(t/\varepsilon - a)^{N+1}}{N!} \int_0^1 X_p^{(N+1)}(s(t-\varepsilon a))(1-s)^N ds,
\end{aligned}$$

and it involves

$$\int_{\frac{t}{\varepsilon}}^{+\infty} X_p(t-\varepsilon a)\varrho(a)da = \sum_{i=0}^{N} \varepsilon^i X_p^{(i)}(0) \int_{\frac{t}{\varepsilon}}^{+\infty} \frac{(t/\varepsilon - a)^i}{i!} \varrho(a)da + S_{N,3},$$

where

$$S_{N,3} := \varepsilon^{N+1} \int_{\frac{t}{\varepsilon}}^{+\infty} \frac{(t/\varepsilon - a)^{N+1}}{N!} \int_0^1 X_p^{(N+1)}(s(t-\varepsilon a))(1-s)^N ds \varrho(a)da, \tag{3.17}$$

and

$$\begin{aligned}
|S_{N,3}| &\leq \varepsilon^N \frac{1}{N!} \|X_p^{(N)}\|_\infty \int_{\frac{t}{\varepsilon}}^{+\infty} \left(\frac{t}{\varepsilon} - a\right)^N \varrho(a)da \\
&\leq \varepsilon^N \frac{1}{N!} \|X_p^{(N)}\|_\infty \sum_{k=0}^{N+1} C_k^{N+1} \int_{\frac{t}{\varepsilon}}^{+\infty} \left(\frac{t}{\varepsilon}\right)^k (-a)^{N-k+1} \varrho(a)da \leq c\varepsilon^{N+1}\mu_{N+1}
\end{aligned} \tag{3.18}$$

Which then yields that past-correctors should be added as :

$$W_N(t) := \sum_{i=0}^{N} \varepsilon^i \frac{X_p^{(i)}(0)}{i!} \tilde{w}_i(t),$$

where $\tilde{w_i}(t) := w_i(t/\varepsilon)$ and w_i satisfies (3.11). ∎

LEMMA 1. *If $\mu_{j+k+1} < \infty$, then one has :*

$$x_{j,k}(0) = \begin{cases} \frac{\mu_k}{\mu_0}, & \text{if } j = 0, \\ 0, & \text{otherwise}, \end{cases}$$

and $x_{j,k}(t) \to \mu_{j+1+k}/((j+1)\mu_1)$ when $t \to \infty$.

Proof. The resolvent associated to (3.9), satisfies :

$$r(t) - (r \star k)(t) = k(t),$$

where

$$k(a) := \varrho(a)/\mu_0, \quad (r \star k)(t) := \int_0^t r(t-\tau)k(\tau)d\tau \tag{3.19}$$

and it can be decomposed [5, Theorem 7.4.1, p.201] as

$$r(t) = \frac{\mu_0}{\mu_1} + \gamma(t),$$

where the function $\gamma \in L^1(\mathbb{R}_+)$. Moreover, the resolvent being defined the solution $x_{j,k}$ is computed explicitly and reads :

$$x_{j,k} = \xi_{j,k} + \xi_{j,k} \star r = \xi_{j,k} + \xi_{j,k} \star (\mu_0/\mu_1 + \gamma) = \xi_{j,k} + \xi_{j,k} \star \gamma + \frac{\mu_0}{\mu_1} \int_0^t \xi_{j,k}(s)ds.$$

Thus the leading term in $x_{j,k}$ when t grows large is the last integral. Indeed

$$\frac{\mu_0}{\mu_1} \int_{\mathbb{R}_+} t^j \int_t^\infty a^k k(a) da dt = \frac{1}{\mu_1} \int_{\mathbb{R}_+} \left(\int_0^a t^j dt \right) a^k \varrho(a) da = \frac{1}{(j+1)} \frac{\mu_{j+k+1}}{\mu_1}$$

and one has :

$$x_{j,k} - \frac{1}{(j+1)} \frac{\mu_{j+k+1}}{\mu_1} \in L^1(\mathbb{R}_+).$$

which we define as the formal expression $x_{j,k}(t) \to \mu_{j+1+k}/((j+1)\mu_1)$ when $t \to \infty$. ∎

LEMMA 2. *Under the same assumptions as in the previous Lemma, the microscopic functions w_ℓ are discontinuous at $t = 0$, for all $\ell \geq 0$:*

$$w_\ell(0^+) = (-1)^\ell \frac{\mu_\ell}{\mu_0}, \quad w_\ell(0^-) = 0,$$

and and $w_\ell(t) \to ((-1)^\ell \mu_{\ell+1})/((\ell+1)\mu_1)$ when $t \to \infty$.

Proof. Using (3.11), we can easily show the discontinuity of the correctors w_ℓ at $t = 0$. By the same arguments as proof of Lemma 1, one has that :

$$\begin{aligned} \lim_{t\to\infty} w_\ell(t) &= \frac{1}{\mu_1} \int_{\mathbb{R}_+} \int_t^\infty (t-a)^\ell \varrho(a) da dt = \frac{1}{\mu_1} \int_{\mathbb{R}_+} \left(\int_0^a (t-a)^\ell dt \right) \varrho(a) da \\ &= \frac{(-1)^\ell}{(\ell+1)} \frac{\mu_{\ell+1}}{\mu_1}, \end{aligned}$$

which ends the proof. ∎

Lemma 3. *Under the previous results, we obtain the error estimate*

$$\left| X_\varepsilon(0^+) - \tilde{X}_{\varepsilon,N}(0^+) \right| \lesssim \varepsilon^N.$$

Proof. By definition, one has :

$$\tilde{X}_{\varepsilon,N}(0^+) = \sum_{i=0}^{N-1} \varepsilon^i X_i(0) + Y_N(0) + Z_N(0) + W_N(0^+),$$

then one has

$$Y_N(0) := \frac{1}{\mu_0} \sum_{\ell=1}^{N} \varepsilon^\ell \sum_{k=1}^{\ell} \frac{(-1)^{k+1}}{k!} X_{\ell-k}^{(k)}(0) \mu_k = \varepsilon \frac{f(0)}{\mu_0}$$

where we used that $\tilde{x}_{j,k}(0) = 0$ for all $j \neq 0$ and (3.7). By definition,

$$Z_N(0) = -\sum_{i=0}^{N-1} \varepsilon^i X_i(0) \tilde{x}_{0,0}(0) = -\sum_{i=0}^{N-1} \varepsilon^i X_i(0),$$

and compensates the first terms of the sum, then

$$W_N(0) = \frac{1}{\mu_0} \sum_{i=0}^{N} \varepsilon^i \frac{X_p^{(i)}(0)}{i!} (-1)^i \mu_i.$$

Then one observes that

$$X_\varepsilon(0) = \varepsilon \frac{f(0)}{\mu_0} + \int_{\mathbb{R}_+} X_p(-\varepsilon a) \frac{\varrho(a)}{\mu_0} da$$

and so the Taylor expansion of the last term ends the proof. ◻

3.2. Matching inner and outer expansions. So far the initial conditions of the outer expansion are not defined. For this sake, we write the inner expansion's limit when $t \to \infty$. This gives :

$$\lim_{t\to\infty} Y_N(t) = \sum_{q=1}^{m} \sum_{k=1}^{q} \frac{(-1)^{k+1}}{(m-q+1)!k!} X_{q-k}^{(k+m-q)}(0) \mu_{m-q+k+1} =: s_1,$$

together with :

$$\lim_{t\to\infty} Z_N(t) = -\sum_{m=1}^{N} \varepsilon^m \sum_{q=0}^{m-1} X_q^{(m-q)} \frac{\mu_{m-q+1}}{(m-q+1)!\mu_1} - \sum_{i=0}^{N-1} \varepsilon^i X_i(0)$$

and

$$\lim_{t\to\infty} W_N(t) = \sum_{i=0}^{N} \varepsilon^i \frac{X_p^{(i)}(0)}{(i+1)!} (-1)^i \frac{\mu_{i+1}}{\mu_1} =: s_3.$$

As we do not want the inner expansion to interfere with the outer expansion, we gather the powers of ε and define the initial conditions of the outer expansion so that

$$\lim_{t\to\infty} (Y_N(t) + Z_N(t) + W_N(t)) = 0$$

which then gives :

Theorem 1. *The macroscopic Ansatz should be given the initial conditions : for* $m=0$*,* $X_0(0)=X_p(0)$ *while for* $m\in\{1,\dots,N-1\}$

$$\mu_1 X_m(0)=(-1)^m X_p^{(m)}(0)\frac{\mu_{m+1}}{((m+1)!)}-\sum_{q=0}^{m-1}X_q^{(m-q)}(0)\frac{\mu_{m-q+1}}{(m-q+1)!}$$
$$+\sum_{q=1}^{m}\sum_{k=1}^{q}\frac{(-1)^{k+1}}{(m-q+1)!k!}X_{q-k}^{(k+m-q)}(0)\mu_{m-q+k+1}.$$

3.3. Error estimates. In this section, we give an error estimate between X_ε, the solution of

$$\begin{cases}\dfrac{1}{\varepsilon}\displaystyle\int_{\mathbb{R}_+}\left(X_\varepsilon(t)-X_\varepsilon(t-\varepsilon a)\right)\rho(a)da=f(t), & t>0,\\ X_\varepsilon(t)=X_p(t) & t\le 0,\end{cases}\tag{3.20}$$

and the asymptotic expansion $\tilde{X}_{\varepsilon,N}$ given by (3.1). This result is based on the application of a comparison principle [4, Chap. 9, Section 8] and the construction of a super solution U_N such that $U_N\ge\left|X_\varepsilon-\tilde{X}_{\varepsilon,N}\right|$ and $U_N\lesssim\varepsilon^N$. The following lemma is required in order to apply the latter comparison principle :

Lemma 4. *Under the Assumptions 1,* $K_\varepsilon(\tilde{a}):=\frac{1}{\varepsilon\mu_0}\varrho\left(\frac{\tilde{a}}{\varepsilon}\right)$ *satisfies :*

$$\|K_\varepsilon\|_{B^\infty(0,T)}:=\operatorname*{ess\,sup}_{t\in(0,T)}\int_0^t|K_\varepsilon(\tilde{a},t)|\,d\tilde{a}<1.$$

Proof. For almost every $t\in(0,T)$,

$$0\le\int_0^t|K_\varepsilon(\tilde{a},t)|\;d\tilde{a}=\frac{\int_0^{\frac{t}{\varepsilon}}\varrho(a)\;da}{\int_{\mathbb{R}_+}\varrho(a)\;da}\le\frac{\int_0^{\frac{T}{\varepsilon}}\varrho(a)\;da}{\int_{\mathbb{R}_+}\varrho(a)\;da}.\tag{3.21}$$

But, by definition, for every fixed ε there exists a compact set $M\subset(T/\varepsilon,\infty)$ such that $\varrho(a)>0$ for almost every $a\in M$ so that

$$\int_{\mathbb{R}_+}\varrho(a)da-\int_0^{\frac{T}{\varepsilon}}\varrho(a)\;da=\int_{\frac{T}{\varepsilon}}^{\infty}\varrho(a)da\ge\int_M\varrho(a)da>0.$$

which ends the proof. ∎

Theorem 2. *Suppose that Assumptions 1 holds, then :*

$$\left\|X_\varepsilon-\tilde{X}_{\varepsilon,N}\right\|_{C([0,T])}\lesssim\varepsilon^N.$$

where $\tilde{X}_{\varepsilon,N}$ *is defined in* (3.1) *and* X_ε *solving* (3.20)*.*

Proof. First, we consider the zero order approximation (*i.e.* $N=1$). We denote $\hat{X}_1:=X_\varepsilon-\tilde{X}_{\varepsilon,1}$, it solves :

$$\mathcal{L}_\varepsilon[\hat{X}_1]=S_1:=\sum_{i=1}^{3}S_{1,i},$$

where $|S_1| \leq \varepsilon K_1$ (see estimates (3.14), (3.16) and (3.18) for $N = 1$). We construct a super-solution U_1 such that

$$\mathcal{L}_\varepsilon\left[\left|\hat{X}_1\right|\right] \leq \mathcal{L}_\varepsilon[U_1], \quad \text{and } \left|\hat{X}_1(0)\right| \leq U_1(0).$$

We set

$$U_1(t) := \varepsilon\left(c_1 + tc_2 - \varepsilon c_3 \tilde{w}_1(t)\right),$$

where $\tilde{w}_1$ solves (3.11) with $\ell = 1$. As the resolvent associated to (3.11) is non-negative, applying the comparison principle [5, Propostion 8.1 and Lemma 8.2], shows that $\tilde{w}_1(t) \leq 0$ for $t > 0$. Then

$$\mathcal{L}_\varepsilon[U_1] = c_1\Xi_{0,0}(t) + \varepsilon c_2\mu_1 + c_2\int_{\frac{t}{\varepsilon}}^{\infty}(t - \varepsilon a)\varrho(a)da - \varepsilon c_3\mathcal{L}_\varepsilon[\tilde{w}_1]$$

$$= c_1\Xi_{0,0}(t) + \varepsilon c_2\mu_1 + \varepsilon c_2\int_{\frac{t}{\varepsilon}}^{\infty}\left(\frac{t}{\varepsilon} - a\right)\varrho(a)da - \varepsilon c_3\int_{\frac{t}{\varepsilon}}^{\infty}\left(\frac{t}{\varepsilon} - a\right)\varrho(a)da \geq \varepsilon c_2\mu_1.$$

The last inequality being true when $c_1 \geq 0$ and $c_2 = c_3$. Then, one tunes $c_2 \geq K_1/\mu_1$ so that

$$\mathcal{L}_\varepsilon\left[\left|\hat{X}_1\right|\right] \leq |S_1| \leq \varepsilon K_1 \leq \varepsilon\mu_1 c_2 \leq \mathcal{L}_\varepsilon[U_1],$$

and the constant c_1 is chosen such that $\left|\hat{X}_1(0)\right| \leq \varepsilon c_1 \leq \varepsilon c_1 + \varepsilon^2 c_3\frac{\mu_1}{\mu_0} = U_1(0)$.

More generally, for any N, one sets $U_N := \varepsilon^N(c_1 + c_2 t - \varepsilon c_3\tilde{w}_1)$ and the result follows the same : choosing $c_1 := \left|\hat{X}_N(0)\right|$, $c_2 := K_N/\mu_1$ where $|S_N| \leq \varepsilon^N K_N$ and $c_3 = c_2$. ∎

4. The kernel is time-dependent.

Assumptions 2. *Assume the kernel ρ_ε solves (1.2) and that the data satisfies :*

a) the off-rate ζ is in $C([0,T]; L^\infty(\mathbb{R}_+))$ and there exists a non-increasing $m \in L^1(\mathbb{R}_+;(1+a)^3)$ such that

$$-\frac{m'(a)}{m(a)} \leq \zeta(a,t) \leq \zeta_{\max}, \quad \text{a.e. } a \in \mathbb{R}_+.$$

b) the birth-rate $\beta \in C(\mathbb{R}_+)$ is such that

$$0 < \beta_{\min} \leq \beta(t) \leq \beta_{\max}.$$

c) the initial condition $\rho_I \in BV(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+,(1+a)^3)$ satisfies

$$\rho_I \leq cm(a), \ \textit{for almost every } \ a \in \mathbb{R}_+.$$

Whereas X_ε solves (1.1) and the data satisfy
d) the source term f belongs to $C^2(\mathbb{R}_+)$.

4.1. The linkages' asymptotic expansion. Now we assume that ρ_ε solves the ε, t, a-dependent problem (1.2). We introduce its first order (with respect to ε) asymptotic approximation :

$$\tilde{\rho}_\varepsilon(a,t) := \rho_0(a,t) - \tilde{\mathfrak{r}}_0(a,t) - \varepsilon\rho_1(a,t) \tag{4.1}$$

where ρ_0 is the zeroth order macroscopic limits given by (1.4), ρ_1 is the first order macroscopic limits given by (1.8) and $\tilde{\mathfrak{r}}_0 := \mathfrak{r}_0(a, t/\varepsilon)$ where the initial layer $\mathfrak{r}_0$ solve (1.9).

4.1.1. Outer expansions of ρ_ε.

PROPOSITION 4. *Let Assumptions 2 hold, then there exists generic constants $c_j > 0$ and $c_{j,1} > 0$, such that ρ_0 (resp. ρ_1) solution of (1.4) (resp. (1.8)) satisfies*

$$|\rho_j(a,t)| \leq c_j(1+a)^{2j} m(a), \text{ for } j \in \{0,1\}.$$

In a generic way, for all $j \in \{0,1\}$ and $k \in \mathbb{N}$, if $\zeta \in W^{k,\infty}(\mathbb{R}_+ \times \mathbb{R}_+)$ then :

$$\left|\partial^k_{t^k} \rho_j(a,t)\right| \leq c_{j,k}(1+a)^{2j+k} m(a).$$

The proof uses equations (1.4) and (1.8) together with Assumptions 2 and is postponed to Appendix B.

4.1.2. The initial layer. One considers the problem (1.9) and defines $x(t) := \mathfrak{r}_0(0,t)$, then by using Duhamel's principle this problem can be rewritten as

$$\begin{cases} x + k \star x = b, \\ k(a) := \beta \exp\left(-\displaystyle\int_0^a \zeta(\tau,0)d\tau\right), \\ b(t) := -\beta \displaystyle\int_t^\infty \mathfrak{r}_0(a-t,0) \exp\left(-\displaystyle\int_a^{a-t} \zeta(\tau,0)d\tau\right) da. \end{cases} \tag{4.2}$$

As a consequence of the Paley-Wiener theorem [4, Theorem 4.1] and the fact that k is a decreasing function of a, [4, p.264] :

THEOREM 3. *If k is a decreasing non-negative kernel such that $k \in L^1(\mathbb{R}_+)$, then the resolvent associated to (1.9) satisfies : $r + r \star k = k$ and $r \in L^1(\mathbb{R}_+)$.*

PROPOSITION 5. *Let Assumptions 2 hold. If moreover, $\mathfrak{r}_0(\cdot,0) \in L^1(\mathbb{R}_+,(1+a)^2)$ and that there exists a constant $c > 0$ such that*

$$\mathfrak{r}_0(a,0) \leq c m(a),$$

then $x = \mathfrak{r}_0(0,\cdot) \in L^1_t(\mathbb{R}_+;(1+t)^2) \cap L^\infty(\mathbb{R}_+)$ and

$$\mathfrak{r}_0 \in L^1(\mathbb{R}_+ \times \mathbb{R}_+;(1+a)),$$

and there exists another constant $c' > 0$ such that

$$\mathfrak{r}_0(a,t) \leq c' m(a).$$

Proof. Using the Assumption 2.a) on ζ and on the data, one has

$$|b(t)| \leq \beta \int_t^\infty \mathfrak{r}_0(a-t,0) \frac{m(a)}{m(a-t)} da \leq c \int_t^\infty m(a) da$$

which is bounded since $m \in L^1(\mathbb{R}_+)$ and it is also integrable because so is the first moment of m. Writing then that

$$x = b - r \star b$$

and since L^1 is an algebra for the convolution, $x \in L^1(\mathbb{R}_+)$. Because, b is bounded and $r \in L^1$, $x \in L^\infty(\mathbb{R}_+)$ as well. Then in a similar way, using Duhamel's principle, one has :

$$\mathfrak{r}_0(a,t) := \begin{cases} \mathfrak{r}_0(0,t-a) \exp\left(-\int_0^a \zeta(\tilde{a},0)d\tilde{a}\right), & \text{if } t > a, \\ \mathfrak{r}_0(a-t,0) \exp\left(-\int_{a-t}^a \zeta(\tilde{a},0)d\tilde{a}\right), & \text{otherwise }, \end{cases}$$

so that

$$|\mathfrak{r}_0(a,t)| \leq \begin{cases} \|x\|_{L^\infty} \frac{m(a)}{m(0)}, & \text{if } t > a, \\ \frac{|\mathfrak{r}_0(a-t,0)|}{m(a-t)} m(a) \leq c m(a), & \text{otherwise }, \end{cases}$$

which gives c' in the last estimates of the claim. Next we consider :

$$\begin{aligned} \int_{\mathbb{R}_+} |\mathfrak{r}_0(a,t)| da \leq & \int_0^t |\mathfrak{r}_0(0,t-a)| \exp\left(-\int_0^a \zeta(\tilde{a},0) d\tilde{a}\right) da \\ & + \int_t^\infty |\mathfrak{r}_0(a-t,0)| \exp\left(-\int_{a-t}^a \zeta(\tilde{a},0) d\tilde{a}\right) da \\ & \leq \int_0^t |\mathfrak{r}_0(0,t-a)| \frac{m(a)}{m(0)} da + \int_t^\infty \mathfrak{r}_0(a-t,0) \frac{m(a)}{m(a-t)} da \\ & = \frac{1}{m(0)} \left\{ (|\mathfrak{r}_0(0,\cdot)| \star m)(t) + \int_{\mathbb{R}_+} \frac{|\mathfrak{r}_0(\tilde{a},0)|}{m(\tilde{a})} m(\tilde{a}+t) d\tilde{a} \right\} \\ & = \frac{1}{m(0)} (|x| \star m)(t) + c \int_t^\infty m(a) da \end{aligned}$$

Then using that $L^1(\mathbb{R}_+)$ is an algebra for the convolution, and that $m \in L^1(\mathbb{R}_+, (1+a))$, one concludes that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathfrak{r}_0(a,t) da dt < \frac{\|m\|_{L^1(\mathbb{R}_+,(1+a))}}{m(0)} \left(\|\mathfrak{r}_0(0,\cdot)\|_{L^1(\mathbb{R}_+)} + c m(0) \right)$$

the same holds for the first moment as well.

Setting $q_1(a,t) := t\mathfrak{r}_0(a,t)$ and $y(t) := q_1(0,t)$, one has

$$\begin{aligned} y(t) = & -\beta(0) \int_0^t y(t-a) \exp\left(-\int_0^a \zeta(s,0) ds\right) da \\ & - \beta(0) \int_0^t a x(t-a) \exp\left(-\int_0^a \zeta(s,0) ds\right) da \\ & - \beta \int_t^\infty t \mathfrak{r}_0(a-t,0) \exp\left(-\int_{a-t}^a \zeta(s) ds\right) da \end{aligned}$$

i.e. $y + k \star y = b_x$. Assuming that $\int_{\mathbb{R}_+} (1+a)^2 m(a) da < \infty$ shows that $b_x \in L^1(\mathbb{R}_+)$. Then as above,

$$\begin{aligned} \int_0^t |q_1(a,t)| \, da \leq & \int_0^t (|q_1(0,t-a)| + a \, |x(t-a)|) \exp\left(-\int_0^a \zeta(s,0) ds\right) da \\ & \leq C \, |q_1(0,\cdot)| \star m + |x(0,\cdot)| \star (\cdot) \, m(\cdot) \end{aligned}$$

together with

$$\int_t^\infty |q_1(a,t)| \, da \leq t \int_t^\infty \mathfrak{r}_0(a-t,0) \exp\left(-\int_{a-t}^a \zeta(\tau,0) d\tau\right) da \leq tc \int_t^\infty m(a) da$$

both left-hand sides are then $L^1_t(\mathbb{R}_+)$ functions in time, provided that $m \in L^1_a(\mathbb{R}_+, (1+a)^2)$. For the next step, one works similarly and obtains that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} t^2 \mathfrak{r}_0(a,t) da dt < \infty$$

since m is in $L^1_a(\mathbb{R}_+, (1+a)^3)$. $\square$

COROLLARY 1. *Under the Assumptions 2, one has $x(t) := \mathfrak{r}_0(0,t) \to 0$ when t grows large and the first moment can be estimated as follows :*

$$\int_{\mathbb{R}_+} (1+a)\, |\mathfrak{r}_0(a,t)|\, da \leq o(1) + c \int_t^{\infty} (1+a) m(a) da,$$

where $o(1)$ denotes small when t grows large.

Proof. Using Lyapunov's functional (1.5), one has that

$$\int_{\mathbb{R}_+} |\mathfrak{r}_0(a,t)|\, da \leq \infty$$

which, thanks to the boundary condition, provides that $x(t)$ is bounded on $\mathbb{R}_+$. This shows using Duhamel's principle that $\mathfrak{r}_0 \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$. Moreover, defining the discrete differences

$$D_t^h \mathfrak{r}_0(a,t) \mathfrak{r}_0 := \frac{\mathfrak{r}_0(a,t+h) - \mathfrak{r}_0(a,t)}{h}$$

it solves the problem :

$$\begin{cases} (\partial_t + \partial_a + \zeta(a,0))\, D_t^h \mathfrak{r}_0 = 0, & a>0,\ t>0, \\ D_t^h \mathfrak{r}_0(0,t) = -\beta(0) \displaystyle\int_{\mathbb{R}_+} D_t^h \mathfrak{r}_0(a,t) da, & a=0,\ t>0, \\ D_t^h \mathfrak{r}_0(a,0) = \dfrac{\mathfrak{r}_0(a,h) - \mathfrak{r}_0(a,0)}{h}, & a>0,\ t=0. \end{cases} \tag{4.3}$$

Thanks to Assumptions 2, one shows that $\mathfrak{r}_0(\cdot,0) \in BV(\mathbb{R}_+)$. Moreover, one has that

$$\left|D_t^h x(t)\right| = \left|D_t^h \mathfrak{r}_0(0,t)\right| \leq \beta(0) \int_{\mathbb{R}_+} \left|D_t^h \mathfrak{r}_0(a,t)\right| da \leq \beta(0) \mathcal{H}[D_t^h \mathfrak{r}_0(a,t)]$$

similarly as [6, Theorem 3.2], by using Gronwall's Lemma, we obtain that

$$\mathcal{H}[D_t^h \mathfrak{r}_0(\cdot,t)] \leq \mathcal{H}[D_t^h \mathfrak{r}_0(\cdot,0)].$$

It suffices then to prove that the initial term $\mathcal{H}[D_t^h \mathfrak{r}_0(\cdot,0)]$ is bounded. Indeed, we have

$$\mathcal{H}[D_t^h \mathfrak{r}_0(a,0)] = \int_{\mathbb{R}_+} \left| \frac{\mathfrak{r}_0(a,h) - \mathfrak{r}_0(a,0)}{h} \right| da + \left| \frac{\mu_{\mathfrak{r}_0}(h) - \mu_{\mathfrak{r}_0}(0)}{h} \right| := I_1 + I_2$$

where $\mu_{\mathfrak{r}_0}(t) := \int_{\mathbb{R}_+} \mathfrak{r}_0(a,t) da$. For the first term, we split the integral in two parts

$$\begin{aligned} I_1 &= \int_0^h \left| \frac{\mathfrak{r}_0(a,h) - \mathfrak{r}_0(a,0)}{h} \right| da + \int_h^{+\infty} \left| \frac{\mathfrak{r}_0(a,h) - \mathfrak{r}_0(a,0)}{h} \right| da \\ &= \frac{1}{h} \int_0^h \left| \mathfrak{r}_0(0,h-a) \exp\left(- \int_0^a \zeta(\tilde{a},0) d\tilde{a} \right) - \mathfrak{r}_0(a,0) \right| da \\ &+ \frac{1}{h} \int_h^{+\infty} \left| \mathfrak{r}_0(a-h,0) \exp\left(- \int_{a-h}^a \zeta(\tilde{a},0) d\tilde{a} \right) - \mathfrak{r}_0(a,0) \right| da =: I_{1,1} + I_{1,2} \end{aligned}$$

where we used Duhamel's principle. It easy to see that the first term

$$I_{1,1} \lesssim \frac{1}{h} \int_0^h da \left(|\beta(0)|\, \|\mathfrak{r}_0\|_{L_t^\infty L_a^1} + \|\mathfrak{r}_0(a,0)\|_{L^1(\mathbb{R}_+)} \right) < \infty.$$

Concerning $I_{1,2}$, one splits the integral adding and subtracting intermediate terms

$$\begin{aligned} I_{1,2} &\leq \frac{1}{h}\int_h^{+\infty} \left|(\mathfrak{r}_0(a-h,0)-\mathfrak{r}_0(a,0))\exp\left(-\int_{a-h}^{a}\zeta(\tilde{a},0)d\tilde{a}\right)\right| da \\ &+ \frac{1}{h}\int_h^{+\infty}\left|\mathfrak{r}_0(a,0)\exp\left(-\int_{a-h}^{a}\zeta(\tilde{a},0)d\tilde{a}\right)-\mathfrak{r}_0(a,0)\right| da \\ &\lesssim TV(\mathfrak{r}_0(\cdot,0)) + C\|\mathfrak{r}_0(a,0)\|_{L^1(\mathbb{R}_+)} \end{aligned}$$

where TV denotes total variation of $\mathfrak{r}_0(\cdot,0)$ [2]. For the second term I_2 noting that

$$|\partial_t \mu_{\mathfrak{r}_0}(t)| \leq (\zeta_{\max}+\beta_{\max})\,\|\mathfrak{r}_0\|_{L^\infty_t L^1_a},$$

one obtains

$$I_2 \leq (\zeta_{\max}+\beta_{\max})\,\|\mathfrak{r}_0\|_{L^\infty_t L^1_a},$$

and finally, we obtain that $\mathcal{H}[D_t^h\mathfrak{r}_0(a,t)] < \infty$, for all $t>0$, which shows that $x \in Lip(\mathbb{R}_+)$. Since $x\in L^1(\mathbb{R}_+)$ (see Proposition 5), this implies that $\lim_{t\to+\infty} x(t) = 0$. Now, we consider

$$\begin{aligned} J(t) := \int_0^t (1+a)\,|\mathfrak{r}_0(a,t)|\,da &\leq \int_0^t |x(t-a)|\,(1+a)m(a)da \\ &= \left(\int_0^{t/2} + \int_{t/2}^t\right)|x(t-a)|\,(1+a)m(a)da =: J_1+J_2. \end{aligned}$$

For every $\delta>0$ there exists η_1 such that $t>\eta_1$,implying that

$$\sup_{s\in(\frac{t}{2},t)} |x(s)| < \frac{\delta}{\left(2\int_{\mathbb{R}_+}(1+a)m(a)da\right)}$$

which shows that $J_1<\delta/2$. On the other hand, there exists η_2 such that $t>\eta_2$ which implies that

$$J_2 \leq \|x\|_{L^\infty(\mathbb{R}_+)}\int_{t/2}^t (1+a)m(a)da < \delta/2,$$

by Lebesgue's Theorem (since the integral of $(1+a)m(a)$ is finite). These arguments show that $J(t)$ vanishes when t grows large. On the other hand :

$$\int_t^\infty (1+a)\,|\mathfrak{r}_0(a,t)|\,da \leq c\int_t^\infty (1+a)m(a)da,$$

which is an initial layer. □

4.1.3. Error estimates for the linkage's density. We define the difference : $\hat{\rho}_\varepsilon(a,t) := \rho_\varepsilon(a,t) - \tilde{\rho}_\varepsilon(a,t)$ where $\tilde{\rho}_\varepsilon$ is defined by (4.1). We obtain :

Theorem 4. *Under Assumptions 2, one has that*

$$\mathcal{H}[\hat{\rho}_\varepsilon](t) \leq o_\varepsilon(1), \quad \text{a.e. } t\in\mathbb{R}_+ \tag{4.4}$$

where $\mathcal{H}[u](t) := \int_{\mathbb{R}_+}|u(a)|\,da + \left|\int_{\mathbb{R}_+} u(a)da\right|$.

Proof. We write the system satisfied by $\hat{\rho}_\varepsilon$:

$$\begin{cases} (\varepsilon\partial_t + \partial_a + \zeta(a,t))\hat{\rho}_\varepsilon(a,t) = -\varepsilon\left(\zeta(a,t) - \zeta(a,0)\right)\tilde{\mathfrak{r}}_0(a,t) - \varepsilon^2 \partial_t \rho_1 \\ \hat{\rho}_\varepsilon(0,t) = -\,\beta(t)\int_{\mathbb{R}_+} \hat{\rho}_\varepsilon(a,t)da - (\beta(t) - \beta(0))\int_{\mathbb{R}_+} \tilde{\mathfrak{r}}_0(a,t)da \\ \hat{\rho}_\varepsilon(a,0) = -\varepsilon\rho_1(a,0) \end{cases} \tag{4.5}$$

Then following the same steps as in [7], one has that

$$\begin{aligned} &\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}[\hat{\rho}_\varepsilon](t) + \int_{\mathbb{R}_+} \zeta(a,t)\left\{|\hat{\rho}_\varepsilon(a,t)| + \hat{\rho}_\varepsilon(a,t)\,\mathrm{sgn}\left(\int_{\mathbb{R}_+}\hat{\rho}_\varepsilon(\tilde{a},t)d\tilde{a}\right)\right\}da \\ &\quad\le 2\varepsilon^2\int_{\mathbb{R}_+}|\partial_t\rho_1(a,t)|\,da + 2\int_{\mathbb{R}_+}|\zeta(a,t) - \zeta(a,0)|\,|\tilde{\mathfrak{r}}_0(a,t)|\,da \\ &\quad + |\beta(t) - \beta(0)|\int_{\mathbb{R}_+}|\tilde{\mathfrak{r}}_0(a,t)|\,da \end{aligned}$$

which after integration in time provides :

$$\begin{aligned} \mathcal{H}[\hat{\rho}_\varepsilon](t) \le& \mathcal{H}[\hat{\rho}_\varepsilon](0) + 2\varepsilon\int_0^t\int_{\mathbb{R}_+}|\partial_t\rho_1(a,s)|\,dads \\ &+ 2\int_0^{\frac{t}{\varepsilon}}\int_{\mathbb{R}_+}\left|\zeta(a,\varepsilon\tilde{t}) - \zeta(a,0)\right|\left|\mathfrak{r}_0(a,\tilde{t})\right|dad\tilde{t} \\ &+ 2\int_0^{\frac{t}{\varepsilon}}\left|\beta(\varepsilon\tilde{t}) - \beta(0)\right|\int_{\mathbb{R}_+}\left|\mathfrak{r}_0(a,\tilde{t})\right|dad\tilde{t}. \end{aligned}$$

Now, here the crucial point is that, thanks to Lebesgue's Theorem, the last two terms of the right-hand side do tend to zero as ε goes to zero. ∎

Corollary 2. *Let Assumptions 2 hold, then one has*

$$\int_{\mathbb{R}_+}(1+a)\,|\rho_\varepsilon(a,t) - \rho_0(a,t)|\,da \le o_\varepsilon(1) + \int_{\frac{t}{\varepsilon}}^\infty (1+a)m(a)da.$$

Proof. Considering the system solved by the difference $e(a,t) = \rho_\varepsilon(a,t) - \rho_0(a,t)$

$$\begin{cases} (\varepsilon\partial_t + \partial_a + \zeta(a,t))e(a,t) = \varepsilon\partial_t\rho_0(a,t), & a > 0,\ t > 0, \\ e(0,t) = -\beta(t)\int_{\mathbb{R}_+}\left(\rho_\varepsilon(\tilde{a},t) - \rho_0(\tilde{a},t) - \tilde{\mathfrak{r}}_0(\tilde{a},t)\right)da & \\ \qquad\qquad - \beta(t)\int_{\mathbb{R}_+}\tilde{\mathfrak{r}}_0(a,t)da, & a = 0,\ t > 0, \\ e(a,0) = \rho_I(a) - \rho_0(a,0), & a > 0,\ t = 0. \end{cases} \tag{4.6}$$

It satisfies (4.6) in the sense of characteristics, namely

$$e(a,t) = \begin{cases} e\left(0, t - \varepsilon a\right)\exp(-\int_{-a}^0 \zeta(a+s, t+\varepsilon s)ds) + \\ \quad +\varepsilon\int_{-a}^0 \partial_t\rho_0(a+s, t+\varepsilon s)\exp(-\int_s^0 \zeta(a+\tau, t+\varepsilon\tau)d\tau)ds, & a < t/\varepsilon, \\ e\left(a - t/\varepsilon, 0\right)\exp(-\int_{t/\varepsilon}^0 \zeta(a+s, t+\varepsilon s)ds) + \\ \quad +\varepsilon\int_{-t/\varepsilon}^0 \partial_t\rho_0(a+s, t+\varepsilon s)\exp(-\int_s^0 \zeta(a+\tau, t+\varepsilon\tau)d\tau)ds, & a > t/\varepsilon. \end{cases} \tag{4.7}$$

One has that

$$\int_0^{+\infty}(1+a)e(a,t)da = \int_0^{t/\varepsilon}(1+a)e(a,t)da + \int_{t/\varepsilon}^{+\infty}(1+a)e(a,t)da \\ := I_1 + I_2. \tag{4.8}$$

We treat each term separately because they correspond to the two cases of Duhamel's formula (4.7) :

$$\begin{aligned} I_1(t) \le & \int_0^{\frac{t}{\varepsilon}}(1+a)\,|e(0,t-\varepsilon a)|\exp\left(-\int_{-a}^0 \zeta(a+s,t+\varepsilon s)ds\right)da \\ & + \varepsilon\int_0^{\frac{t}{\varepsilon}}(1+a)\int_{-a}^0 \partial_t\rho_0(a+s,t+\varepsilon s)\exp\left(-\int_s^0 \zeta(a+\tau,t+\varepsilon\tau)d\tau\right)dsda. \end{aligned}$$

Using Theorem 4 and Corollary 1, one has that :

$$|e(0,t-\varepsilon a)| \lesssim o_\varepsilon(1) + \int_{\frac{t}{\varepsilon}-a}^{\infty} m(a)da,$$

which thanks to Proposition 4 gives that

$$\begin{aligned} I_1(t) \le & o_\varepsilon(1)\int_0^{\frac{t}{\varepsilon}}(1+a)m(a)da + \int_0^{\frac{t}{\varepsilon}}(1+a)m(a)\int_{\frac{t}{\varepsilon}-a}^{\infty} m(\tilde{a})d\tilde{a}da \\ & + \varepsilon c_{0,1}\int_{\mathbb{R}_+}(1+a)^3 m(a)da, \end{aligned}$$

using similar argument as in the proof of Corollary 1, one shows that

$$\int_0^{\frac{t}{\varepsilon}}(1+a)m(a)da\int_{\frac{t}{\varepsilon}-a}^{\infty} m(\tilde{a})d\tilde{a}da \le o_\varepsilon(1),$$

since $(1+a)m(a)$ is integrable and by Lebesgue's Theorem $\int_t^\infty m(a)da \to 0$ as t grows large. On the other hand,

□

$$\begin{aligned} I_2 \le & \int_{\frac{t}{\varepsilon}}^{\infty}(1+a)\left|e(a-\frac{t}{\varepsilon},0)\right|\exp\left(-\int_{-\frac{t}{\varepsilon}}^0 \zeta(a+s,t+\varepsilon s)ds\right)da \\ & + \varepsilon\int_{\frac{t}{\varepsilon}}^{\infty}(1+a)\int_{-\frac{t}{\varepsilon}}^0 |\partial_t\rho_0(a+s,t+\varepsilon s)|\exp\left(-\int_s^0 \zeta(a+\tau,t+\varepsilon\tau)d\tau\right)dsda \\ \le & \, c\int_{\frac{t}{\varepsilon}}^{\infty}(1+a)m(a)da + \varepsilon\int_{\mathbb{R}_+}(1+a)^3 m(a)da. \end{aligned}$$

4.2. Convergence results for the position.

THEOREM 5. *Under Assumptions 2, if ρ_ε is a solution of (1.2), ρ_0 solves (1.4), X_ε is a solution of (1.1) and X_0 solves (3.4) then*

$$\|\rho_\varepsilon - \rho_0\|_{L^1((0,T);L^1(\mathbb{R}_+,(1+a)))} \le o_\varepsilon(1), \quad \|X_\varepsilon - X_0\|_{C([0,T])} \le o_\varepsilon(1).$$

Proof. Setting $u_\varepsilon(a,t) := X_\varepsilon(t) - X_\varepsilon(t-\varepsilon a)$ where $X_\varepsilon(t) = X_p(t)$ when $t<0$, u_ε solves in a weak sense [8] :

(4.9)
$$\begin{cases} (\varepsilon\partial_t + \partial_a) u_\varepsilon = \partial_t X_\varepsilon = \dfrac{1}{\mu_{0,\varepsilon}} \left\{ \varepsilon \partial_t f + \displaystyle\int_{\mathbb{R}_+} u_\varepsilon(a,t)\zeta(a,t)\rho_\varepsilon(a,t)da \right\}, & a>0,\ t>0, \\ u_\varepsilon(0,t) = 0, & a=0,\ t>0, \\ u_\varepsilon(a,0) = u_I(a) := \dfrac{X_\varepsilon(0) - X_p(-\varepsilon a)}{\varepsilon}, & a>0,\ t=0, \end{cases}$$

since problem (1.1) can be expressed in a integro-differential equation :

$$\mu_{0,\varepsilon}(t)\partial_t X_\varepsilon = \varepsilon \partial_t f + \int_{\mathbb{R}_+} \left(\frac{X_\varepsilon(t) - X_\varepsilon(t-\varepsilon a)}{\varepsilon} \right) \zeta(a,t)\rho_\varepsilon(a,t)da.$$

Following [8, Theorem 6.1], one has that

$$\int_{\mathbb{R}_+} |u_\varepsilon(a,t)|\rho_\varepsilon(a,t)da \quad \leq \int_0^t |\partial_t f(\tau)|\, d\tau + |f(0)| + L_{X_p}\mu_{1,\max} =: c_1$$

Moreover, one has also the bound :

$$\|\partial_t X_\varepsilon\|_{L^\infty(0,T)} \leq \frac{1}{\mu_{0,\min}} \left\{ \varepsilon \|\partial_t f\|_{L^\infty(0,T)} + \zeta_{\max} c_1 \right\} =: c_2$$

which provides thanks to Ascoli-Arzella that there exists a converging sub-sequence X_ε in $C([0,T])$. Moreover, it $t > \varepsilon a$, using Duhamel's principle,

$$|u_\varepsilon(a,t)| \leq \frac{1}{\varepsilon}\int_{t-\varepsilon a}^t |\partial_t X_\varepsilon(\tau)|\, d\tau \leq c_2 a$$

whereas if $t < \varepsilon a$, by similar arguments,

$$|u_\varepsilon(a,t)| \leq \frac{t}{\varepsilon} c_2 + \left| \frac{f(0)}{\mu_{0,\min}} \right| + L_{X_p} \frac{\mu_{1,\max}}{\mu_{0,\min}}$$

Thus, $u_\varepsilon(a,t)/(1+a) \in L^\infty(\mathbb{R}_+ \times 0, T)$ uniformly with respect to ε. These results provide that u_ε weak-* converges in $L^\infty(\mathbb{R}_+ \times (0,T); (1+a)^{-1})$ to u_0 a weak solution of

$$\begin{cases} \partial_a u_0 = \partial_t X_0 = \dfrac{1}{\mu_{0,0}} \displaystyle\int_{\mathbb{R}_+} u_0 \zeta(a,t)\rho_0(a,t)da \\ u_0(0,t) = 0 \end{cases} \tag{4.10}$$
□

which shows that $u_0(a,t) = aX_0(t)$. Since (1.1) reads as

$$A_\varepsilon := \int_0^T \int_{\mathbb{R}_+} u_\varepsilon(a,t)\rho_\varepsilon(a,t)da\varphi(t)dt = \int_0^T f(t)\varphi(t)dt, \quad \forall \varphi \in L^1(0,T),$$

ans since $\partial_t X_\varepsilon \overset{\star}{\rightharpoonup} \partial_t X_0$ weak-$\star$ in $L^\infty(0,T)$ together with

$$\rho_\varepsilon \to \rho_0 \text{ in } L^1(\mathbb{R}_+ \times (0,T); (1+a)), \quad u_\varepsilon \overset{\star}{\rightharpoonup} u_0 \text{ in } L^\infty(\mathbb{R}_+ \times (0,T); (1+a)^{-1})$$

one concludes that

$$A_\varepsilon \to \int_0^T \int_{\mathbb{R}_+} u_0(a,t)\rho_0(a,t)da\varphi(t)dt = \int_0^T \mu_{0,1}(t)\partial_t X_0(t)\varphi dt = \int_0^T f(t)\varphi(t)dt.$$

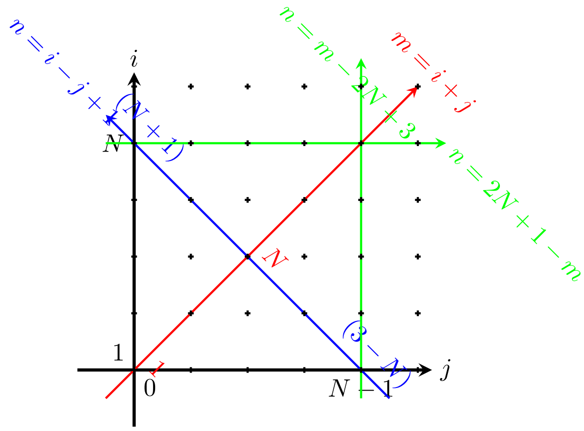

Fig. A.1: The index change from (i,j) to (m,n) (here as an example $N=5$).

Appendix A. Some summations over integers.

PROPOSITION 6. *Assume that $N \geq 2$, and define the sum S as follows*

$$S := \sum_{i=1}^{N} \sum_{k=1}^{i} \sum_{j=0}^{N-k} a_{i,j,k}$$

where $a_{i,j,k}$ is a sequence of real numbers, then this sum is in fact equal to

$$S = \left(\sum_{m=1}^{N} \sum_{q=1}^{m} \sum_{k=1}^{q} + \sum_{m=N+1}^{2N-1} \sum_{q=m+1-N}^{N} \sum_{k=1}^{q+N-m} \right) a_{q,m-q,k}$$

Proof. We separate the case where $i < N$ and the case where $i = N$. In the first case, one has, by the following computation, that

$$s := \sum_{i=1}^{N-1} \sum_{k=1}^{i} \sum_{j=0}^{N-k} a_{i,j,k} = \sum_{i=1}^{N-1} \sum_{j=0}^{N-i-1} \sum_{k=1}^{i} a_{i,j,k} + \sum_{i=1}^{N-1} \sum_{j=N-i}^{N-1} \sum_{k=1}^{N-j} a_{i,j,k},$$

because

$$\begin{cases} 0 \leq j \leq N-i-1, \quad k \in \{1,\dots,i\} \Rightarrow k \in \{1,\dots,\min(i,N-j)\}, \\ N-i \leq j \leq N-1, \quad k \in \{1,\dots,N-j\} \Rightarrow k \in \{1,\dots,\min(i,N-j)\}. \end{cases}$$

One has that

$$s = \sum_{i=1}^{N-1} \sum_{j=0}^{N-1} \sum_{k=1}^{\min(i,N-j)} a_{i,j,k}.$$

On the other hand, when $i = N$, a simple check shows that

$$s' := \sum_{k=1}^{N} \sum_{j=0}^{N-k} a_{N,j,k} = \sum_{j=0}^{N-1} \sum_{k=1}^{N-j} a_{N,j,k},$$

and then one remarks simply that $\min(i = N, N - j) = N - j$ which gathering the terms gives that

$$S = s + s' = \sum_{i=1}^{N} \sum_{j=0}^{N-1} \sum_{k=1}^{\min(i,N-j)} a_{i,j,k}.$$

The previous sum can be rewritten as :

$$S = \left(\sum_{m=1}^{N} \sum_{n=3-m}^{m+1} + \sum_{m=N+1}^{2N-1} \sum_{n=m-2N+3}^{2N-m+1} \right) \sum_{k=1}^{\min(i,N-j)} a_{i(m,n),j(m,n),k} \chi_{\{m,n \in \mathbb{N}^2\}}(m,n),$$

where $m := i + j$ and $n := i - j + 1$ and the inverse transform should provide integer values $i(m,n)$ and $j(m,n)$ (see Fig. A.1). When $m > N$, one needs to bound the summation on n in an interval depending on m (see Fig. A.1). Indeed, when $i = N$, we write :

$$n = N - j + 1, \quad m = N + j, \quad \Rightarrow n = 2N - m + 1,$$

while if $j = N - 1$,

$$n = i - N + 2, \quad m = i + N - 1, \quad \Rightarrow n = m - 2N + 3,$$

a simple check shows that $n \leq 2N - 1 \Leftrightarrow m - 2N + 3 \leq 2N - m + 1$.

Then since the indicatrix function is not zero when $[n + m - 1]_2 = 0$, there exists $q \in \mathbb{Z}$ such that

$$n + m - 1 = 2q \Leftrightarrow n = 1 + 2q - m$$

so that the summation with respect to n can be exchanged with a summation over q. When $n \in \{3-m, m+1\}$, $q \in \{1, m\}$, and similarly when $n \in \{m-2N+3, 2N-m+1\}$, $q \in \{m+1-N, N\}$. Moreover $q = i$ and $j = m - q$. Thus the previous sum becomes :

$$S = \left(\sum_{m=1}^{N} \sum_{q=1}^{m} + \sum_{m=N+1}^{2N-1} \sum_{q=m+1-N}^{N} \right) \sum_{k=1}^{\min(q,N-(m-q))} a_{q,m-q,k}$$

Since $\min(q, q+N-m) = q + \min(0, N-m) = q$ as soon as $m \leq N$ this gives the first term. If $m \in \{N+1, \dots, 2N-1\}$, then $N - m \leq -1$ and $\min(q, q+N-m) = q+N-m$ which ends the proof. ∎

Proposition 7. *In the same way as above*

$$S' := \sum_{i=0}^{N-1} \sum_{j=1}^{N-i} a_{i,j} = \sum_{m=1}^{N} \sum_{q=0}^{m-1} a_{q,m-q}$$

Proof. Again we perform the change of variables $m = i + j$ and $q = i$ and we proceed as above. ∎

Appendix B. Proof of Proposition 4.

Proof. First, for $j = 0$, ρ_0 is explicitly given by

$$\rho_0(a,t) = \rho_0(0,t) \exp\left(- \int_0^a \zeta(\tilde{a},t) d\tilde{a} \right) \leq \rho_0(0,t) \frac{m(a)}{m(0)} \leq \frac{\beta_{\max}\zeta_{\max}}{\zeta_{\max} + \beta_{\min}} \frac{m(a)}{m(0)}$$

which gives c_0. Similarly, $\partial_t \rho_0(a,t)$ it is explicit and reads :

$$\partial_t \rho_0(a,t) = \partial_t \rho_0(0,t) \exp\left(-\int_0^a \zeta(\tilde{a},t)d\tilde{a}\right) - \int_0^a \exp\left(-\int_\tau^a \zeta(\tilde{a},t)d\tilde{a}\right) \partial_t \zeta(\tau,t)\rho_0(\tau,t)d\tau,$$

where

$$\partial_t \rho_0(0,t) = \frac{g(t)}{1+\beta(t)\int_0^{+\infty} \exp\left(-\int_\tau^a \zeta(\tilde{a},t)d\tilde{a}\right)da}$$

and

$$g(t) = \beta'(t)\left(1-\mu_0(t)\right)\int_0^{+\infty} \exp\left(-\int_\tau^a \zeta(\tilde{a},t)d\tilde{a}\right)da - \int_0^{+\infty}\int_0^a \exp\left(-\int_\tau^a \zeta(\tilde{a},t)d\tilde{a}\right)\partial_t \zeta(\tau,t)\rho_0(\tau,t)d\tau da.$$

So that

$$|\partial_t \rho_0(a,t)| \leq |\partial_t \rho_0(0,t)| \frac{m(a)}{m(0)} + m(a)\|\zeta\|_{W^{1,\infty}} \int_0^a \frac{\rho_0(\tau,t)}{m(\tau)}d\tau \leq (k_1 + k_2 a)m(a) \leq c'(1+a)m(a)$$

where

$$k_1 := C\left(\frac{\zeta_{\max}}{\beta_{\max}+\zeta_{\max}}, \|\beta\|_{W^{1,\infty}}, \|m\|_{L^1(\mathbb{R}_+)}, \|\zeta\|_{W^{1,\infty}}\right), \quad k_2 = c_0 \frac{\|\zeta\|_{W^{1,\infty}}}{m(0)}$$

and $c_{0,1} := \max(k_1, k_2)$. Now, for $j=1$, ρ_1 can be given explicitly by

$$\rho_1(a,t) = \rho_1(0,t)\exp\left(-\int_0^a \zeta(\tilde{a},t)d\tilde{a}\right) - \int_0^a \exp\left(-\int_\tau^a \zeta(\tilde{a},t)d\tilde{a}\right)\partial_t \rho_0(\tau,t)d\tau,$$

where

$$\rho_1(0,t) = \frac{h(t)}{1+\beta(t)\int_0^{+\infty}\exp\left(-\int_\tau^a \zeta(\tilde{a},t)d\tilde{a}\right)da}$$

such that

$$|h(t)| = \left|\int_{\mathbb{R}_+}\int_0^a \exp\left(-\int_\tau^a \zeta(\tilde{a},t)d\tilde{a}\right)\partial_t \rho_0(\tau,t)d\tau da\right| \leq \int_{\mathbb{R}_+}\int_0^a \frac{m(a)}{m(\tau)}\left|\partial_t \rho_0(\tau,t)\right| d\tau da \leq c_{0,1}\int_{\mathbb{R}_+}(1+a)^2 m(a)da,$$

and finally, we obtain that

$$|\rho_1(a,t)| \leq k_1' \frac{m(a)}{m(0)} + \int_0^a c_{0,1}(1+\tau)m(a)d\tau \leq \max(k_1', c_{0,1})(1+a)^2 m(a)$$

where

$$k_1' := C\left(\frac{\zeta_{\max}}{\beta_{\max}+\zeta_{\max}}, \int_{\mathbb{R}_+}(1+a)^2 m(a)da\right).$$

Similarly, we can prove that

$$|\partial_t \rho_1| \leq c_{1,1}(1+a)^3 m(a),$$

and the generic way can be deduced by induction. ∎

Acknowledgments. This research was partly funded by Cofund Math in Paris, co-funded by the European Commission under the Horizon2020 research and innovation programme, Marie Sklodowska-Curie grant agreement No 101034255.

REFERENCES

[1] S. Allouch and V. Milišić, *Friction mediated by transient elastic linkages: extension to loads of bounded variation*, Journal of Integral Equations and Applications, (2021).

[2] J. Appell, J. Banas, and N. J. M. Díaz, *Bounded variation and around*, in Bounded Variation and Around, de Gruyter, 2013.

[3] B. Grec, B. Maury, N. Meunier, and L. Navoret, *A 1D model of leukocyte adhesion coupling bond dynamics with blood velocity*, J. Theor. Biol., 452 (2018), pp. 35–46.

[4] G. Gripenberg, S. O. Londen, and O. Staffans, *Frontmatter*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1990, pp. i–iv.

[5] G. Gripenberg, S.-O. Londen, and O. Staffans, *Volterra integral and functional equations*, vol. 34 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1990.

[6] V. Milisic, *Initial layer analysis for a linkage density in cell adhesion mechanisms*, in CIMPA School on Mathematical Models in Biology and Medicine, vol. 62 of ESAIM Proc. Surveys, EDP Sci., Les Ulis, 2018, pp. 108–122.

[7] V. Milišić and D. Oelz, *On the asymptotic regime of a model for friction mediated by transient elastic linkages*, J. Math. Pures Appl. (9), 96 (2011), pp. 484–501, https://doi.org/10.1016/j.matpur.2011.03.005.

[8] V. Milišić and D. Oelz, *On a structured model for the load dependent reaction kinetics of transient elastic linkages*, SIAM J. Math. Anal., 47 (2015), pp. 2104–2121, https://doi.org/10.1137/130947052, http://dx.doi.org/10.1137/130947052.

[9] V. Milišić and D. Oelz, *Tear-off versus global existence for a structured model of adhesion mediated by transient elastic linkages*, Commun. Math. Sci., 14 (2016), pp. 1353–1372, https://doi.org/10.4310/CMS.2016.v14.n5.a7, http://dx.doi.org/10.4310/CMS.2016.v14.n5.a7.

[10] V. Milisic and D. Oelz, *From memory to friction : convergence of regular solutions towards the heat equation with a non-constant coefficient*, ArXiv e-prints, (2017), https://arxiv.org/abs/1706.02650.

[11] V. Milišić and C. Schmeiser, *Asymptotic limits for a nonlinear integro-differential equation modelling leukocytes' rolling on arterial walls*, Nonlinearity, 35 (2021), p. 843.

[12] D. Oelz and C. Schmeiser, *Derivation of a model for symmetric lamellipodia with instantaneous cross-link turnover*, Archive for Rational Mechanics and Analysis, 198 (2010), pp. 963–980, https://doi.org/10.1007/s00205-010-0304-z.

[13] D. Oelz and C. Schmeiser, *How do cells move? mathematical modelling of cytoskeleton dynamics and cell migration*, in Cell mechanics: from single scale-based models to multiscale modelling, A. Chauviere, L. Preziosi, and C. Verdier, eds., Chapman and Hall / CRC Press, to appear, 2009.

[14] D. Oelz, C. Schmeiser, and V. Small, *Modelling of the actin-cytoskeleton in symmetric lamellipodial fragments*, Cell Adhesion and Migration, 2 (2008), pp. 117–126.

[15] W. P. Ziemer, *Weakly differentiable functions*, vol. 120 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.