ESAIM: PROCEEDINGS AND SURVEYS, Vol. ?, 2017, 1-10
Editors: Will be set by the publisher

INITIAL LAYER ANALYSIS FOR A LINKAGE DENSITY IN CELL ADHESION MECHANISMS *

Vuk Milisic[1]

Abstract. In this paper we present a non local age structured equation involved in cell motility modeling [4, 8, 10]. It models the evolution of a density of linkages of a point submitted to adhesion. It depends on an asymptotic parameter ε representing the characteristic age of linkages. Here we introduce a new initial layer term in the asymptotic expansion wrt ε. This improves error estimates obtained in [4]. Moreover, we study the convergence of the time derivative of this density and show how a singular term appears when ε goes to zero. We show convergence, in the tight topology of measures, to the time derivative of the limit solution and a Dirac mass supported on the initial half-axis. In order to illustrate theoretical results, direct numerical simulations are performed and compared to the asymptotic expansion for various values of ε.

Résumé. Dans cet article, nous analysons un problème structuré en âge avec un terme non local de saturation. Ce problème apparait dans la modélisation de la motilité et des mécanismes d'adhérence cellulaires [4, 8, 10]. L'équation dépend d'un paramètre asymptotique ε représentant l'âge caractéristique des liaisons. Ici, nous introduisons un nouveau terme de couche initiale dans le développement asymptotique par rapport à ε. Ceci améliore les estimations d'erreur obtenues dans [4]. En outre, nous étudions la convergence de la dérivée en temps de la densité des liaisons et montrons comment un terme singulier apparait quand ε devient nul. Nous montrons la convergence, dans la topologie étroite des mesures, vers la somme de la dérivée temporelle de la solution limite et d'une masse de Dirac supportée par le demi-axe initial. Des simulations numériques directes sont effectuées pour diverses valeurs d'ε et comparées au développement asymptotique afin d'illustrer les résultats théoriques.

Introduction

This work is related to the mathematical modeling of cell motility. Originally, a mechanical description of a network of actin filaments allowed to model the lamellopodium and was presented in [8–10]. From these, D. Oelz and the author extracted a toy model presented and extensively analyzed from the mathematical point of view in [4]. The model describes the motion of a single adhesion point submitted to some ascribed external force. This motion is described by z_ε, a position variable solving a Volterra equation

$$\begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_0^\infty (z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a))\,\rho_\varepsilon(a,t)\,da = f(t)\,, & t \geq 0\,, \\ z_\varepsilon(t) = z_p(t)\,, & t < 0\,, \end{cases} \tag{1}$$

* *Mohammed Boubekeur performed the numerical simulations illustrating the main results of the paper.*

[1] Laboratoire Analyse & Géométrie, Université Villeteuse Paris 13, France

coupled with a density of linkages ρ_ε weighting the adhesions already set and creating the pull-back force exerted from these (see (2) below). Then various mathematical problems were handled extending [4] : in [5], the authors weakened some of the hypotheses and rephrased the model in new variables, while in [6] the non-linear coupling was exposed and blow-up in finite time vs global existence was proved.

More recently, a model closer to the original description in [8] is analyzed [3]. This led to new questions, this work aims to answer one of those. We focus, here, on the age structured model governing the density of linkages ρ_ε. The age distribution $\rho_\varepsilon = \rho_\varepsilon(a,t)$ is the solution of the following system :

$$\begin{cases} \varepsilon\partial_t\rho_\varepsilon + \partial_a\rho_\varepsilon + \zeta_\varepsilon\,\rho_\varepsilon = 0\,, & a>0\,,\ t>0\,,\\ \rho_\varepsilon(0,t) = \beta_\varepsilon(t)\,(1-\mu_{0,\varepsilon})\,, & a=0,\ t>0\,,\\ \rho_\varepsilon(a,0) = \rho_I(a)\,, & a>0,\ t=0, \end{cases} \tag{2}$$

where $\mu_{k,\varepsilon}(t) := \int_0^\infty a^k\rho_\varepsilon(\tilde a,t)\,d\tilde a$ and the on-rate of bonds is a given coefficient β_ε times a factor, that takes into account saturation of the moving binding site with linkages. The off-rate ζ_ε is a prescribed function representing the death rate of the population for a given age a at a time t. The limit function $\rho_0 := \lim_{\varepsilon\to 0}\rho_\varepsilon$ is explicitly given by

$$\rho_0(a,t) = \frac{1}{\frac{1}{\beta_0(t)} + \int_0^\infty \exp\left(-\int_0^b \zeta_0(\tilde a,t)\,d\tilde a\right)db}\exp\left(-\int_0^a \zeta_0(\tilde a,t)\,d\tilde a\right), \tag{3}$$

being the solution of

$$\begin{cases} \partial_a\rho_0 + \zeta_0\,\rho_0 = 0\,, & t\geq 0\,,\quad a>0\,,\\ \rho_0(a=0,t) = \beta_0(t)\left(1-\displaystyle\int_0^\infty \rho_0(\tilde a,t)\,d\tilde a\right)\,, & t\geq 0\,. \end{cases} \tag{4}$$

The characteristic curves associated to (2) are straight lines parallel to $t=\varepsilon a$. Close to $S_0 := \{(a,t)\in \mathbb{R}_+\times\{0\}\}$, there is a set $S_\varepsilon := \{(a,t)\in(\mathbb{R}_+)^2\ ;\ \varepsilon a > t\}$, where the initial condition is transported and when ε goes to zero, S_ε collapses to S_0. The paper focusses precisely on the behavior of ρ_ε in this area. Namely in [4], the convergence of ρ_ε towards ρ_0 was shown in $C((0,T];L^1(\mathbb{R}_+))\cap L^1((0,T)\times\mathbb{R}_+)$: an *a priori* estimate was obtained leading to

$$\|\rho_\varepsilon(\cdot,t)-\rho_0(\cdot,t)\|_{L^1(\mathbb{R}_+)} \leq C\exp(-\zeta_{\min}t/\varepsilon) + o_\varepsilon(1). \tag{5}$$

where $\zeta_{\min}$ is the strictly positive lower bound of ζ_ε. Here, we enrich the asymptotic expansion with a supplementary term. We solve the *initial layer* problem : find $\tilde\rho_0$ solving

$$\begin{cases} \partial_{\tilde t}\tilde\rho_0 + \partial_a\tilde\rho_0 + \zeta_0(a,0)\tilde\rho_0 = 0, & (a,\tilde t)\in(\mathbb{R}_+)^2,\\ \tilde\rho_0(0,\tilde t) = -\beta_0(0)\displaystyle\int_{\mathbb{R}_+}\tilde\rho_0(a,\tilde t)da, & a=0,\ \tilde t>0,\\ \tilde\rho_0(a,0) = \rho_I(a)-\rho_0(a,0) =: \tilde\rho_I(a), & a>0, \tilde t=0. \end{cases} \tag{6}$$

This improves the error estimates above. Indeed for any $t\geq 0$, one has now :

$$\|\rho_\varepsilon(\cdot,t)-\rho_0(\cdot,t)-\tilde\rho_0(\cdot,t/\varepsilon)\|_{L^1(\mathbb{R}_+)} \lesssim o_\varepsilon(1).$$

The initial layer is a sort of *microscopic* lifting of the initial condition of the difference $\rho_\varepsilon-\rho_0$, it removes the exponential decay occurring in S_ε, which is visible on the first term of the rhs in (5). As the boundary term in (2) is non local, the analysis is not straightforward, and relies strongly on the specific energy functional introduced in [4].

Since ε mutiplies $\partial_t\rho_\varepsilon$ in (2), it is interesting to investigate to which space the time derivative $\partial_t\rho_\varepsilon$ belongs uniformly wrt ε. Another question of interest is to characterize its limit when ε goes to zero. We show in

Theorem 3.2 that $\partial_t \rho_\varepsilon$ belongs to the dual of $C_b([0,T]\times\mathbb{R}_+)$ (bounded continuous functions on $[0,T]\times\mathbb{R}_+$) uniformly wrt ε. Moreover, it tends to $\partial_t\rho_0$ and a Dirac mass supported by S_0 in the tight topology of measures (see Theorem 3.3 for a precise claim).

The paper is organized as follows : in Section 1 we present some notations, the main assumptions and useful results from [4], in Section 2, we analyse the initial layer, provide basic existence results and show that it is of bounded variation in time, so that the time derivative is related to a bounded Radon measure. We show some limits involving this measure used later on. In Section 3, we construct the zero order asymptotic expansion and show error estimates, we then prove that $\partial_t\rho_\varepsilon$ is also associated to a finite Radon measure and finally we provide error estimates in the total variation norm and show the main claim in Proposition 3.2 and Theorem 3.3. Numerical simulations are performed in order to illustrate these results in Section 4

1. Notations, hypotheses and previous results

1.1. Some notations

We set $Q_T := \mathbb{R}_+^* \times (0,T)$ and $\overline{Q}_T := \mathbb{R}_+ \times [0,T]$. The space of signed, locally bounded Radon measures $M^1_{\mathrm{loc}}(\overline{Q_T})$ is by Riesz' Theorem identified as the space of linear forms on $C_c(\overline{Q_T})$, the space of continuous functions with compact support in $\overline{Q_T}$. We call $C_0(\overline{Q_T})$ the space of continuous functions vanishing at infinity. The space of signed bounded Radon measures is denoted $M^1(\overline{Q_T}) = C_0(\overline{Q_T})^*$ (for more details cf [1, 7] and references therein). We define $C_b(\overline{Q_T})$, the space of bounded continuous functions on $\overline{Q_T}$. The absolute value applied to a measure denotes the total variation measure, *i.e.* if $\lambda \in M^1_{\mathrm{loc}}(\overline{Q_T})$ then by the Hahn decomposition, $\lambda = \lambda_+ - \lambda_-$ and $|\lambda| = \lambda_+ + \lambda_-$ where $\lambda_\pm$ are positive measures. For any function u defined a.e. $(a,t) \in Q_T$, we define the discrete time derivative operator D_t^τ

$$D_t^\tau u(a,t) := \frac{u(a,t+\tau) - u(a,t)}{\tau} \tag{7}$$

where τ is a small positive parameter.

In what follows, we put ourselves in a similar context as in [4]. For this sake we recall assumptions and main results useful for the rest of the paper.

1.2. Main assumptions

Assumptions 1.1. *The dimensionless parameter $\varepsilon > 0$ is assumed to induce two families of chemical rate functions that satisfy:*

(i) For any $T > 0$ the function β_ε is Lipschitz in $[0,T]$ (the Lipschitz constant is denoted β_{Lip}) and ζ_ε is in $\mathrm{Lip}_t([0,T]; L^\infty_a(\mathbb{R}_+))$ (resp. ζ_{Lip}).

(ii) For limit functions $\beta_0 \in W^{2,\infty}([0,T])$ and $\zeta_0 \in W^{2,\infty}([0,T]; L^\infty_a(\mathbb{R}_+))$, moreover it holds that

$$\|\zeta_\varepsilon - \zeta_0\|_{\mathrm{Lip}_t([0,T];L^\infty(\mathbb{R}_+))} \to 0 \quad \text{and} \quad \|\beta_\varepsilon - \beta_0\|_{W^{1,\infty}([0,T])} \to 0$$

as $\varepsilon \to 0$.

(iii) We also assume that there are upper and lower bounds such that

$$0 < \zeta_{\min} \le \zeta_\varepsilon(a,t) \le \zeta_{\max} \quad \text{and} \quad 0 < \beta_{\min} \le \beta_\varepsilon(t) \le \beta_{\max}$$

for all $\varepsilon > 0$, $a \ge 0$ and $t > 0$.

The initial data for the density model (2) satisfies some hypotheses that we sum up here:

Assumptions 1.2. *The initial condition $\rho_I \in L^\infty_a(\mathbb{R}_+)$ satisfies*

(i) positivity

$$\rho_I(a) \geq 0 \ , \qquad a.e. \ in \ \mathbb{R}_+ \ ,$$

moreover, one has also that the total initial population satisfies

$$0 < \int_{\mathbb{R}_+} \rho_I(a) da < 1 \ .$$

(ii) boundedness of higher moments,

$$0 < \int_{\mathbb{R}_+} a^p \rho_I(a) \ da \leq c_p \ , \quad for \ p = 1, 2 \ ,$$

where c_p are positive constants depending only on p.

(iii) further regularity : we assume that $\partial_a \rho_I \in L^1(\mathbb{R}_+)$, which together with the first hypotheses on ρ_I implies that $\rho_I \in W^{1,1}(\mathbb{R}_+)$.

1.3. **Useful existing results**

In this setting, one has existence and uniqueness as stated in Theorem 2.1 [4] recalled here for sake of self-compliance.

Theorem 1.1. *Let assumptions 1.1 and 1.2 hold, then for every fixed ε there exists a unique solution $\rho_\varepsilon \in C(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty((\mathbb{R}_+)^2)$ of the problem* (2)*. We say that ρ_ε is a mild solution since it satisfies* (2) *in the sense of characteristics, namely*

$$\rho_\varepsilon(a,t) = \begin{cases} \beta_\varepsilon(t-\varepsilon a)\left(1 - \int_0^\infty \rho_\varepsilon(\tilde{a}, t-\varepsilon a)\, d\tilde{a}\right) \times \\ \qquad\qquad \times \exp\left(-\int_0^a \zeta_\varepsilon(\tilde{a}, t-\varepsilon(a-\tilde{a}))\, d\tilde{a}\right) \ , & when \ a < t/\varepsilon \ , \\ \rho_I(a - t/\varepsilon) \exp\left(-\frac{1}{\varepsilon}\int_0^t \zeta_\varepsilon((\tilde{t}-t)/\varepsilon + a, \tilde{t})\, d\tilde{t}\right) \ , & if \ a \geq t/\varepsilon \ . \end{cases} \tag{8}$$

Moreover, it is a weak solution as well since it satisfies

$$\begin{aligned} \int_0^\infty \int_0^T \rho_\varepsilon(a,t) \left\{(\varepsilon\partial_t + \partial_a + \zeta_\varepsilon)\, \varphi(a,t) + \beta_\varepsilon(t)\varphi(0,t)\right\} \, dt \, da - \varepsilon \int_0^\infty \rho_\varepsilon(a,T)\varphi(a,T) \, da \\ = \int_0^T \beta_\varepsilon(t)\, \varphi(0,t) \, dt - \varepsilon \int_0^\infty \rho_I(a)\varphi(a,0) \, da = 0 \ , \end{aligned} \tag{9}$$

for every $T > 0$ and every test function $\varphi \in C^\infty((\mathbb{R}_+)^2) \cap L^\infty((\mathbb{R}_+)^2)$.

Lemma 1.1. *Let assumptions 1.1 and 1.2 hold, then the unique solution $\rho_\varepsilon \in C(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty((\mathbb{R}_+)^2)$ of the problem* (2) *from Theorem 1.1 satisfies*

$$\rho_\varepsilon(t,a) \geq 0 \quad a.e. \ in \quad (\mathbb{R}_+)^2 \quad and$$

$$\mu_{0,\min} \leq \mu_{0,\varepsilon}(t) < 1 \ , \quad \forall t \in \mathbb{R}_+ \quad where \quad \mu_{0,\min} := \min\left(\mu_{0,\varepsilon}(0), \frac{\beta_{\min}}{\beta_{\min} + \zeta_{\max}}\right). \tag{10}$$

Furthermore the following results on the convergence of ρ_ε as ε tends to 0 have been obtained. We define the functional

$$\mathcal{H}[u] := \left|\int_0^\infty u(a) \ da\right| + \int_0^\infty |u(a)| \ da \ , \tag{11}$$

and we obtain

Lemma 1.2. *Let $\zeta_{\min} > 0$ be the lower bound to $\zeta_\varepsilon(a,t)$ according to assumption 1.1, then it holds for all $t \geq 0$ that*

$$\mathcal{H}[\rho_\varepsilon(\cdot,t) - \rho_0(\cdot,t)] \leq \mathcal{H}[\rho_{\varepsilon,I} - \rho_0(\cdot,0)]e^{\frac{-\zeta_{\min} t}{\varepsilon}} + \frac{2}{\zeta_{\min}} \left\| \|\mathcal{R}_\varepsilon\|_{L^1_a(\mathbb{R}_+)} + |M_\varepsilon| \right\|_{L^\infty_t(\mathbb{R}_+)} \tag{12}$$

with $\mathcal{R}_\varepsilon := -\varepsilon\partial_t\rho_0 - \rho_0(\zeta_\varepsilon - \zeta_0)$ and $M_\varepsilon := (\beta_\varepsilon - \beta_0)\left(1 - \int_0^\infty \rho_0\, da\right)$.

As a consequence we conclude

Theorem 1.2. *Let ρ_ε be the solution to the system (2) according to Theorem 1.1 and let the ρ_0 be as defined in (3), then it holds that*

$$\rho_\varepsilon \to \rho_0 \quad in \quad C_{\text{loc}}(]0,\infty); L^1(\mathbb{R}_+)) \quad as \quad \varepsilon \to 0\,.$$

Remark 1.1. *Note that in general $\rho_{\varepsilon,I}$ does not converge to $\rho_0(\cdot,0)$ in L^1_a as $\varepsilon \to 0$. An initial layer will be observable and its profile will be shaped like a multiple of $e^{\frac{-\zeta_{\min} t}{\varepsilon}}$, which is again a consequence of Lemma 1.2.*

2. The initial layer

The limit solution ρ_0 can be seen as a crude approximation of ρ_ε for a fixed ε. Since ρ_0 does not depend on ρ_I, in a certain way it has *forgotten* the past. This is why we introduce a microscopic corrector solving (6).

2.1. Existence uniqueness and *a priori* estimates

The same analytic tools as above are applied in order to prove existence and uniqueness of $\tilde{\rho}_0$ on compact time intervals. We detail the global existence and boundedness.

Theorem 2.1. *Under hypotheses 1.1 and 1.2, there exists a unique solution $\tilde{\rho}_0 \in C(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^1(\mathbb{R}_+ \times \mathbb{R}_+) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$.*

Proof. Local existence in $C([0,T]; L^1(\mathbb{R}_+))$ is easy and follows the same lines as in Theorem 1.1. The global existence in $L^1(\mathbb{R}_+ \times \mathbb{R}_+)$ is more involved and we detail it here. Using the functional $\mathcal{H}$ one has that

$$\partial_t \mathcal{H}[\tilde{\rho}_0(\cdot,t)] + \zeta_{\min}\mathcal{H}[\tilde{\rho}_0(\cdot,t)] \leq 0$$

which by applying Grönwall's Lemma gives that

$$\mathcal{H}[\tilde{\rho}_0(\cdot,t)] \leq \exp(-\zeta_{\min}t)\mathcal{H}[\tilde{\rho}_0(\cdot,0)] = \exp(-\zeta_{\min}t)\mathcal{H}\left[\rho_I(\cdot) - \rho_0(\cdot,0)\right] \tag{13}$$

and this integrated in time gives the $L^1_t(\mathbb{R}_+)$ bound in time for $\mathcal{H}[\tilde{\rho}_0(t,\cdot)]$ this with the definition of $\mathcal{H}$ proves that $\tilde{\rho}_0 \in L^1(\mathbb{R}_+ \times \mathbb{R}_+)$ and that $\tilde{\mu}_0(t) := \int_{\mathbb{R}_+} \tilde{\rho}_0(a,t)da$ is in $L^1([0,T])$. Inequality (13) also proves that $\tilde{\mu}_0 \in L^\infty(\mathbb{R}_+)$ and thus $\tilde{\rho}_0 \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ since the initial and boundary values are bounded. Since on any compact interval in time $\tilde{\rho}_0$ is continuous with values in $L^1_a(\mathbb{R}_+)$, the claim follows. □

Corollary 2.1. *Under hypotheses 1.1 and 1.2, the unique solution $\tilde{\rho}_0 \in L^1(\mathbb{R}_+ \times \mathbb{R}_+)$ satisfies also*

$$\mathcal{H}[\tilde{t}\tilde{\rho}_0(\cdot,\tilde{t})] \in L^1_{\tilde{t}}(\mathbb{R}_+).$$

Proof. The proof is a consequence of (13), which multiplied by t and integrated in time provides the claim. □

2.2. The time derivative $\partial_t\tilde{\rho}_0$ is a signed bounded Radon measure

Theorem 2.2. *If $\zeta_\varepsilon(\cdot,0)$ is a function bounded from below, the time derivative of the initial layer $\partial_t\tilde{\rho}_0$ can be identified with a finite Radon measure $\lambda_{\partial_t\tilde{\rho}_0}$ and one has*

$$|\lambda_{\partial_t\tilde{\rho}_0}|((\mathbb{R}_+)^2) = \limsup_{\tau\to 0} \|D_t^\tau\tilde{\rho}_0\|_{L^1((\mathbb{R}_+)^2)} < \infty,$$

where the absolute value of the measure denotes its total variation. One has also that

$$\|(1+t)D_t^\tau\tilde{\rho}_0\|_{L^1((\mathbb{R}_+)^2)} < \infty$$

The proof is very similar to the proof of Theorem 3.2, and is left to the reader.

Proposition 2.1. *Under hypotheses 1.2, $\partial_t\tilde{\rho}_0$ extends to a linear continuous form on $C_b(\overline{Q_T})$ and the tight convergence holds up to a subsequence $(\tau_k)_{k\in\mathbb{N}}$:*

$$\lim_{k\to\infty}\int_{Q_T}\phi(a,t)D_t^{\tau_k}\tilde{\rho}_0 dadt = \int_{Q_T}\phi(a,t)d\lambda_{\partial_t\tilde{\rho}_0}(a,t)$$

for all functions $\phi\in C_b(\overline{Q_T})$.

Proof. The convergence occurs in the weak topology, *i.e.* for all $\phi\in C_0(\overline{Q_T})$ (cf p.11 Chap. 4, [7]). The continuous linear form extends to all $\phi\in C_b(\overline{Q_T})$, (see Lemma 3.3.2, p.11 [7]). The convergence occurs also in the tight topology of measures. Indeed, we consider

$$\int_0^T\int_R^\infty |D_t^\tau\tilde{\rho}_0|dadt \le \int_0^T\int_{R-T}|D_a^\tau\tilde{\rho}_I(a)| + C|\tilde{\rho}_I(a)|da \lesssim \|\tilde{\rho}_I\|_{W^{1,1}((R-T,\infty))}$$

By Lebesgues theorem as R goes to infinity the norm goes to zero. This shows that for every $\delta>0$ there exists a compact set $K=[0,T]\times[0,R]\in\overline{Q_T}$ s.t.

$$\left|\lambda_{D_t^\tau\tilde{\rho}_0}\right|(\overline{Q_T}\setminus K) < \delta, \quad \forall\tau\in(\tau_0,0],$$

which implies that $\lambda_{D_t^\tau\tilde{\rho}_0}$ tends towards $\lambda_{\partial_t\tilde{\rho}_0}$ in the tight topology up to the extraction of a subsequence (cf Lemma 3.4.5, p.14 [7]). □

Moreover one has

Proposition 2.2. *Under the same hypotheses as above, one has*

$$\lim_{\varepsilon\to 0}\int_{Q_{\frac{T}{\varepsilon}}}\varphi(a)d\lambda_{\partial_t\tilde{\rho}_0}(a,t) = -\int_{\mathbb{R}_+}\varphi(a)\tilde{\rho}_I(a)da$$

for every $\varphi\in C_b(\mathbb{R}_+)$.

Proof. Because $\tilde{\rho}_0$ is of bounded variation wrt time

$$\limsup_{\tau\to 0}\int_0^{T/\varepsilon}\int_{\mathbb{R}_+}|\varphi(a)|\,|D_t^\tau\tilde{\rho}_0(a,t)|\,da \le |\lambda_{\partial_t\tilde{\rho}_0}|\,\|\varphi\|_{L^\infty(\mathbb{R}_+)} < \infty$$

and thus

$$\limsup_{\tau\to 0}\int_0^{\frac{T}{\varepsilon}}\left|D_t^\tau\int_{\mathbb{R}_+}\varphi(a)\tilde{\rho}_0(a,t)da\right|dt \le \limsup_{\tau\to 0}\int_0^{\frac{T}{\varepsilon}}\int_{\mathbb{R}_+}|\varphi(a)|\,|D_t^\tau\tilde{\rho}_0(a,t)|\,dadt < \infty.$$

This shows that $q(t) := \int_{\mathbb{R}_+} \varphi(a)\tilde{\rho}_0(a,t)da$ is a function of bounded variation, thus there exists a Radon measure $\lambda_{\partial_t q}$ associated to the time derivative of q. The integral $\int_0^{T/\varepsilon} d\lambda_{\partial_t q}$ coincides with the Riemann-Stieltjes integral. Thus integration by parts holds, leading to

$$\int_0^{T/\varepsilon} d\lambda_{\partial_t q} = q(T/\varepsilon) - q(0) = \int_{\mathbb{R}_+} \varphi(a)\tilde{\rho}_0(a,T/\varepsilon)da - \int_{\mathbb{R}_+} \varphi(a)\tilde{\rho}_I(a)da.$$

On the other hand, D_t^τ and integration commute giving that

$$\int_{Q_{\frac{T}{\varepsilon}}} \varphi(a) D_t^\tau \tilde{\rho}_0(a,t)dadt = \int_0^{\frac{T}{\varepsilon}} D_t^\tau \left\{ \int_{\mathbb{R}_+} \varphi(a)\tilde{\rho}_0(a,t)da \right\} dt.$$

By Proposition 2.1, the lhs tightly converges to its limit, whereas the rhs converges weakly, leading to

$$\int_0^{\frac{T}{\varepsilon}} \int_{\mathbb{R}_+} \varphi(a) d\lambda_{\partial_t \tilde{\rho}_0}(a,t) = \int_0^{\frac{T}{\varepsilon}} d\lambda_{\left(\int_{\mathbb{R}_+} \varphi(a)\tilde{\rho}_0(a,t)da\right)},$$

which thus gives that

$$\int_0^{\frac{T}{\varepsilon}} \int_{\mathbb{R}_+} \varphi(a) d\lambda_{\partial_t \tilde{\rho}_0}(a,t) = \int_{\mathbb{R}_+} \varphi(a)\tilde{\rho}_0(a,T/\varepsilon)da - \int_{\mathbb{R}_+} \varphi(a)\tilde{\rho}_I(a)da.$$

The first term in the rhs can be estimated as

$$\left| \int_{\mathbb{R}_+} \varphi(a)\tilde{\rho}_0(a,T/\varepsilon)da \right| \leq \|\varphi\|_{L^\infty(\mathbb{R}_+)} \|\tilde{\rho}_0(\cdot,t)\|_{L^1(\mathbb{R}_+)} \leq C\exp(-T/\varepsilon),$$

which vanishes as ε goes to zero since $T > 0$. □

3. A complete zero order approximation

3.1. Error estimates on the density of linkages

Simple computations on the explicit solution ρ_0 of (4) given by (3) lead to

Lemma 3.1. *Under hypotheses 1.1 and 1.2, one has*

$$\max(|\rho_0(a,t)|, |\partial_t \rho_0(a,t)|, |\partial_{tt}\rho_0(a,t)|) \leq C\left(\|\beta_0\|_{W^{2,\infty}([0,T])}, \|\zeta_0\|_{W^{2,\infty}([0,T];L^\infty(\mathbb{R}_+))} \right) \exp(-\zeta_{\min} a),$$

for all (a,t) in Q_T.

We denote by $\tilde{\rho}_{0,\varepsilon}(a,t) := \tilde{\rho}_0(a,t/\varepsilon)$ and $\tilde{\mu}_{0,\varepsilon}(t) := \int_{\mathbb{R}_+} \tilde{\rho}_0(a,t/\varepsilon)da = \tilde{\mu}_0(t/\varepsilon)$. Now we define an approximation of ρ_ε solving (2),

$$\tilde{\rho}_\varepsilon(a,t) := \rho_0(a,t) + \tilde{\rho}_{0,\varepsilon}(a,t),$$

and we compute error estimates when accounting for the initial layer $\tilde{\rho}_0$.

Theorem 3.1. *Under hypotheses 1.1 and 1.2, one has for every fixed time $t > 0$:*

$$\mathcal{H}[(\rho_\varepsilon - \tilde{\rho}_\varepsilon)(\cdot,t)] \leq o_\varepsilon(1)$$

with respect to ε. Thus $\rho_\varepsilon - \tilde{\rho}_\varepsilon$ tends to zero in $L^\infty((0,T);L^1(\mathbb{R}_+))$ strongly when ε goes to zero.

Proof. We set $\hat{\rho}_\varepsilon(a,t) := \rho_\varepsilon(a,t) - \tilde{\rho}_\varepsilon(a,t)$ and write the problem it satisfies :

$$\begin{cases} \varepsilon\partial_t\hat{\rho}_\varepsilon + \partial_a\hat{\rho}_\varepsilon + \zeta_\varepsilon(a,t)\hat{\rho}_\varepsilon = \mathcal{R}_\varepsilon\ , & a>0\,,\ t>0\,, \\ \hat{\rho}_\varepsilon(a=0,t) = -\beta_\varepsilon(t)\displaystyle\int_0^\infty \hat{\rho}_\varepsilon(\tilde{a},t)\,d\tilde{a} + M_\varepsilon\ , & t>0\,, \\ \hat{\rho}_\varepsilon(a,t=0) = 0, & a\geq 0\,, \end{cases} \tag{14}$$

with

$$\begin{cases} \mathcal{R}_\varepsilon := -\varepsilon\partial_t\rho_0 - \rho_0(\zeta_\varepsilon-\zeta_0) + (\zeta_0(a,0)-\zeta_\varepsilon(a,t))\,\tilde{\rho}_{0,\varepsilon}, \\ M_\varepsilon := (\beta_\varepsilon-\beta_0)\left(1-\displaystyle\int_0^\infty \rho_0\,da\right) - (\beta_\varepsilon(t)-\beta_0(0))\,\tilde{\mu}_{0,\varepsilon}. \end{cases}$$

Applying the same estimates as above one concludes that :

$$\varepsilon\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}[\hat{\rho}_\varepsilon(\cdot,t)] + \zeta_{\min}\mathcal{H}[\hat{\rho}_\varepsilon(\cdot,t)] \leq \int_{\mathbb{R}_+} |\mathcal{R}_\varepsilon|da + |M_\varepsilon| =: I_1 + I_2.$$

Now we decompose the rests inside the rhs

$$\begin{aligned} I_1 &\leq \varepsilon\int_{\mathbb{R}_+} |\partial_t\rho_0|\,da + \|\zeta_\varepsilon-\zeta_0\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}_+)} + I_{1,1} \leq o_\varepsilon(1) + I_{1,1}, \\ I_2 &\leq \|\beta_\varepsilon-\beta_0\|_{L^\infty_t(\mathbb{R}_+)} + I_{2,1} \leq o_\varepsilon(1) + I_{2,1}, \end{aligned}$$

where the latter terms are due essentially to errors introduced by our new initial layer :

$$I_{1,1}(t) := \int_{\mathbb{R}_+} |(\zeta_0(a,0)-\zeta_\varepsilon(a,t))\,\tilde{\rho}_0(a,t/\varepsilon)|\,da, \quad I_{2,1}(t) := |(\beta_\varepsilon(t)-\beta_0(0))\,\tilde{\mu}_0(t/\varepsilon)|\,.$$

Using Grönwall's Lemma and the fact that $\mathcal{H}[\hat{\rho}_\varepsilon(\cdot,0)] = 0$ one has

$$\mathcal{H}[\hat{\rho}_\varepsilon(\cdot,t)] \leq \frac{1}{\varepsilon}\int_0^t \left\{\exp\left(-\zeta_{\min}\frac{(t-s)}{\varepsilon}\right)(I_{1,1}(s)+I_{2,1}(s))\right\}ds + o_\varepsilon(1)$$

Now we detail the second term in the rhs, the first follows the same arguments :

$$\begin{aligned} \frac{1}{\varepsilon}\int_0^t \exp\left(-\zeta_{\min}\frac{(t-s)}{\varepsilon}\right) I_{2,1}(s)ds &\leq \int_0^{t/\varepsilon} \{|\beta_\varepsilon(\varepsilon\tilde{s})-\beta_0(\varepsilon\tilde{s})| + |\beta_0(\varepsilon\tilde{s})-\beta_0(0)|\}\,|\tilde{\mu}_0(\tilde{s})|\,d\tilde{s} \\ &\leq \|\beta_\varepsilon-\beta_0\|_{L^\infty(0,T)}\int_{\mathbb{R}_+}|\tilde{\mu}_0(\tilde{s})|\,d\tilde{s} + \varepsilon\|\beta_0'\|_{L^\infty(0,T)}\int_{\mathbb{R}_+}\tilde{s}\,|\tilde{\mu}_0(\tilde{s})|\,d\tilde{s} \\ &\leq o_\varepsilon(1)\|\mathcal{H}[\tilde{\rho}_\varepsilon]\|_{L^1(\mathbb{R}_+)} + \varepsilon\|\beta_0'\|_{L^\infty(0,T)}\big\|\mathcal{H}[\tilde{t}\tilde{\rho}_\varepsilon]\big\|_{L^1_{\tilde{t}}(\mathbb{R}_+)} \leq o_\varepsilon(1) \end{aligned}$$

where we assumed that β_ε is uniformly Lipschitz in $[0,T]$ according to hypotheses 1.1, and we used Theorem 2.1 and its Corollary 2.1. □

Remark 3.1. *This result is to be compared with Lemma 1.2 and Theorem 1.2. It shows that the addition of an initial layer improves the convergence result for small times since if $t\sim\varepsilon$ the first term in the rhs of* (12) *is of order 1.*

Remark 3.2. *This result is useful since it shows that we found an approximation of the actual initial layer depending only on the data ζ_0, β_0 and ρ_I at $t=0$.*

3.2. **Variation in time**

Considering (2), ε multiplies $\partial_t \rho_\varepsilon$. As we are interested in the convergence of ρ_ε when ε goes to zero, one could ask where does $\partial_t \rho_\varepsilon$ belongs uniformly wrt ε and to which limit does it tend. To this aim we consider the system satisfied by $D_t^\tau \rho_\varepsilon$ (the operator is defined as in (7)) :

$$\begin{cases} (\varepsilon\partial_t + \partial_a + \zeta_\varepsilon) D_t^\tau \rho_\varepsilon = -\left(D_t^\tau \zeta_\varepsilon\right) \rho_\varepsilon(a,t), & (a,t) \in Q_T \\ D_t^\tau \rho_\varepsilon(0,t) = -\beta_\varepsilon \displaystyle\int_{\mathbb{R}_+} D_t^\tau \rho_\varepsilon(t,\tilde{a}) d\tilde{a} + \left(D_t^\tau \beta_\varepsilon\right)(1-\mu_{0,\varepsilon}), & a = 0,\ t > 0 \\ D_t^\tau \rho_\varepsilon(a,0) = \dfrac{\rho_\varepsilon(a,\tau) - \rho_I(a)}{\tau}, & a > 0,\ t = 0 \end{cases} \tag{15}$$

Theorem 3.2. *Under hypotheses 1.1 and 1.2, and for every fixed τ small enough, one has :*

$$\mathcal{H}\left[D_t^\tau \rho_\varepsilon(\,\cdot, t)\right] \leq C_1 \exp(-\zeta_{\min} t/\varepsilon)/\varepsilon + C_2$$

which gives the uniform bound in ε :

$$\|D_t^\tau \rho_\varepsilon\|_{L^1((0,T)\times\mathbb{R}_+)} \leq C_3.$$

where the constants C_i are independent on ε, for $i \in \{1,2,3\}$.

Proof. One uses $\mathcal{H}$, the functional introduced above and gets using Grönwall's Lemma that

$$\mathcal{H}\left[D_t^\tau \rho_\varepsilon(\,\cdot, t)\right] \leq \exp(-\zeta_{\min} t/\varepsilon)\mathcal{H}\left[D_t^\tau \rho_\varepsilon(\,\cdot, 0)\right] + 2\left(\zeta_{\text{Lip}} + \beta_{\text{Lip}}\right)/\zeta_{\min} \tag{16}$$

The main point of the proof is the control of the initial term $\mathcal{H}\left[D_t^\tau \rho_\varepsilon(\,\cdot, 0)\right]$ as a function of ε and τ.

$$\mathcal{H}\left[D_t^\tau \rho_\varepsilon(\,\cdot, 0)\right] = \int_{\mathbb{R}_+} \left|\frac{\rho_\varepsilon(a,\tau) - \rho_I(a)}{\tau}\right| da + \left|\frac{\mu_{0,\varepsilon}(\tau) - \mu_{0,\varepsilon}(0)}{\tau}\right| =: I_1 + I_2$$

The first term decomposes in two parts

$$\begin{aligned} I_1 =& \int_0^{\tau/\varepsilon} \left|\frac{\rho_\varepsilon(a,\tau) - \rho_I(a)}{\tau}\right| da + \int_{\tau/\varepsilon}^{\infty} \left|\frac{\rho_\varepsilon(a,\tau) - \rho_I(a)}{\tau}\right| da \\ =& \frac{1}{\tau}\int_0^{\tau/\varepsilon} \left|\beta_\varepsilon(\tau-\varepsilon a)(1-\mu_{0,\varepsilon}(\tau-\varepsilon a)\exp\left(-\int_{-a}^0 \zeta_\varepsilon(a+s,\tau+\varepsilon s)ds\right) - \rho_I(a)\right| da \\ &+ \frac{1}{\tau}\int_{\tau/\varepsilon}^{\infty} \left|\rho_I(a-\tau/\varepsilon)\exp\left(-\int_{-\tau/\varepsilon}^0 \zeta_\varepsilon(a+s,\tau+\varepsilon s)ds\right) - \rho_I(a)\right| da = I_{1,1} + I_{1,2} \end{aligned}$$

where we used the method of characteristics. One splits the first term adding and subtracting intermediate terms

$$I_{1,1} \lesssim \frac{1}{\tau}\int_0^{\tau/\varepsilon} d\tau \|\rho_\varepsilon\|_{L^\infty(Q_T)} \lesssim \frac{1}{\varepsilon}$$

As for $I_{1,2}$, one has :

$$I_{1,2} \leq \frac{1}{\varepsilon}\left(\text{TV}(\rho_I) + C\|\rho_I\|_{L^1(\mathbb{R}_+)}\right) \lesssim \frac{1}{\varepsilon}\|\rho_I\|_{W^{1,1}(\mathbb{R}_+)}$$

where TV denotes total variation of ρ_I [11]. In a similar way, as $|\varepsilon\partial_t \mu_{0,\varepsilon}(t)| \leq C$, one obtains

$$I_2 \leq \frac{1}{\varepsilon}\left(\beta_{\max} + \zeta_{\max}\right),$$

which ends the first part of the proof, then integrating (16) in time gives the other claim. □

Proposition 3.1. *There exists a unique regular measure $\lambda_{\partial_t\rho_\varepsilon} \in M^1(\overline{Q_T})$ associated to $\partial_t\rho_\varepsilon$. Moreover $\partial_t\rho_\varepsilon$ is also a linear functional on $C_b(\overline{Q_T})$, for a fixed ε, one has*

$$\int_{Q_T} \varphi(a,t) D_t^\tau \rho_\varepsilon(a,t) da dt \to \int_{Q_T} \varphi(a,t) d\lambda_{\partial_t\rho_\varepsilon}, \quad \forall \varphi \in C_b(\overline{Q_T}).$$

when τ goes to zero (up to a subsequence).

Proof. By Theorem 1.7.2. [2], there exists a finite Radon measure s.t.

$$|\lambda_{\partial_t\rho_\varepsilon}|(Q_T) = \limsup_{\tau\to 0} \int_{Q_T} |D_t^\tau \rho_\varepsilon| \, da dt < \infty.$$

The distribution $\partial_t\rho_\varepsilon$ belongs to $M^1(\overline{Q_T})$. By similar arguments as in Proposition 2.1, one extends $D_t^\tau\rho_\varepsilon$ and $\partial_t\rho_\varepsilon$ into a linear continuous map on $C_b(\overline{Q_T})$, and shows the tight convergence. □

3.3. **Error estimates for the time derivative of the density of linkages**

Now we prove the key point of this paper :

Proposition 3.2. *Under assumptions 1.1 and 1.2, setting $\hat\rho_\varepsilon := \rho_\varepsilon - \rho_0 - \tilde\rho_{0,\varepsilon}$, one has that*

$$\left|\lambda_{\partial_t\rho_\varepsilon} - \lambda_{\partial_t\rho_0} - \lambda_{\partial_t\tilde\rho_{0,\varepsilon}}\right|(Q_T) = \limsup_{\tau\to 0} \|D_t^\tau \hat\rho_\varepsilon\|_{L^1(Q_T)} \sim o_\varepsilon(1)$$

Proof. The proof follows the same line as above, nevertheless since it is a crucial estimate, we give the details of the computations. One applies the discrete time operator D_t^τ to $\hat\rho_\varepsilon$ solving (14) and uses the functional $\mathcal{H}$ in the same spirit as above. This leads to

$$\begin{aligned}
\frac{\varepsilon}{2}\partial_t \mathcal{H}[D_t^\tau\hat\rho_\varepsilon(\cdot,t)] + \frac{\zeta_{\min}}{2}\mathcal{H}[D_t^\tau\hat\rho_\varepsilon(\cdot,t)] &\le \int_{\mathbb{R}_+} |D_t^\tau\zeta_\varepsilon||\hat\rho_\varepsilon| + |D_t^\tau\mathcal{R}_\varepsilon| da + |D_t^\tau\beta_\varepsilon|\mathcal{H}[\hat\rho_\varepsilon(\cdot,t)] + |D_t^\tau M_\varepsilon| \\
&\lesssim (\zeta_{\mathrm{Lip}} + \beta_{\mathrm{Lip}})\,\mathcal{H}[\hat\rho_\varepsilon(\cdot,t)] + \varepsilon\|\partial_{tt}\rho_0\|_{L^1(\mathbb{R}_+)} + \int_{\mathbb{R}_+} |\partial_t((\zeta_\varepsilon - \zeta_0)\rho_0)| da + |\partial_t(\beta_\varepsilon - \beta_0)| \\
&+ (\zeta_{\mathrm{Lip}} + \beta_{\mathrm{Lip}}) \int_{\mathbb{R}_+} |\tilde\rho_{0,\varepsilon}(a,t)|\, da + \int_{\mathbb{R}_+} |\zeta_\varepsilon(a,t) - \zeta_0(a,0)||D_t^\tau\tilde\rho_{0,\varepsilon}| da + |\beta_\varepsilon(t) - \beta_0(0)|\,|D_t^\tau\tilde\mu_{0,\varepsilon}| \\
&\lesssim o_\varepsilon(1) + \underbrace{\int_{\mathbb{R}_+} |\zeta_\varepsilon(a,t) - \zeta_0(a,0)||D_t^\tau\tilde\rho_{0,\varepsilon}| da + |\beta_\varepsilon(t) - \beta_0(0)|D_t^\tau\tilde\mu_{0,\varepsilon}|}_{=:I_\tau(t)} + \mathcal{H}[\tilde\rho_{0,\varepsilon}(\cdot,t)].
\end{aligned}$$

Applying Grönwall's Lemma, one obtains

$$\mathcal{H}[D_t^\tau\hat\rho_\varepsilon] \lesssim \exp(-\zeta_{\min}t/\varepsilon)\mathcal{H}[D_t^\tau\hat\rho_\varepsilon(\cdot,0)] + o_\varepsilon(1) + \frac{1}{\varepsilon}\int_0^t (I_\tau(s) + \mathcal{H}[\tilde\rho_{0,\varepsilon}(\cdot,s)]) \exp\left(-\zeta_{\min}(t-s)/\varepsilon\right) ds, \qquad (17)$$

which integrated once again in time provides :

$$\int_0^T \mathcal{H}[D_t^\tau\hat\rho_\varepsilon] dt \le \varepsilon\mathcal{H}[D_t^\tau\hat\rho_\varepsilon(\cdot,0)] + o_\varepsilon(1) + \frac{1}{\varepsilon}\int_0^T\int_0^t (I_\tau(s) + \mathcal{H}[\tilde\rho_{0,\varepsilon}(\cdot,s)]) \exp\left(-\zeta_{\min}(t-s)/\varepsilon\right) ds dt$$

again as in the proof of Theorem 3.1, one uses scaling arguments and obtain :

$$\frac{1}{\varepsilon}\int_0^T\int_0^t I_\tau(s) \exp\left(-\zeta_{\min}(t-s)/\varepsilon\right) ds dt \le o_\varepsilon(1)\|(1+t)D_t^\tau\tilde\rho_0\|_{L^1((\mathbb{R}_+)^2)} \sim o_\varepsilon(1).$$

In the same way, for the term containing $\mathcal{H}[\tilde{\rho}_{0,\varepsilon}(\cdot,t)]$ above,

$$\frac{1}{\varepsilon}\int_0^T\int_0^t \mathcal{H}[\tilde{\rho}_0(\cdot,s/\varepsilon)]\exp\left(-\zeta_{\min}(t-s)/\varepsilon\right)dsdt \lesssim \varepsilon\int_0^{T/\varepsilon}\mathcal{H}[\tilde{\rho}_0(\cdot,s)]ds \lesssim \varepsilon\|\mathcal{H}[\tilde{\rho}_0]\|_{L^1\mathbb{R}_+)}.$$

Now we concentrate on what remains near S_0

$$\mathcal{H}[D_t^\tau\hat{\rho}_\varepsilon(\cdot,0)] \leq \frac{2}{\tau}\left\{\int_0^{\tau/\varepsilon}|\hat{\rho}_\varepsilon(a,\tau)|da + \int_{\tau/\varepsilon}^\infty|\hat{\rho}_\varepsilon(a,\tau)|da\right\} =: J_1+J_2.$$

We use the method of characteristics in order to express J_1 and J_2 as functions of initial and boundary values.

$$J_1 \leq \frac{2}{\tau}\left\{\int_0^{\tau/\varepsilon}\left(|\hat{\rho}_\varepsilon(0,\tau-\varepsilon a)| + \int_{-a}^0|\mathcal{R}_\varepsilon(a+s,\tau+\varepsilon s)|ds\right)da\right\} =: J_{1,1}+J_{1,2}.$$

The boundary term is given by (14), leading to

$$\begin{aligned}
J_{1,1} &\leq \frac{1}{\tau}\int_0^{\tau/\varepsilon}\beta_{\max}\mathcal{H}[\hat{\rho}_\varepsilon(\cdot,\tau-\varepsilon a)]da + \frac{1}{\tau}\int_0^{\tau/\varepsilon}\|\beta_\varepsilon-\beta_0\|_{L^\infty(0,T)} + |\beta_\varepsilon(\tau-\varepsilon a)-\beta_0(0)|\,|\tilde{\mu}_{0,\varepsilon}(\tau-\varepsilon a)|\,da\\
&\lesssim \frac{o_\varepsilon(1)}{\varepsilon} + \frac{1}{\tau}\int_0^{\tau/\varepsilon}|\beta_0(\tau-\varepsilon a)-\beta_0(0)|\,|\tilde{\mu}_{0,\varepsilon}(\tau-\varepsilon a)|\,da \lesssim \frac{o_\varepsilon(1)}{\varepsilon} + \frac{1}{\varepsilon\tau}\int_0^\tau|\beta_0(s)-\beta_0(0)|\,|\tilde{\mu}_{0,\varepsilon}(s)|\,ds\\
&\lesssim \frac{o_\varepsilon(1)}{\varepsilon} + \frac{\beta_{\mathrm{Lip}}}{\varepsilon\tau}\int_0^\tau s\exp(-\zeta_{\min}s/\varepsilon)ds \leq \frac{o_\varepsilon(1)}{\varepsilon}+C,
\end{aligned}$$

the latter constant depends neither on ε nor on τ. Then

$$\begin{aligned}
J_2 \leq& \frac{1}{\tau}\int_{\tau/\varepsilon}^\infty\int_{-\tau/\varepsilon}^0|\mathcal{R}_\varepsilon(a+s,\tau+\varepsilon s)|dsda \leq \frac{1}{\tau}\int_0^{\tau/\varepsilon}\|\mathcal{R}_\varepsilon(\cdot,\tau-\varepsilon s)\|_{L^1(\mathbb{R}_+)}ds\\
&\lesssim \frac{o_\varepsilon(1)}{\varepsilon} + \frac{1}{\tau}\int_0^{\tau/\varepsilon}\int_{\mathbb{R}_+}|\zeta_0(\tau-\varepsilon s,a)-\zeta_0(a,0)|\,|\tilde{\rho}_{0,\varepsilon}(a,\tau-\varepsilon s)|dads\\
&\lesssim \frac{o_\varepsilon(1)}{\varepsilon} + \frac{1}{\varepsilon\tau}\int_0^{\tau}\int_{\mathbb{R}_+}|\zeta_0(s,a)-\zeta_0(a,0)|\,|\tilde{\rho}_{0,\varepsilon}(a,s)|dads\\
&\lesssim \frac{o_\varepsilon(1)}{\varepsilon} + \frac{1}{\tau}\int_0^{\tau/\varepsilon}\int_{\mathbb{R}_+}|\zeta_0(\varepsilon s,a)-\zeta_0(a,0)|\,|\tilde{\rho}_0(a,s)|dads\\
&\lesssim \frac{o_\varepsilon(1)}{\varepsilon} + \frac{\varepsilon\zeta_{\mathrm{Lip}}}{\tau}\int_0^{\tau/\varepsilon}s\left(\int_{\mathbb{R}_+}|\tilde{\rho}_0(a,s)|da\right)ds \lesssim \frac{o_\varepsilon(1)}{\varepsilon} + \|\mathcal{H}[\tilde{\rho}_0]\|_{L_t^\infty(\mathbb{R}_+)}
\end{aligned}$$

the latter term being bounded (cf. Theorem 2.1). A similar approach provides the same bound for $J_{1,2}$. This finally proves that

$$\limsup_{\tau\to0}\int_0^T\mathcal{H}[D_t^\tau\hat{\rho}_\varepsilon]dt \leq o_\varepsilon(1)$$

which is the claim. □

Now we combine the latter result together with Proposition 2.2 in order to obtain

Theorem 3.3. *Under hypotheses 1.1 and 1.2, when ε goes to zero, one has that*

$$\partial_t\rho_\varepsilon \Rightarrow \partial_t\rho_0 - \delta_{t=0}\left(\rho_I(a)-\rho_0(a,0)\right)$$

in the sense of tight convergence, i.e.

$$\int_{Q_T} \varphi(a,t)\lambda_{\partial_t \rho_\varepsilon}(a,t) \to \int_{Q_T} \varphi(a,t)\partial_t \rho_0(a,t) da dt - \int_{\mathbb{R}_+} \varphi(a,0)(\rho_I(a) - \rho_0(a,0)) da$$

for all $\varphi \in C_b(\overline{Q_T})$.

Proof. The convergence of measures in the total variation norm is stronger and implies tight convergence. Moreover, by a triangular inequality, one has :

$$\begin{aligned}
&\left| \int_{Q_T} \varphi(a,t)\lambda_{\partial_t \rho_\varepsilon}(a,t) - \int_{Q_T} \varphi(a,t)\partial_t \rho_0(a,t) da dt + \int_{\mathbb{R}_+} \varphi(a,0)(\rho_I(a) - \rho_0(a,0)) da \right| \\
\leq &\left| \int_{Q_T} \varphi(a,t)\lambda_{\partial_t \rho_\varepsilon}(a,t) - \int_{Q_T} \varphi(a,t)\partial_t \rho_0(a,t) da dt - \int_{Q_T} \varphi(a,t) d\lambda_{\partial_t \tilde{\rho}_{0,\varepsilon}}(a,t) \right| \\
&+ \left| \int_{Q_T} \varphi(a,t) d\lambda_{\partial_t \tilde{\rho}_{0,\varepsilon}}(a,t) + \int_{\mathbb{R}_+} \varphi(a,0)(\rho_I(a) - \rho_0(a,0)) da \right| \leq o_\varepsilon(1) + I
\end{aligned}$$

Expressing again the term containing $\lambda_{\partial_t \tilde{\rho}_{0,\varepsilon}}$ as the limit of the discrete operator D_t^τ, one gets :

$$\int_{Q_T} \varphi(a,t) d\lambda_{\partial_t \tilde{\rho}_{0,\varepsilon}}(a,t) = \int_{Q_{\frac{T}{\varepsilon}}} \varphi(a,\varepsilon\tilde{t}) d\lambda_{\partial_{\tilde{t}} \tilde{\rho}_0}(a,\tilde{t})$$

Because $\lambda_{\partial_t \tilde{\rho}_0}$ is a signed measure on $(\mathbb{R}_+)^2$ one has that

$$\left| \int_{Q_{\frac{T}{\varepsilon}}} \left(\varphi(a,\varepsilon\tilde{t}) - \varphi(a,0)\right) d\lambda_{\partial_t \tilde{\rho}_0}(a,\tilde{t}) \right| \leq \int_{Q_{\frac{T}{\varepsilon}}} \left|\varphi(a,\varepsilon\tilde{t}) - \varphi(a,0)\right| d|\lambda_{\partial_t \tilde{\rho}_0}|(a,\tilde{t})$$

The difference $\left|\varphi(a,\varepsilon\tilde{t}) - \varphi(a,0)\right| \chi_{Q_{\frac{T}{\varepsilon}}}(a,t)$ is a bounded function converging pointwisely to zero everywhere in $(\mathbb{R}_+)^2$ and $\lambda_{\partial_t \tilde{\rho}_0}$ is a finite measure, so applying Vitali's convergence Theorem proves that the rhs tends to zero as ε goes to zero. Thus

$$I \leq \left| \int_{Q_{\frac{T}{\varepsilon}}} \left(\varphi(a,\varepsilon\tilde{t}) - \varphi(a,0)\right) d\lambda_{\partial_t \tilde{\rho}_0}(a,\tilde{t}) \right| + \left| \int_{Q_{\frac{T}{\varepsilon}}} \varphi(a,0) d\lambda_{\partial_t \tilde{\rho}_0}(a,\tilde{t}) + \int_{\mathbb{R}_+} \varphi(a,0)(\rho_I(a) - \rho_0(a,0)) da \right|$$

tends to zero as well thanks to Proposition 2.2. □

4. A numerical illustration

4.1. **Numerical discretization**

Let's consider a bounded age domain $A = [0, A_{max}]$. It is discretized into control volumes $A_i = [a_{i-\frac{1}{2}}, a_{i+\frac{1}{2}}]$ with uniform size $\Delta a = a_{i+\frac{1}{2}} - a_{i-\frac{1}{2}}$. The number of age cels is denoted $N_a = A/\Delta a$. The time domain is subdivided into intervals $[t_n, t_{n+1}]$ with time step $\Delta t = t_{n+1} - t_n$. An upwind finite volume scheme is used in

order to approximate the resolution of (2). So the numerical scheme reads :

$$\begin{cases} \varepsilon \dfrac{\rho_{\varepsilon,i}^{n+1} - \rho_{\varepsilon,i}^{n}}{\Delta t} + \dfrac{\rho_{\varepsilon,i}^{n} - \rho_{\varepsilon,i-1}^{n}}{\Delta a} + \zeta_{\varepsilon,i}^{n+1}\, \rho_{\varepsilon,i}^{n+1} = 0\,, \quad n \geq 0\,,\ i > 0\,, \\ \rho_{\varepsilon,0}^{n+1} = \beta_{\varepsilon}^{n}\,(1 - \mu_{\varepsilon}^{n})\,, \qquad\qquad n \geq 0\,, \\ \rho_{\varepsilon,i}^{0} = \rho_{I,i}\,, \qquad\qquad i \geq 0\,, \end{cases} \tag{18}$$

where $\rho_{\varepsilon,i}^{n} \sim \dfrac{1}{\Delta a \Delta t} \displaystyle\int_{t_n}^{t_{n+1}} \int_{a_{i-\frac{1}{2}}}^{a_{i+\frac{1}{2}}} \rho_{\varepsilon}(\tilde{a}, t) d\tilde{a} dt$ is a piecewise constant approximation of ρ_{ε}, while $\zeta_{\varepsilon,i}^{n} := \zeta_{\varepsilon}(t_n, a_i)$, $\beta_{\varepsilon}^{n} := \beta_{\varepsilon}(t_n)$, $\rho_{I,i} := \rho_I(a_i)$ and $\mu_{\varepsilon}^{n} = \mathfrak{S}(\rho_{\varepsilon,.}, i, N_a)$. The latter function denotes the trapezoidal rule approximating the integral :

$$\mathfrak{S}(\rho_{\varepsilon,.}^{n}, i, N_a) := \frac{\Delta a}{2} \sum_{i=0}^{N_a - 1} \left(\rho_{\varepsilon,i}^{n} + \rho_{\varepsilon,i+1}^{n}\right).$$

The approximation of (3) is given by:

$$\rho_{0,i}^{n} = \frac{1}{\frac{1}{\beta_0^n} + \mathfrak{S}(\exp(-\mathfrak{S}(\zeta_{0,.}^{n}, j, i)), i, N_a)} \exp\left(-\mathfrak{S}(\zeta_{0,.}^{n}, k, i)\right), \tag{19}$$

where $\rho_{0,i}^{n} = \rho_0(t_n, a_i)$, $\beta_0^n = \beta_0(t_n)$. For the approximation of (6), the time domain is subdivided into intervals $[\tilde{t}_n, \tilde{t}_{n+1}]$ with time step $\Delta\tilde{t} = \tilde{t}_{n+1} - \tilde{t}_n$. Again the upwind finite volume scheme is used. The numerical scheme is

$$\begin{cases} \dfrac{\tilde{\rho}_{0,i}^{n+1} - \tilde{\rho}_{0,i}^{n}}{\Delta\tilde{t}} + \dfrac{\tilde{\rho}_{0,i}^{n} - \tilde{\rho}_{0,i-1}^{n}}{\Delta a} + \zeta_{0,i}^{0}\, \tilde{\rho}_{0,i}^{n+1} = 0\,, \quad n \geq 0\,,\ i > 0\,, \\ \tilde{\rho}_{0,0}^{n+1} = -\beta_0^0 \mathfrak{S}(\tilde{\rho}_{0,.}^{n}, i, N_a)\,, \qquad\qquad n \geq 0\,, \\ \tilde{\rho}_{0,i}^{0} = \rho_{I,i} - \rho_{0,i}\,, \qquad\qquad i \geq 0\,, \end{cases} \tag{20}$$

where $\tilde{\rho}_{0,i}^{n} \sim \dfrac{1}{\Delta a \Delta t} \displaystyle\int_{t-n}^{t_{n+1}} \int_{a_{i-\frac{1}{2}}}^{a_{i+\frac{1}{2}}} \tilde{\rho}_0(\tilde{a}, \tilde{t}^n) d\tilde{a}$, $\zeta_{0,i}^{0} = \zeta_0(0, a_i)$and $\beta_0^0 = \beta_0(0)$.

Using the approximations (18), we compute ρ_{ε} for different values of ϵ. And ρ_{ε}, ρ_0 and $\tilde{\rho}_0$ using (19) and (20). The time steps are defined as $\Delta t = \epsilon \Delta a$ and $\Delta\tilde{t} = \Delta a$, we obtain $\tilde{\rho}_{\varepsilon}^{n} = \rho_0^n + \tilde{\rho}_0^n$.

4.2. **Results**

The convergence results is calculated with a discrete formulation of (11) defined as:

$$H_{\Delta}[u] := |\mathfrak{S}(u., i, N_a)| + \mathfrak{S}(|u|., i, N_a), \tag{21}$$

where u is a real sequence $(u_i)_{i \in \mathbb{N}}$.

4.2.1. *Pure initial layer*

In the simulations, one uses the on and off rates $\beta_{\varepsilon}(t) = \beta_{\min} + (\sin(2\pi t))_+$ and $\zeta_{\varepsilon}(a, t) = (1 + a)(1 + t)$, while the initial data is $\rho_I(a) := \exp(-a)$. $A_{\max} := 1$, $\Delta a := 1e - 3$ and $\Delta t := \varepsilon \Delta a$.

Numerical results agree in a close manner to theoretical estimates stated in Theorems 1.2 and 3.1, for the L^1 and L^∞ norms either for $\rho_{\varepsilon} - \rho_0$ or for the complete expansion $\rho_{\varepsilon} - \rho_0 - \tilde{\rho}_{0,\varepsilon}$ (cf Fig. 1). We see that even in the $L^1(0, T)$ norm the complete zero order approximation is closer to $\rho_{\varepsilon,.}^{n}$ although the convergence order is similar.

More interestingly (cf Fig. 2), when considering the discrete time derivative *i.e.* $D_t^{\Delta} \rho_{\varepsilon,.}^{n} := ((\rho_{\varepsilon,i}^{n+1} - \rho_{\varepsilon,i}^{n})/\Delta t)_{(i,n) \in \mathbb{N}^2}$, one obtains explosion of $D_t^{\Delta}\left(\rho_{\varepsilon,.}^{n} - \rho_{0,.}^{n}\right)$ in the L^∞ norm for small values of ε, which shows how

the time derivative behaves close to the origin. This also shows that one cannot expect better results from theoretical estimates then the convergences exhibited above. When including the initial layer, in Proposition 3.2, $\mathcal{H}[D_t^\tau(\rho_\varepsilon - \rho_0 - \tilde{\rho}_{0,\varepsilon})]$ is bounded in the $L^\infty(0,T)$ norm by $o_\varepsilon(1)/\varepsilon$. As in this numerical test $\beta_\varepsilon = \beta_0$, $\zeta_\varepsilon = \zeta_0$, in fact, we only consider the error due to the initial layer which is of order ε. This leads to the uniform boundedness of $\mathcal{H}[D_t^\tau(\rho_\varepsilon - \rho_0 - \tilde{\rho}_{0,\varepsilon})]$ wrt ε (cf fig. 2 right).

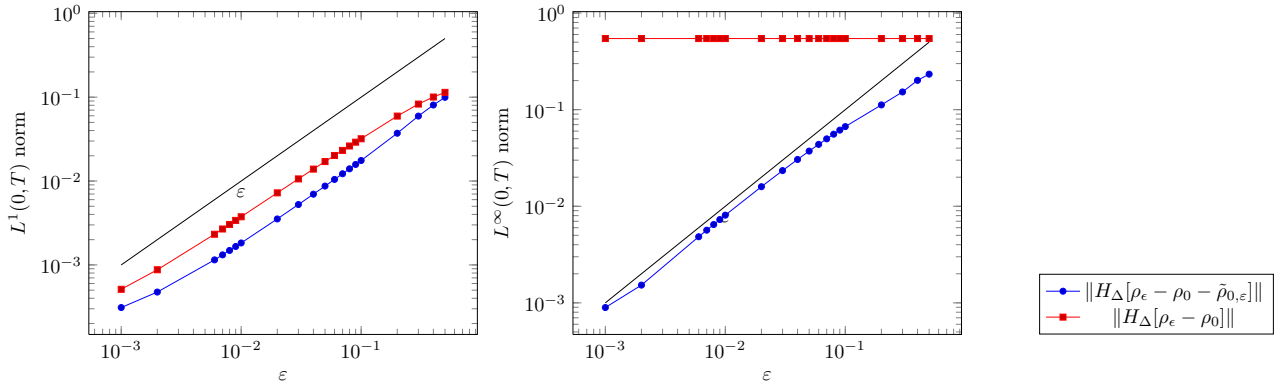

FIGURE 1. Error estimates approximating ρ_ε by either ρ_0 or by $\rho_0 + \tilde{\rho}_{0,\varepsilon}$ in different norms.

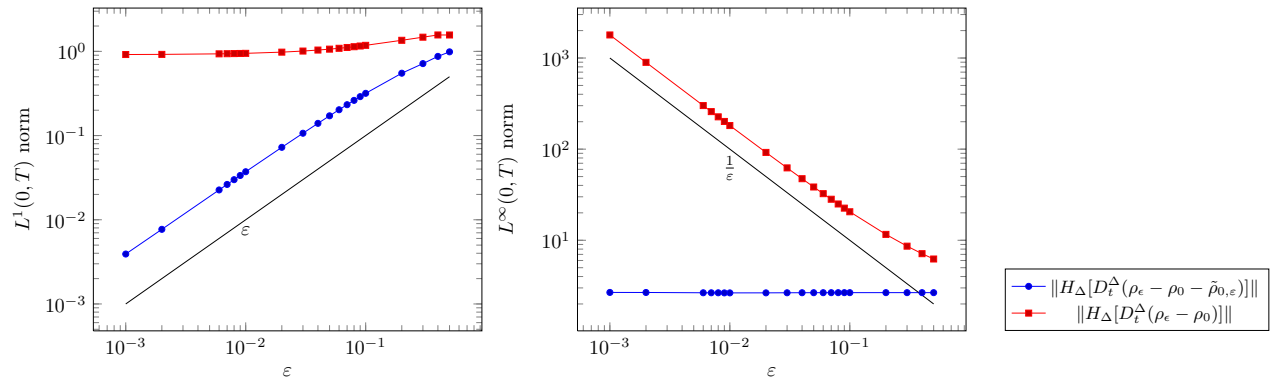

FIGURE 2. Error estimates approximating $D_t^\tau \rho_\varepsilon$ by either $D_t^\tau \rho_0$ or $D_t^\tau(\rho_0 + \tilde{\rho}_{0,\varepsilon})$ in different norms. Here $\tau = \Delta t$.

4.2.2. *Perturbing the on and off rates*

While in the second set of simulations, we perturb the data in order to test the accuracy of our estimates.

$$\begin{cases} \beta_\varepsilon(t) = \beta_{\min} + (\sin(2\pi t))_+ + \sqrt{\varepsilon}(\cos(2k\pi t))_+ \\ \zeta_\varepsilon(a,t) = (1+a)(1+t) + \sqrt{\varepsilon}(\cos(2k\pi t))_+ \end{cases}$$

while the initial data is still $\rho_I(a) := \exp(-a)$.

When the initial layer is not part of the asymptotic expansion, the errors related to the layer dominate the perturbation. At the contrary, when the initial layer is included, the $\sqrt{\varepsilon}$ perturbation becomes perceptible in the error estimates (cf figs. 3 and 4).

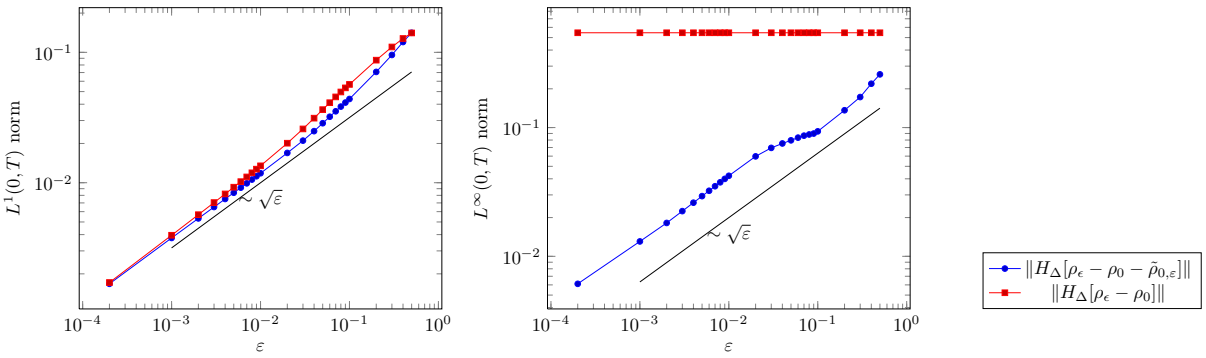

FIGURE 3. Error estimates approximating ρ_ε by either ρ_0 or by $\rho_0 + \tilde{\rho}_{0,\varepsilon}$ in different norms.

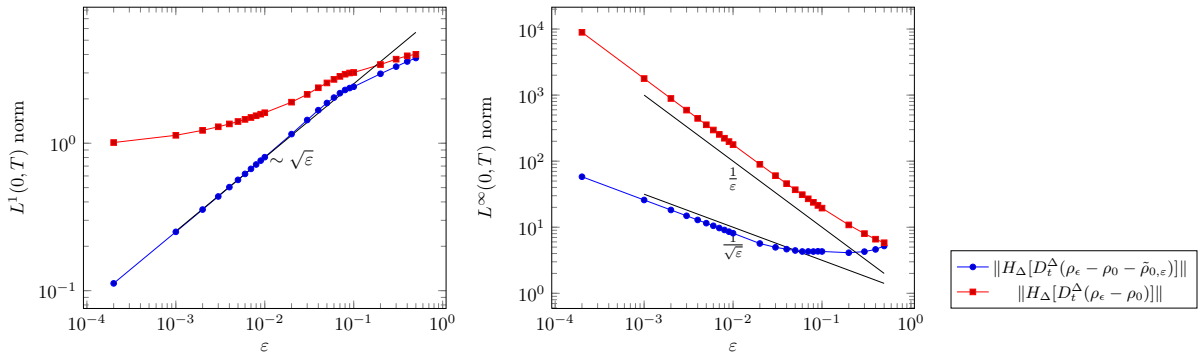

FIGURE 4. Error estimates approximating $D_t^\tau \rho_\varepsilon$ by either $D_t^\tau \rho_0$ or $D_t^\tau(\rho_0 + \tilde{\rho}_{0,\varepsilon})$ in different norms. Here $\tau = \Delta t$.

REFERENCES

[1] N. Bourbaki. *Integration. I. Chapters 1–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2004. Translated from the 1959, 1965 and 1967 French originals by Sterling K. Berberian.

[2] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 2010.

[3] V. Milišić and D. Oelz. From memory to friction : the case of the heat equation. In preparation.

[4] V. Milišić and D. Oelz. On the asymptotic regime of a model for friction mediated by transient elastic linkages. *J. Math. Pures Appl. (9)*, 96(5):484–501, 2011.

[5] V. Milišić and D. Oelz. On a structured model for the load dependent reaction kinetics of transient elastic linkages. *SIAM J. Math. Anal.*, 47(3):2104–2121, 2015.

[6] V. Milišić and D. Oelz. Tear-off versus global existence for a structured model of adhesion mediated by transient elastic linkages. *Commun. Math. Sci.*, 14(5):1353–1372, 2016.

[7] S. Mischler. Cours d'analyse fonctionnelle et EDP. 2007-2008.

[8] D. Oelz and C. Schmeiser. Derivation of a model for symmetric lamellipodia with instantaneous cross-link turnover. *Archive for Rational Mechanics and Analysis*, 198(3):963–980, 2010. cited By 3.

[9] D. Oelz and C. Schmeiser. How do cells move? mathematical modelling of cytoskeleton dynamics and cell migration. In A. Chauviere, L. Preziosi, and C. Verdier, editors, *Cell mechanics: from single scale-based models to multiscale modelling*. Chapman and Hall / CRC Press, to appear, 2009.

[10] D. Oelz, C. Schmeiser, and V. Small. Modelling of the actin-cytoskeleton in symmetric lamellipodial fragments. *Cell Adhesion and Migration*, 2:117–126, 2008.

[11] W. P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.