Multi-scale blood flow modelling for stented arteries: theoretical and numerical results

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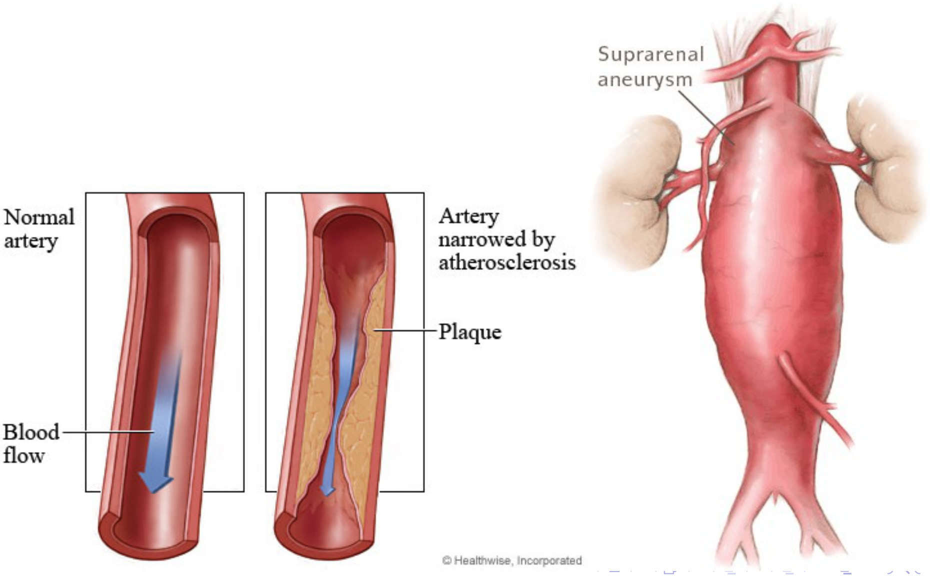
Outline

Two common pathologies of the cardio-vascular system



Industrial context

Cardiatis®: conception and comercialisation of metallic wired multi-layer stents ▸ Image 3D

- A new technology
  One controls
  - The # of layers
  - Their connectivity
- In vivo experiments
  1. on mini-pigs show :no thrombus up to 6 months ▸ Dissection pictures
  2. on humans : ▸ Microscopy pictures
- Multi-Scale phenomenon lying on:
  - Hemodynamics
  - Chemical reactions between blood flow and the surrounding wires and tissues

Theoretical & numerical study of hemodynamics

Problem description

- Geometrical properties
  - Femoral artery diameter:$\emptyset_{\mathrm{A}} = 6mm$
  - Total thickness of the stent : $\epsilon = 0.25mm$
  - Thickness of a single wire: $\epsilon = 0.04mm$
  - Red blood cell diameter: $\emptyset_{\mathrm{RC}} = 0.008mm$

$$\frac{\epsilon}{\emptyset_A} = \frac{0.25}{6} \sim 4\%$$

  stent $\sim$ periodic rugous wall in a straight cilindrical geometry
- The blood flow is composed of
  - Steady state part: Poiseuille profile
  - Plus a pulsatile periodic perturbation: Womersley profile
- We consider here the Poiseuille profile

Objectives and references

-We aim to

- understand the dynamics of flows in rugous channels
  $\Longrightarrow$ Boundary layer correctors
- Avoid heavy discretisations related to the rugous wall
  $\Longrightarrow$ Wall laws
- Include the micro scales in the macro Poiseuille profile
  $\Longrightarrow$ Multi-scale aspects

Use of **assymptotic expansions** adapted for the **perturbed boundaries**.

Main references

N. Neuss, M. Neuss-Radu, and A. Mikelić.
Effective laws for the poisson equation on domains with curved oscillating boundaries.
Applicable Analysis, 2006.

Y. Achdou, P. Le Tallec, F. Valentin, and O. Pironneau,
Constructing wall laws with domain decomposition or asymptotic expansion techniques
Comput. Methods Appl. Mech. Eng. 1998

Derivation of the Navier Stokes equations

The continuity equation

- ω_0 subdomain of Ω, γ_0 boundary of ω_0
- ρ density
- decrease of mass per time unit:
- total mass exiting from ω_0 through γ_0: $\int_{\gamma_0} \rho \mathbf{u} \cdot \mathbf{n} d\gamma_0$
- $\mathbf{n}$ outward normal vector
- $\mathbf{u}$ flow's velocity
- Mass balance

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$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_{\omega_0} \rho dx = \int_{\gamma_0} \rho \mathbf{u} \cdot \mathbf{n} d\gamma_0$$

- From the divergence theorem

$$\int_{\omega_0} \partial_t \rho + \mathrm{div}\,(\rho \mathbf{u}) dx = 0$$

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$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega_0} \rho dx = \int_{\gamma_0} \rho \mathbf{u} \cdot \mathbf{n} d\gamma_0$$

- Since ω_0 is arbitrary

$$\partial_t \rho + \mathrm{div}\,(\rho \mathbf{u}) = 0$$

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The momentum equation

- Newton's law to a moving element of volume ω

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\omega \rho \mathbf{u} dx = \int_\omega \rho \mathbf{f}\, dx + \int_\gamma \mathbf{S}\, d\gamma$$

- $\mathbf{f}$ denotes a density of volume forces
- $\mathbf{S}$ a density of surface forces per surface unit
- $\Delta t = t' - t$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\omega \rho \mathbf{u} dx = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\int_{\omega'} \rho \mathbf{u}(x', t') dx' - \int_\omega \rho \mathbf{u}(x,t) dx \right)$$

- a material point x' at time t' corresponding to (x, t)

$$x' = x + \Delta t \mathbf{u}(x,t) + 0(\Delta t^2).$$

- change of variables

$$\int_{\omega'} \rho \mathbf{u}(x', t') dx' = \int_\omega (\rho \mathbf{u})(x + \Delta t \mathbf{u}, t + \Delta t) \left| \det \nabla_x x' \right| dx$$

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$$\int_{\omega'}\rho\mathbf{u}(x',t')dx' = \int_{\omega}(\rho\mathbf{u})(x+\Delta t\mathbf{u}, t+\Delta t)(1+\Delta t\,\mathrm{div}\,\mathbf{u})dx$$

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The momentum equation II

- first order Taylor expansion in Δt, and $\Delta t \to 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\omega \rho \mathbf{u} dx = \int_\omega \left[\partial_t(\rho \mathbf{u}) + (\mathrm{div}\, u)\rho \mathbf{u} + (\mathbf{u} \cdot \nabla)\rho \mathbf{u} \right] dx$$

- Taking into account the continuity equation

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where

$$v \cdot \nabla w = \sum_{j=1}^{N} v_j \frac{\partial_j w_i}{xj}$$

- ω arbitrary, eqs become pointwise:

$$\rho(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}) - \mathrm{div}\, \boldsymbol{\sigma} = \rho \mathbf{f}$$

- Viscous stresses,

$$\sigma := -p\mathrm{Id} + \sigma', \quad \sigma' := 2\mu \left[\mathbf{D}(\mathbf{u}) - \frac{1}{3} \mathrm{div}\, \mathbf{u} \mathrm{Id} \right], \quad 2\mathbf{D}(\mathbf{u}) := \nabla \mathbf{u} + \nabla \mathbf{u}^T$$

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The full Navier-Stokes system

Newtonian incompressible viscous fluids with constant density

- Incompressibility: fixed volume the contiuity equation reduces to

$$\operatorname{div} \mathbf{u} = 0$$

- momentun equation reduces to

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} + \nabla p = \rho \mathbf{f}$$

- Dimensionless formulation of the Navier-Stokes equations setting

$$x' = x/L, t' = (U/L)t, \mathbf{u}' = \mathbf{u}/U, p' = \frac{p}{\rho U^2}, \mathbf{f}' = (L/U^2)\mathbf{f}$$

gives

$$\begin{cases} \partial_t \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{u}' - \frac{1}{\Re} \Delta \mathbf{u}' + \nabla p = \mathbf{f} \\ \operatorname{div} \mathbf{u}' = 0 \end{cases}$$

where $\Re$ is the Reynolds number

$$\Re := \rho \frac{UL}{\mu}$$

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The stokes system

Consider a flow:

- steady
- linearized around $\mathbf{u} \equiv 0$
- low reynolds number $\Re \sim 1$
- you obtain The Stokes system
- complement with boundary conditions

$$\mathbf{u} = \mathbf{g}_D, \quad \sigma_{\mathbf{u},p} \cdot \mathbf{n} = \mathbf{g}_N,$$

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Variations on boundary conditions

- variational form $\forall \mathbf{v} \in \mathcal{D}(\overline{\Omega})$:

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} dx - \int_\Omega p \mathrm{div}\, \mathbf{v} dx$$
$$+ \int_{\partial\Omega} ((p\mathrm{Id} - \nabla \mathbf{u}) \cdot \mathbf{n}, \mathbf{v}) ds = \int_\Omega \mathbf{f} \cdot \mathbf{v}$$

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- Complete Dirichlet: test space $\mathbf{v} = 0$
- Partial dirichlet
  1. Dirichlet on normal velocity

$$\mathbf{v} \cdot \mathbf{n} = 0, \qquad \partial_n \mathbf{u} \cdot \tau + \mu(\mathbf{u} \cdot \tau) = g, \quad \mu \geq 0$$

  2. Dirichlet on tangent velocity

- Natural boundary conditions

$$-(\nabla \mathbf{u} - p\mathrm{Id}) \cdot \mathbf{n} = \mathrm{M}\mathbf{u} + \mathbf{g}, \quad \forall \mathrm{M} \in \mathcal{M}^+_{2,2}(\mathbb{R})$$

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Abstract problem

Define

- the Banach spaces
$$X, Y$$
- the operators
$$A : X \to Y', \quad B : X \to Y$$
- solve the problem, find u, p s.t.
$$\begin{cases} Au + B^T p & = f \\ Bu & = g \end{cases}$$

We study in an abstract formalism

1. the well-posednes
2. the continuity wrt data

Framework

- fundamental results for linear bijective operators in Banach spaces
- classical

Brezis. H
Analyse fonctionnelle
Masson

Yosida
Functional analysis
Springer

A. Ern and J.-L. Guermond.
Theory and Practice of Finite Elements, volume 159 of *Applied Mathematical Series*.
Springer-Verlag

Preliminary results

- V and W Banach spaces
- A an application $A \in \mathcal{L}(V, W)$
- $\mathcal{N}(A)$ kernel
- $\mathcal{R}(A)$ rank
- $V/\mathcal{R}(A)$ quotiented space

$$v \equiv w \Leftrightarrow v - w \in \mathcal{N}(A), \quad \|v\|_{V/\mathcal{N}(A)} \leq inf_{w \in \mathcal{N}(A)} \|v + w\|_V$$

Theorem 1.1

- *$V/\mathcal{N}(A)$ is a Banach space*
- *$\overline{A} : V/\mathcal{N}(A) \to \mathcal{R}(A)$ s.t.*

$$\overline{A}\overline{v} = Av$$

$\overline{A}$ bijective

Kernels Ranks and Adjoint operators

- V Banach space
- $M \subset V, N \subset V'$

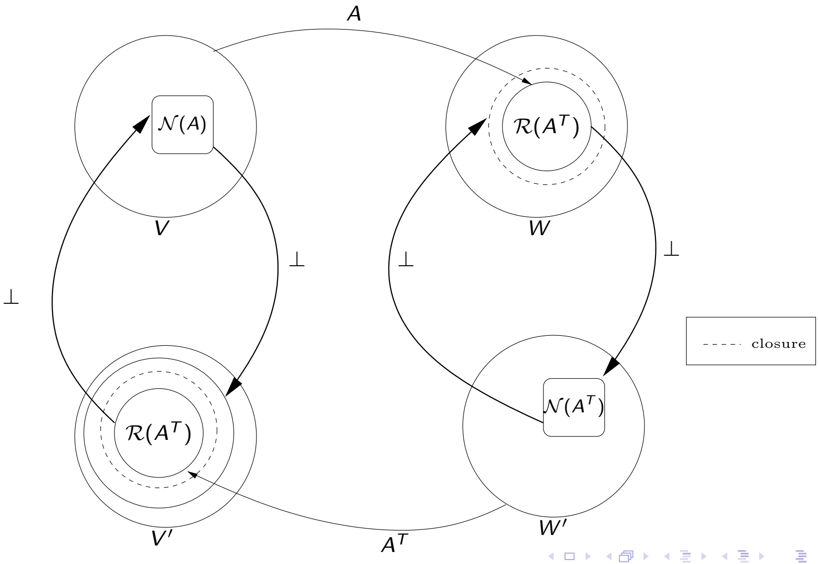
$$M^{\perp} := \{v' \in V';\ \forall m \in M, < v', m >_{V',V} = 0\}$$
$$N^{\perp} := \{v \in V;\ \forall n' \in N, < n', v >_{V',V} = 0\}$$

Theorem 1.2

For $A \in \mathcal{L}(V;W)$, the following properties hold

1. $\mathcal{N}(A) = (\mathcal{R}(A^T))^{\perp}$
2. $\mathcal{N}(A^T) = (\mathcal{R}(A))^{\perp}$
3. $\overline{\mathcal{R}(A)} = (\mathcal{N}(A^T))^{\perp}$
4. $\overline{\mathcal{R}(A^T)} \subset (\mathcal{N}(A))^{\perp}$

Kernels Ranks and Adjoint operators



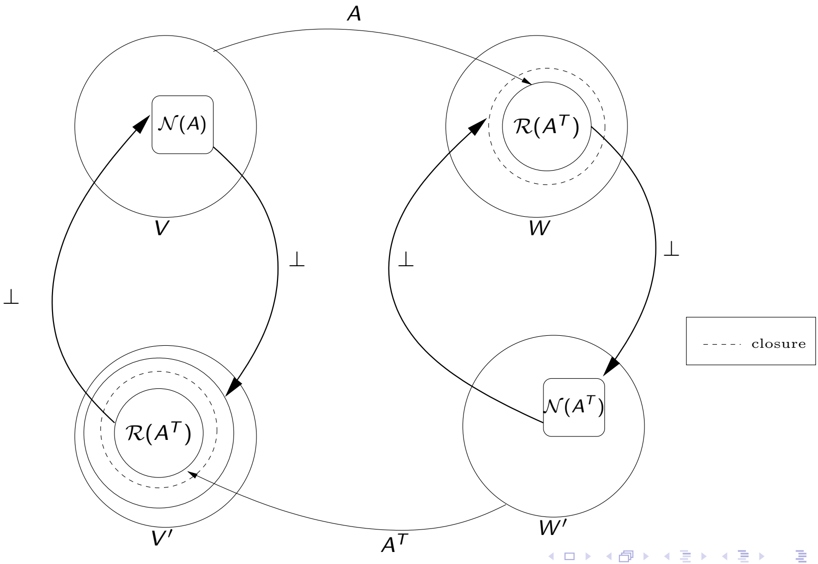
Closed range

Theorem 1.2

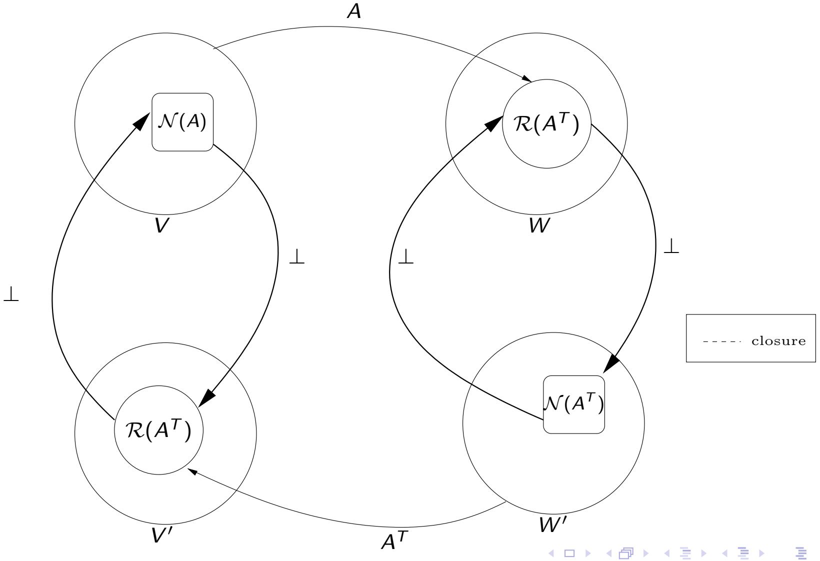
For $A \in \mathcal{L}(V; W)$, the following properties are equivalent

1. $\mathcal{R}(A)$ *is closed*
2. $\mathcal{R}(A^T)$ *is closed*
3. $\mathcal{R}(A) = (\mathcal{N}(A^T))^{\perp}$
4. $\mathcal{R}(A^T) = (\mathcal{N}(A))^{\perp}$

Closed range



Closed range



Also

Lemma 1.1

If $A \in \mathcal{L}(V;W)$, the following propositions are equivalent

- $\mathcal{R}(A)$ *closed*
- $\exists \alpha > 0$ *s.t.* $\forall w \in \mathcal{R}(A)$, $\exists v_w \in V$ *s.t.*

$$Av_w = w, \quad \alpha \|v_w\|_V \leq \|w\|_W$$

Proof.

$\mathcal{R}(A)$ closed $\implies A : V \to \mathcal{R}(A)$ surjective.
Then apply the open mapping theorem on $A : V \to \mathcal{R}(A)$ □

Also

Theorem 1.2 (Petree Tartar)

Hypotheses:

- X, Y, Z *Banach spaces*
- $A \in \mathcal{L}(X, Y)$ *injective*
- $T \in \mathcal{L}(X, Z)$ *compact*
- *There exists* $c > 0$ *s.t.*

$$c\|x\|_X \leq \|Ax\|_Y + \|Tx\|_Z$$

Conclusion : there exists α *s.t.*

$$\forall x \in X, \quad \alpha\|x\|_X \leq \|Ax\|_Y$$

Proof.

By contradiction: suppose $\exists x_n \in X$ s.t. $\|x_n\|_X = 1$ and $\|Ax\| \to 0$ □

The inf-sup condition

Surjectivity sometimes tedious instead possible characterisation:

Lemma 1.1

Hypotheses:

- *V and W Banach spaces*
- *V reflexive*

then the following claim are $\sim$

(i) $\exists \alpha \in \mathbb{R}_+$ *s.t.* $\forall w \in W$, $\exists v_w \in V$ *s.t.* $A v_w = w$ *and* $\alpha \|v_w\|_V \leq \|w\|_W$

(ii) *The inf-sup condition*

$$\inf_{w' \in W'} \sup_{v \in V} \frac{< A^T w', v >}{\|v_w\|_V \|w\|_W} \geq \alpha$$

Proof.

$(i) \implies (ii)$ easy , reverse cf Ern-Guermond $\square$

Surjective operators

Lemma 1.2

$A \in \mathcal{L}(V, W)$ *then the following assertions are* $\sim$

1. A^T *surjective*
2. A *injective and* $\mathcal{R}(A)$ *closed*
3. $\exists \alpha > 0$ *s.t.* $\forall v \in V \; \alpha \|v\|_V \leq \|Av\|_W$

Lemma 1.3

$A \in \mathcal{L}(V, W)$ *then the following assertions are* $\sim$

1. A *surjective*
2. A^T *injective and* $\mathcal{R}(A^T)$ *closed*
3. $\exists \alpha > 0$ *s.t.* $\forall w \in W' \; \alpha \|w'\|_{W'} \leq \left\|A^T w'\right\|_{V'}$

Onto mappings

Theorem 1.3

$A \in \mathcal{L}(V, W)$ bijective iff

$$\begin{cases} A^T : W' \to V' \text{ injective} \\ \forall v \in V \quad \|v\|_V \leq \alpha \|Av\|_W \end{cases}$$

Proof.

A surjective $\Leftrightarrow$ A^T injective and $\mathcal{R}(A^T)$ closed
$\mathcal{R}(A^T)$ closed $\Leftrightarrow$ $\mathcal{R}(A)$ closed
$\mathcal{R}(A)$ closed and A injective $\Leftrightarrow \exists \alpha > 0$ s.t. $\forall v \in V \; \alpha \|v\|_V \leq \|Av\|_W$ $\square$

Note

A bijective Banach operator iff

- A injective
- $\mathcal{R}(A)$ closed
- A^T injective

Saddle point problems

- X, M Banachs
- $A : X \to X'$
- $B : X \to M$
- Given $(f, g) \in X' \times M)$ find $(u, p) \in X \times M'$ solving

$$\begin{cases} Au + B^T p & = f \\ Bu & = g \end{cases} \tag{1}$$

- $\mathcal{N}(B)$ kernel of B
- $\pi A : \mathcal{N}(B) \to \mathcal{N}(B)'$ s.t.

$$< \pi Au, v > = < Au, v >, \quad \forall u, v \in \mathcal{N}(B)$$

Theorem 1.4

Problem (1) *is well-posed iff*

1. $\pi A : \mathcal{N}(B) \to \mathcal{N}(B)'$ *isom*
2. $B : X \to M$ *surjective*

Proof of theorem 1.4

Necessary conditions pbm well posed $\implies$ 1 and 2 (part I)

- B surjective ?
  $h \in M$ denote (u, p) solution of (1) with data $(0, h)$. B surjective ok.
- πA surjective ?
  Let $h \in \mathcal{N}(B)'$, Hahn-Banach theorem there exists $\tilde{h} \in X'$ extension of h s.t. (cf Yosida p.102 and 106.)

$$\begin{cases} < \tilde{h}, v > = < h, v >, & \forall v \in \mathcal{N}(B) \\ \left\| \tilde{h} \right\|_{X'} = \| h \|_{\mathcal{N}(B)'} \end{cases}$$

Let (u, p) solution pbm (1) with data $(\tilde{h}, 0) \implies u \in \mathcal{N}(B)$ as

$$< B^T p, v > = < p, Bv > = 0, \quad \forall v \in \mathcal{N}(B)$$

one has

$$< \pi A u, v > = < h, v >, \quad \forall v \in \mathcal{N}(B)$$

thus $\exists u \in \mathcal{N}(B)$ s.t. $\pi A u = h$

Proof of theorem 1.4

Necessary conditions pbm well posed $\implies$ 1 and 2 (part II)

- πA injective ?
  Hypothesis: $< \pi A u, v >= 0, \forall v \in \mathcal{N}(B)$
  then $\pi A u \in \mathcal{N}(B)^{\perp} = \mathcal{R}(B^T)$ (because B surjective)
  $\exists p \in M'$ s.t. $Au = -B^T p$
  thus (u, p) satisfy
  $$\begin{cases} Au + B^T p & = 0 \\ Bu & = 0 \end{cases}$$
  pbm well posed $\implies (u, p) = (0, 0)$

Proof of theorem 1.4

Sufficient conditions 1 and 2 $\Longrightarrow$ pbm well posed (part I)

- Existence ?
  Given (f, g) show that $\exists(u, p)$ solving pbm (1)
  1. B surjective $\Longrightarrow$ $\exists u_g$ s.t. $Bu_g = g$
  2. Question: $\exists \Phi \in \mathcal{N}(B)$ s.t. $A\Phi = f - Au_g$ in $\mathcal{N}(B)'$?
     Answer: yes if $f_A u_g \in \mathcal{N}(B)'$ but $f - Au_g \in X' \subset \mathcal{N}(B)'$
     Set $u = u_g + \Phi$, one then has:
     $$Au - f \in \mathcal{N}(B)^{\perp}$$
     As B surjective $\mathcal{N}(B)^{\perp} = \mathcal{R}(B^T)$ and $\exists p \in M'$ s.t.
     $$Au - f = -B^T p, \quad \text{and } Bu = g$$
     existence ok

Proof of theorem 1.4

Sufficient conditions 1 and 2 $\Longrightarrow$ pbm well posed (part II)

- Uniqueness ?
  $(f, g) := (0, 0)$ Above gives there exists (u, p) s.t.

$$\begin{cases} Au + B^T p & = 0 \\ Bu & = 0 \end{cases}$$

  then $u \in \mathcal{N}(B)$ and $\pi A u = 0 \implies u \equiv 0$
  B surjective $\implies B^T$ injective $\implies p \equiv 0$
  ok

A priori estimates

Lemma 1.4

Conditions (i) and (ii) satisfied then

$$\exists c_i(\alpha, \beta), \quad i \in 1, \dots, 4 \text{ independent on } f, g, u, p \text{ s.t.}$$

$$\begin{aligned} \|u\|_X &\leq c_1 \|f\|_{X'} + c_2 \|g\|_M \\ \|p\|_{M'} &\leq c_3 \|f\|_{X'} + c_4 \|g\|_M \end{aligned}$$

A priori estimates

Proof.

$$\left.\begin{array}{l} \exists u_g \;/\; Bu_g = g \\ B \text{ surjective} \\ X \text{ reflexive} \end{array}\right| \implies \exists \beta > 0 \text{ s.t } \beta \|u_g\|_X \leq \|g\|_M$$

Then solve $A\Phi = f - Au_g$ in $\mathcal{N}(B)'$, A surjective $\implies$

$$\exists \alpha > 0 \text{ s.t. } \alpha \|\Phi\|_X \leq \|f\|_{X'} + \|A\|_{\mathcal{L}(X;X')} \|u_g\|_X$$

As we set $u := \Phi + u_g$

$$\|u\|_X \leq \|\Phi\|_X + \|u_g\|_X$$

B surjective

$$\beta \|p\|_{M'} \leq \left\| B^T p \right\|_{X'}$$

also for p s.t. $B^T p = f - Au$. ok □

Surjectivity of the div operator

Set

$$\mathbf{W}_0^{1,q}(\Omega) := \{v \in L^q(\Omega) \text{ s.t. } D^\alpha v \in L^q(\Omega), \quad v = 0 \text{ on } \partial\Omega\}$$

Theorem 1.5

Hypothesis: let

- *Ω a bounded domain of $\mathbb{R}^n$ s.t.*

$$\Omega = \cup_{k=1}^{N} \Omega_k, \quad N \geq 1$$

  where Ω_k star shaped wrt B_k s.t. $\overline{B}_k \subset \Omega_k$

- *$f \in L^q(\Omega)$ s.t. $\int_\Omega f dx = 0$*

Conclusion: $\exists$ a vector $\mathbf{v} \in \mathbf{W}_0^{1,q}(\Omega)$ s.t.

$$\mathrm{div}\, \mathbf{v} = f, \quad |\mathbf{v}|_{\mathbf{W}_0^{1,q}(\Omega)} \leq c \|f\|_{L^q(\Omega)}$$

Idea of the proof

cf p.115-125 in

Giovanni P. Galdi,
An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I.
Springer-Verlag,

Idea of the proof

1. rescale Ω wrt radius, center in in 0
2. This domain is star-like wrt $\forall$ point of $B(0,1) \subset \Omega$ set

$$\forall \omega \in C_0^\infty(\mathbb{R}^n) \text{ s.t. } \operatorname{supp} \omega \subset B(0,1), \quad \int_B \omega(y) dy = 1$$

one has an explicit formula if $f \in C_0^\infty(\Omega)$

$$\mathbf{v}(x) = \int_\Omega f(y) \left[\frac{x-y}{|x-y|^n} \int_{|x-y|}^\infty \omega \left(y + \xi \frac{x-y}{|x-y|} \right) \xi^{n-1} d\xi \right] dy$$

3. check that rescaled again it satisfies

$$\operatorname{div} \mathbf{v} = f, \quad |\mathbf{v}|_{\mathbf{W}^{1,q}(\Omega)} \leq c |f|_{L^q(\Omega)}$$

4. approximate f by $\{f_m\} \in C_0^\infty(\Omega)$ and set

$$f_m^* := f_m - \varphi \int_\Omega f_m dy, \quad m \in \mathbb{N} \text{ with } \varphi \in C_0^\infty(\Omega), \int_\Omega \varphi = 1$$

extract $\mathbf{v}_{m_k} \rightharpoonup \mathbf{v}$ in $\mathbf{W}^{1,q}(\Omega)$

Idea of the proof

5. As $\Omega = \cup_{k=1}^{N} \Omega_k$, $\exists$ N functions f_k s.t. for $k \in \{1, \dots, N\}$
   (i) $f_k \in L^q(\Omega)$
   (ii) $\mathrm{supp}(f_k) \in \overline{\Omega}_k$
   (iii) $\int_{\Omega_k} f_k dx = 0$
   (iv) $f = \sum_k f_k$
   (v) $\exists C(\Omega_k)$ s.t.
$$\|f_k\|_{L^q(\Omega)} \leq C \|f\|_{L^q(\Omega)}$$

proof: contructive, explicit form wrt f and $\int_{\Omega_k} f dx$

Conclusion

Set

- Ω bounded Lipshitz domain
- $(\mathbf{f}, g) \in \mathbf{H}^{-1}(\Omega) \times L^2(\Omega)_{\int=0}$
- solve the problem: find $(\mathbf{u}, p)$ solving

$$\begin{cases} -\Delta \mathbf{u} + \nabla p & = 0 \text{ in } \Omega \\ \operatorname{div} \mathbf{u} & = 0 \\ \mathbf{u} = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

Theorem 1.6

$\exists!$ pair $(\mathbf{u}, p) \in \mathbf{H}^1_0(\Omega) \times L^2(\Omega)/\mathbb{R}$ solving (2). Moreover one has

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leq C(\alpha, \beta) \left\{ \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + \|g\|_{L^2(\Omega)} \right\}$$

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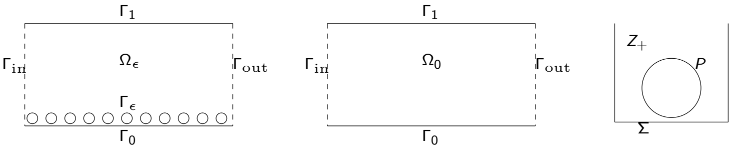
preprint

Notations and Methodology



1. Construction of a complete boundary layer corrector: Ω_ϵ
2. Derivation of wall laws: Ω_0

We denote:

- $P = \partial Q$, Q a body isomorphic to an open ball, regular
- Ω the "smooth domain", Γ^0 the fictitious interface,
- x the slow space variable , $y = \frac{x}{\epsilon}$ the fast one.

The problem

- One aims to solve

$$\begin{cases} -\Delta \mathbf{u}_\epsilon + \nabla p_\epsilon = 0 \text{ in } \Omega_\epsilon \\ \operatorname{div} \mathbf{u}_\epsilon = 0 \\ \mathbf{u}_\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_\epsilon \\ \mathbf{u}_\epsilon \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}} \\ p_\epsilon = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad p_\epsilon = 0 \text{ on } \Gamma_{\mathrm{out}}, \end{cases}$$

The limit solution when $\epsilon \to 0$

- The Poiseuille flow

$$\begin{cases} -\Delta \mathbf{u}_0 + \nabla p_0 = 0 \text{ in } \Omega \\ \operatorname{div} \mathbf{u}_0 = 0 \\ \mathbf{u}_0 = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \\ \mathbf{u}_0 \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}, \\ p_0 = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad p_0 = 0 \text{ on } \Gamma_{\mathrm{out}} \\ \mathbf{u}_0 \neq 0 \text{ on } \Gamma_\epsilon \end{cases}$$

- $(\mathbf{u}_0, p_0)$ is explicit and reads:

$$\begin{cases} \mathbf{u}_0(x) = \dfrac{p_{\mathrm{in}}}{2}(1 - x_2)x_2\mathbf{e}_1, \quad \forall x \in \Omega \\ p_0(x) = p_{\mathrm{in}}(1 - x_1) \end{cases}$$

Theorem 1.7

$$\|\mathbf{u}_\epsilon - \mathbf{u}_0\|_{\mathbf{H}^1(\Omega_\epsilon)^2} + \|p_\epsilon - p_0\|_{L^2(\Omega_\epsilon)} \leq k\sqrt{\epsilon}$$

where the constant k does not depend on ϵ.

Proof.

- set $v := \mathbf{u}_\epsilon - \mathbf{u}_0, q := p_\epsilon - p_0$ they solve

$$\begin{cases} -\Delta \mathbf{v} + \nabla q = 0 \text{ in } \Omega \\ \operatorname{div} \mathbf{v} = 0 \\ \mathbf{v} = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \\ \mathbf{v} \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}, \\ q = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}} \\ \mathbf{v} \neq 0 \text{ on } \Gamma_\epsilon \end{cases}$$

- lift the Dirichlet data on Γ_ϵ set $\mathcal{R}(v) := \mathbf{u}_0 \psi(x_2/\epsilon)$
- use *a priori* estimates
- compute $\|\nabla \mathcal{R}(v)\|_{L^2(\Omega_\epsilon)}$ and conclude

$\square$

- Taylor expansion of $\mathbf{u}$ around $(x_1, 0)$

So where does the error come from ?

- Taylor expansion of $\mathbf{u}$ around $(x_1, 0)$

$$u_{0,1}(x) = u_{0,1}(x_1, 0) + \frac{\partial u_{0,1}}{\partial x_2}(x_1, 0) x_2$$

- The error is ϵ times a microscopic oscilation of first order
- This is corrected by a micorscopic periodic boundary layer

So where does the error come from ?

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$$u_{0,1}(x) = u_{0,1}(x_1, 0) + \epsilon \frac{\partial u_{0,1}}{\partial x_2}(x_1, 0) \quad \frac{x_2}{\epsilon}$$

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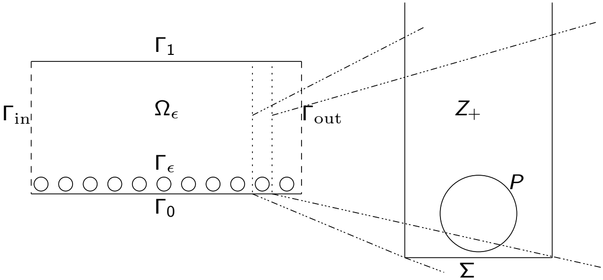

Horizontal correctors

- Microscopic corrector *à la Mikelić*

$$\begin{cases} -\Delta\beta + \nabla\pi = 0 \text{ in } S \\ \operatorname{div}\beta = 0 \\ \beta = -y_2\mathbf{e}_1 \text{ on } P \cup \Sigma \end{cases}$$

- Properties

Proposition 1

$\exists!(\beta,\pi)$, π defined up to a constant, s.t.

$$\nabla\beta \in L^2(S)^4, \quad (\beta - \overline{\beta}(\cdot)) \in L^2(S), \quad \pi \in L^2_{\mathrm{loc}}(S)$$

Moreover, one has:

$$\beta(y) \to \overline{\beta}_+\mathbf{e}_1, \quad y_2 \to +\infty$$

cvg exponential and

$$\begin{cases} \overline{\beta}_2(y_2) = 0, & \forall y_2 \in \mathbb{R} \\ \overline{\beta}_1(y_2) = -\mu(Q) - |\nabla\beta|^2_{L^2(S)} & y_2 > y_{2,P}, \end{cases}$$

where $y_{2,P} := max_{y\in P} y_2$ and $\mu(Q)$ is the volume of the body Q.

Proof

Solve the velocity

- Define the test space:

$$X := \{\mathbf{v} \in \mathbf{L}^2_{\mathrm{loc}}(S),\ \text{s.t. } \nabla \mathbf{v} \in L^2(S)^4, \quad \mathbf{v} = 0 \text{ on } \Sigma \cup P\}$$

- lift the Dirichlet boundary $\tilde{\beta} := \beta - \mathcal{R}(\beta)$
- then $\forall \varphi \in \mathcal{N}(\mathrm{div}\,) \cap X$ one has

$$\int_S \nabla \tilde{\beta} : \nabla \varphi dy = \int_S \nabla \mathcal{R}(\beta) \cdot \nabla \varphi dy$$

- by Lax-Milgram $\exists ! \tilde{\beta} \in \mathcal{N}(\mathrm{div}\,) \cap X$

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Recover the pressure

- To our knoledge no results of surjectivity of div on the undounded strips
- On bounded restrictions $S_k := S\cap]0,1[x]0,k[$ solve
  1. find p solving
  $$\begin{cases} -\Delta p = g, & \text{in } S_l \\ \partial_{\mathbf{n}} p = 0, & \text{on } P \cup \partial S_k \end{cases}$$
  for any g in
  $$M = \left\{ g \in L^2(S_k), \text{ s.t. } \int_{S_k} g dy = 0 \right\}$$
  2. and $\mathbf{w}$ lifts ∇p on P
  $$\begin{cases} \operatorname{div} \mathbf{w} = 0, & \text{in } S_k \\ \mathbf{w} = \nabla p, & \text{on } P \end{cases}$$
- $\nabla : L^2(S_k)/\mathbb{R} \to \mathbf{H}^{-1}_{\Sigma\cup P\cup\{y_2=k\}}(S_k)$ injective

Recover the pressure II

- $S = \cup_k S_k$
- Let $\mathbf{f} \in X'$ such that $< \mathbf{f}, \varphi >= 0 \ \forall \varphi \in \mathcal{N}(\mathrm{div}\,)$, let

$$\text{let } \mathbf{v} \in \mathcal{N}(\mathrm{div}\,_k), \quad \text{set } \tilde{\mathbf{v}} \text{ extension of } \mathbf{v} \text{ on } S \text{ by } 0$$

then $\tilde{\mathbf{v}} \in \mathcal{N}(\mathrm{div}\,)$

$$< \mathbf{f}, \tilde{\mathbf{v}} >= 0, \implies \mathbf{f}|_{S_k} \in \mathcal{N}(\mathrm{div}\,_k)^{\perp} = \mathcal{R}(\nabla_k)$$

- thus $\exists p \in L^2(S_k)/\mathbb{R}$ s.t.

$$\mathbf{f} = \nabla p_k, \text{ on } S_k$$

- S_k increasing sets $p_{k+1} - p_k = C^{st}$ on S_k, choose p_{k+1} s.t. $C^{st} = 0$.
- finaly letting $k \to \infty$

$$\mathbf{f} = \nabla p, \quad p \in L^2_{\mathrm{loc}}(S)$$

Exponential decrease

- So far we proved $\exists!(\beta,\pi) \in X \times L^2_{\mathrm{loc}}$
- At $\{y_2 = L\}$ there exists $\beta(y_1, L) \in H^{\frac{1}{2}}(\{y_2 = L\})$ set $\xi := \mathrm{rot}\beta$

$$\Delta\xi = 0 \text{ on } y_2 > L, \quad \xi = \mathrm{rot}\beta$$

  use the y_1 Fourier transform

$$\xi = \sum_{k=1}^{+\infty} (C_{1,n} sin(2\pi k y_1) + C_{1,n} sin(2\pi k y_1)) e^{-2\pi k y_2}$$

- recover the velocity

$$\Delta\beta = (\nabla\zeta)^{\perp}$$

  which gives

$$\boldsymbol{\beta} = \sum_{k=1}^{\infty} ((\mathbf{D}_{1,n} + \mathbf{C}_{1,n} y_2) \sin(2\pi k y_1) + (\mathbf{D}_{2,n} + \mathbf{C}_{2,n} y_2) \cos(2\pi k y_1)) e^{-2\pi k y_2}$$

  $+$ compatibility condition on $\mathbf{D}_{i,n}, \mathbf{C}_{i,n}$ in order to satisfy $\mathrm{div}\,\beta = 0$
- same story for the pressure

Constants at infinity

- For $\overline{\beta}_2$ integrate the div equation on $S_{\omega,0} := S \cap]0,1[\times]0,\omega[$

$$\int_{S_{\omega,\gamma}} \operatorname{div} \boldsymbol{\beta} dy = 0 = \overline{\beta}_2(\omega) - \int_P y_2 \mathbf{e}_1 \cdot \mathbf{n} ds - \overline{\beta}_2(0)$$

- For $\overline{\beta}_2$: set the "Fundamental solution"

$$\begin{cases} -\Delta I_\nu + \nabla J_\nu = -\delta_{\{y_2=\nu\}} \text{ in } S \\ \operatorname{div} I_\nu = 0 \end{cases}$$

reads

$$I_\nu := \frac{1}{2}|y_2 - \nu|\mathbf{e}_1, \quad J_\nu = 0$$

- Apply the Green formula on $S_{\omega,0}$

$$\begin{aligned} &(-\Delta\boldsymbol{\beta} + \nabla\pi, I_\nu)_{S_{\omega,0}} - (-\Delta I + \nabla J, \boldsymbol{\beta})_{S_{\omega,0}} = \overline{\beta}_1(\nu) \\ &= (-\sigma_{\beta,\pi}\cdot\mathbf{n}, I_\nu)_{\partial S_{\omega,0}} + (\sigma_{I_\nu,J_\nu}\cdot\mathbf{n}, \beta)_{\partial S_{\omega,0}} \\ &= -\frac{1}{2}|\nabla\beta|^2_{L^2(S)^4} + \frac{1}{2}(\partial_{\mathbf{n}} y_2\mathbf{e}_1, y_2\mathbf{e}_1) + \frac{1}{2}\overline{\beta}_1(\omega) \end{aligned}$$

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$$= (-\sigma_{\beta,\pi} \cdot \mathbf{n}, I_\nu)_{\partial S_{\omega,0}} + (\sigma_{I_\nu,J_\nu} \cdot \mathbf{n}, \beta)_{\partial S_{\omega,0}}$$
$$= -\frac{1}{2}|\nabla\beta|^2_{L^2(S)^4} + \frac{1}{2}(\partial_{\mathbf{n}} y_2\mathbf{e}_1, y_2\mathbf{e}_1) + \frac{1}{2}\overline{\beta}_1(\omega)$$

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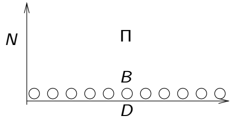
Vertical correctors

- Microscopic corrector

$$\begin{cases} -\Delta w_\beta + \nabla \theta_\beta = 0 \text{ in } \Pi \\ \operatorname{div} w_\beta = 0 \\ w_\beta = 0 \text{ on } D \cup B \\ \left. \begin{array}{l} w_\beta \cdot \boldsymbol{\tau} = \beta_2 \\ \theta_\beta = \pi \end{array} \right\} \text{ on } N \end{cases}$$



- the usual weighted Sobolev space :

$$W_\alpha^{m,p}(\Omega) := \left\{ v \in \mathcal{D}'(\Omega) \text{ s.t. } |D^\lambda v|(1+\rho^2)^{\frac{\alpha+|\lambda|-m}{2}} \in L^p(\Omega),\, 0 \leq |\lambda| \leq m \right\}$$

- Properties

Theorem 1.8

$$\exists!\ (\mathbf{w}, \theta) \in \mathbf{W}_\alpha^{1,2}(\Pi)^2 \times W_\alpha^{0,2}(\Pi) \textit{ if}$$

$$\alpha < 1$$

Proof I

- lift the tangent component of the data
-

Proof I

- lift the tangent component of the data
- then the problem reads

$$\begin{cases} -\Delta w_\beta + \nabla \theta_\beta = \mathbf{f} \text{ in } \Pi \\ \operatorname{div} w_\beta = g \\ w_\beta = 0 \text{ on } D \cup B \\ \left.\begin{array}{l} w_\beta \cdot \boldsymbol{\tau} = 0 \\ \theta_\beta = h \end{array}\right\} \text{ on } N \end{cases}$$

with the spaces:

$$A : \mathbf{W}^{1,2}_\alpha(\Pi) \to \left(\mathbf{W}^{1,2}_{-\alpha}(\Pi)\right)'$$

$$B : \mathbf{W}^{1,2}_\alpha(\Pi) \to \mathbf{W}^{0,2}_\alpha(\Pi)$$

$$B^T : \mathbf{W}^{0,2}_\alpha(\Pi) \to \left(\mathbf{W}^{1,2}_{-\alpha}(\Pi)\right)'$$

the div and ∇ do not map in duality

Proof I

- lift the tangent component of the data
- transform the problem setting

$\rho := (1 + |y|^2)^{\frac{1}{2}}, \quad \mathbb{U} := \rho^\alpha w_\beta, \quad \mathbb{P} := \rho^\alpha \theta_\beta,$

$$\begin{cases} -\mathcal{A}_\alpha \mathbb{U} + \mathcal{B}_\alpha^T \mathbb{P} = \rho^\alpha \mathbf{f} \text{ in } \Pi \\ \mathcal{B}_\alpha \mathbb{U} = \rho^\alpha g \\ \mathbb{U} = 0 \text{ on } B \\ \left. \begin{array}{l} \mathbb{U} \cdot \boldsymbol{\tau} = 0 \\ \mathbb{P} = \rho^\alpha h \end{array} \right\} \text{ on } N \end{cases}$$

where

$$\begin{cases} \mathcal{A}_\alpha \mathbb{U} := -\Delta \mathbb{F} - 2\rho^\alpha \nabla \mathbb{F} \cdot \nabla \dfrac{1}{\rho^\alpha} - \rho^\alpha \Delta \dfrac{1}{\rho^\alpha} \mathbb{F} \\ \mathcal{B}_\alpha \mathbb{U} := \operatorname{div} \mathbb{U} + \rho^\alpha \nabla \left(\dfrac{1}{\rho^\alpha} \right) \cdot \mathbb{U} \end{cases}$$

Proof II

In the new variables the operators act on

$$\begin{cases} \mathcal{A}_\alpha : \mathbf{W}_0^{1,2}(\Pi) \to \mathbf{W}_0^{-1,2}(\Pi) \\ \mathcal{B}_\alpha : \mathbf{W}_0^{1,2}(\Pi) \to W_0^{0,2}(\Pi) \\ \mathcal{B}_\alpha^T : W_0^{0,2}(\Pi) \to \mathbf{W}_0^{-1,2}(\Pi) \end{cases}$$

new change of variables $\mathcal{B}_\alpha$ acts in duality. Check

1. coercivity on the kernel $\forall \mathbb{U} \in \mathbf{W}_0^{1,2}(\Pi) \cap \mathcal{N}(\operatorname{div})$

$$(\mathcal{A}_\alpha \mathbb{U}, \mathbb{U}) \geq \|\nabla \mathbb{U}\|_{L^2(\Pi)} - \alpha^2 \left\| \frac{\mathbb{U}}{\rho} \right\|_{L^2(\Pi)}$$

   use weighted Poincare-Wirtinger and conclude
2. surjection of div follows define a sequence C_n covering Π where

$$C_n := \{y \in \Pi \text{ s.t. if } x = (r, \tilde{\theta}) \quad r \in]2^{n-1}, 2^n[\}, \quad n \geq 1,$$
$$C_0 := B(0,1) \cap \Pi.$$

   on each of them use Galdi's candidate, and conclude

Localizing

1. The corner cut-of Set $\psi_1 := \overline{\psi}(x)$ and $\psi_2 := \overline{\psi}(x-(0,1))$, where $\overline{\psi}$ s.t.

$$\overline{\psi} := \begin{cases} 1 \text{ if } |x| \leq \frac{1}{3} \\ 0 \text{ if } |x| \geq \frac{2}{3} \end{cases}$$

NB: $\partial_{\mathbf{n}} \overline{\psi} = 0$ on $\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}$.

2. The "far from the corner" cut-off
Φ complementary on $\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}} \cup \Gamma_2$ s.t.

$$\begin{cases} \psi + \Phi = 1 \\ \partial_{\mathbf{n}} \Phi = 0, \end{cases} \quad \text{on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}} \cup \Gamma_2$$

for instance $\Phi(x) := 1 - \psi(0, x_2)$ for all $x \in \Omega$.

Macro corrector

$$\begin{cases} \Delta \mathbf{W} + \nabla Z = 0, \quad \text{in } \Omega_\epsilon \\ \operatorname{div} \mathbf{W} = 0 \\ \left. \begin{array}{r} \mathbf{W} \wedge \mathbf{n} = \epsilon \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} (\beta_\epsilon - \overline{\overline{\beta}}) \right\} \wedge \mathbf{n}\, \Phi \\ Z = \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} \beta \pi \right\} \Phi \end{array} \right\} \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \\ \mathbf{W} = 0 \text{ on } \Gamma_\epsilon \\ \mathbf{W} = \epsilon \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} (\beta_\epsilon - \overline{\beta}) \right\} \Phi \text{ on } \Gamma_1 \end{cases}$$

Proposition 2

$\exists\,!$ solution $(\mathbf{W}, Z) \in \mathbf{H}^1(\Omega_\epsilon) \times L^2(\Omega_\epsilon)$, moreover:

$$\|\mathbf{W}\|_{\mathbf{H}^1(\Omega_\epsilon)} + \|Z\|_{L^2(\Omega_\epsilon)} \leq k e^{-\frac{\gamma}{\epsilon}}$$

rate γ and constant k do not depend on ϵ.

define

$$\mathcal{W}_\epsilon(x) := \epsilon \left\{ \psi_1(x) \mathbf{w} \left(\frac{x}{\epsilon} \right) + \psi_2((1,0) - x) \mathbf{w} \left(\frac{(1,0) - x}{\epsilon} \right) \right\} + \mathbf{W}(x),$$

$$\mathcal{Z}_\epsilon(x) := \left\{ \psi_1(x) \theta \left(\frac{x}{\epsilon} \right) + \psi_2((1,0) - x) \theta \left(\frac{(1,0) - x}{\epsilon} \right) \right\} + Z(x),$$

Full boundary layer approximation

Set

$$\mathcal{V}_\epsilon := \mathbf{u}_0 + \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2} (\beta_\epsilon - \overline{\overline{\beta}}) + \mathbf{u}_1 \right\} + \frac{\partial u_{0,1}}{\partial x_2} \mathcal{W}_\epsilon$$

$$\mathcal{P}_\epsilon := p_0 + \left\{ \frac{\partial u_{0,1}}{\partial x_2} \pi_\epsilon + \epsilon p_1 \right\} + \frac{\partial u_{0,1}}{\partial x_2} \mathcal{Z}_\epsilon$$

where

$$\begin{cases} -\Delta \mathbf{u}_1 + \nabla p_1 = 0 \text{ in } \Omega \\ \operatorname{div} \mathbf{u}_1 = 0 \\ \mathbf{u}_1 = 0 \text{ on } \Gamma_1 \\ \mathbf{u}_1 \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}, \\ p_1 = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}} \\ \mathbf{u}_1 = \frac{\partial u_{0,1}}{\partial x_2} \overline{\overline{\beta}} \mathbf{e}_1 \text{ on } \Gamma_0 \end{cases}$$

$(\mathbf{u}_1, p_1)$ give first order macroscopic approximation

Main convergence results

Theorem 1.9

First order error a priori *estimates*

$$\|\mathbf{u}_\epsilon - \mathcal{V}_\epsilon\|_{\mathbf{H}^1(\Omega_\epsilon)} + \|p_\epsilon - \mathcal{P}_\epsilon\|_{L^2(\Omega_\epsilon)} \leq k\epsilon$$

Very weak solutions à la Conca

$$\|\mathbf{u}_\epsilon - \mathcal{V}_\epsilon\|_{L^2(\Omega)} + \|p_\epsilon - \mathcal{P}_\epsilon\|_{H^{-1}(\Omega)/\mathbb{R}} \leq k\epsilon^{\frac{3}{2}^-}$$

At this point the approximation $(\mathcal{V}_\epsilon, \mathcal{P}_\epsilon)$ is multi-scale

Averaged macroscopic approximation

Suppress the oscillating boundary layer, keep the rest:

$$\mathcal{U}_\epsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1$$
$$\mathcal{Q}_\epsilon := p_0 + \epsilon p_1$$

Theorem 1.10

Very weak solutions à la Conca

$$\|\mathbf{u}_\epsilon - \mathcal{U}_\epsilon\|_{L^2(\Omega)} + \|p_\epsilon - \mathcal{Q}_\epsilon\|_{H^{-1}(\Omega)} \leq k\epsilon^{\frac{3}{2}^-}$$

Proof.

Use the triangular inequality

$$\|\mathbf{u}_\epsilon - \mathcal{U}_\epsilon\|_{L^2(\Omega)} \leq \|\mathbf{u}_\epsilon - \mathcal{V}_\epsilon\|_{L^2(\Omega)} + \|\mathcal{V}_\epsilon - \mathcal{U}_\epsilon\|_{L^2(\Omega)}$$

mostly only remains to estimate oscillations

$$\|\mathcal{V}_\epsilon - \mathcal{U}_\epsilon\|_{L^2(\Omega)} \leq \epsilon \left\| \beta - \overline{\overline{\beta}} \right\|_{L^2(\Omega)} + \cdots + O(\epsilon^2) \leq \epsilon^{\frac{3}{2}^-}$$

□

Very weak solutions

Ω bounded set ,

$$\left\{\begin{array}{l} -\Delta \mathbf{v} + \nabla q = G, \\ \quad \mathrm{div}\, \mathbf{v} = 0 \\ \mathbf{v} = \xi \text{ on } \partial\Omega \end{array}\right. \text{in } \Omega \quad \text{and} \quad \left\{\begin{array}{l} -\Delta \mathbf{\Phi} + \nabla \varpi = g, \\ \quad \mathrm{div}\, \mathbf{\Phi} = 0 \\ \mathbf{\Phi} = 0 \text{ on } \partial\Omega \end{array}\right. \text{in } \Omega$$

Suppose that $(\mathbf{v}, q)$ and $(\mathbf{\Phi}, \varpi)$ are regular enough ($\mathbf{H}^2 \times H^1$)

$$(\Delta \mathbf{v} - \nabla q, \mathbf{\Phi})_\Omega - (\Delta \mathbf{\Phi} - \nabla \varpi, \mathbf{v}) = (\sigma_{\mathbf{v},q} \cdot \mathbf{n}, \mathbf{\Phi})_{\partial\Omega} - (\sigma_{\mathbf{\Phi},\varpi} \cdot \mathbf{n}, \mathbf{v})_{\partial\Omega}$$

use the rhs and the BC

$$-(G, \mathbf{\Phi})_\Omega + (g, \mathbf{v})_\Omega = -(\sigma_{\mathbf{\Phi},\varpi} \cdot \mathbf{n}, \xi)_{\partial\Omega}$$

if you can estimate $\mathbf{\Phi}$ as a function of the data g one obtains:

$$|(g, \mathbf{v})_\Omega| \leq \|G\|_{\mathbf{H}^{-1}(\Omega)} \|\mathbf{\Phi}\|_{\mathbf{H}^1(\Omega)} + \|\xi\|_{L^2(\partial\Omega)} \|\sigma_{\mathbf{\Phi},\varpi}\|_{L^2(\partial\Omega)} \leq C \|g\|_{L^2(\Omega)}$$

So

$$\|\mathbf{v}\|_{L^2(\Omega)} = \sup_{g \in L^2(\Omega)} \frac{|(\mathbf{v}, g)|}{\|g\|_{L^2(\Omega)}} \leq \|G\|_{\mathbf{H}^{-1}(\Omega)} + \|\xi\|_{L^2(\partial\Omega)}$$

Very weak solutions

Similarily

$$\begin{cases} -\Delta \mathbf{v} + \nabla q = G, \\ \quad \operatorname{div} \mathbf{v} = H \end{cases} \text{in } \Omega \qquad \text{and} \qquad \begin{cases} -\Delta \mathbf{\Phi} + \nabla \varpi = g, \\ \quad \operatorname{div} \mathbf{\Phi} = h \end{cases} \text{in } \Omega$$

$$\mathbf{v} \cdot \boldsymbol{\tau} = \xi, \quad \sigma_{\mathbf{v},q} \mathbf{n} \cdot \mathbf{n} = \chi \text{ on } \partial\Omega \qquad \mathbf{\Phi} \cdot \boldsymbol{\tau} = 0, \quad \sigma_{\mathbf{\Phi},\varpi} \mathbf{n} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega$$

Suppose that $(\mathbf{v}, q)$ and $(\mathbf{\Phi}, \varpi)$ are regular enough $(\mathbf{H}^2 \times H^1)$

$$\begin{aligned}(\Delta \mathbf{v} - \nabla q, \mathbf{\Phi})_\Omega - (\Delta \mathbf{\Phi} - \nabla \varpi, \mathbf{v}) =& (\sigma_{\mathbf{v},q} \cdot \mathbf{n}, \mathbf{\Phi})_{\partial\Omega} - (\sigma_{\mathbf{\Phi},\varpi} \cdot \mathbf{n}, \mathbf{v})_{\partial\Omega} \\ &+ (q, \operatorname{div} \mathbf{\Phi})_\Omega - (\varpi, \operatorname{div} \mathbf{v})_\Omega\end{aligned}$$

use the rhs and the BC

$$(G, \mathbf{\Phi})_\Omega - (H, \varpi)_\Omega - ((g, \mathbf{v})_\Omega - (h, q)_\Omega) = (\sigma_{\mathbf{\Phi},\varpi} \cdot \mathbf{n}, \xi \boldsymbol{\tau})_{\partial\Omega} - (\mathbf{\Phi} \cdot \mathbf{n} \chi)_{\partial\Omega}$$

if you can estimate $\mathbf{\Phi}, \varpi$ as a function of the data g, h one obtains:

$$\begin{aligned}|(g, \mathbf{v})_\Omega - (h, q)_\Omega| \leq & \|G\|_{\mathbf{H}^{-1}(\Omega)} \|\mathbf{\Phi}\|_{\mathbf{H}^1(\Omega)} + \|H\|_{L^2(\Omega)} \|\boldsymbol{\varpi}\|_{L^2(\Omega)} \\ &+ \|\xi\|_{L^2(\partial\Omega)} \|\sigma_{\mathbf{\Phi},\varpi} \mathbf{n} \boldsymbol{\tau}\|_{L^2(\partial\Omega)} + \|\chi\|_{H^{-1}(\partial\Omega)} \|\mathbf{\Phi} \cdot \mathbf{n} \boldsymbol{\tau}\|_{H^1(\partial\Omega)} \\ \leq & \, C \|g\|_{L^2(\Omega)} + C' \|h\|_{H^1(\Omega)}\end{aligned}$$

So

$$\|\mathbf{v}\|_{L^2(\Omega)} = \sup_{g \in L^2(\Omega)} \frac{|(\mathbf{v}, g)|}{\|g\|_{L^2(\Omega)}} \leq C \qquad \|q\|_{H^{-1}(\Omega)} = \sup_{h \in L^2(\Omega)} \frac{|(q, h)|}{\|h\|_{H^1(\Omega)}} \leq C'$$

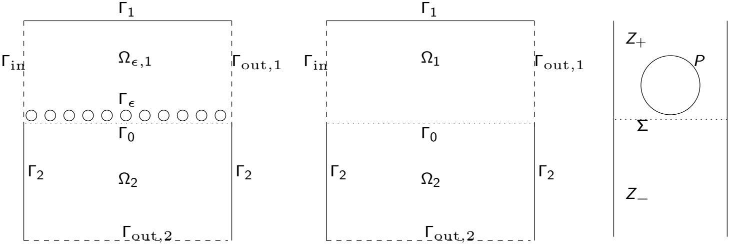
Wall law

- System of equations satisfied by $(\mathcal{U}_\epsilon, \mathcal{Q}_\epsilon)$?

$$
\begin{cases}
-\Delta \mathcal{U}_\epsilon + \nabla \mathcal{Q}_\epsilon = 0 \text{ in } \Omega \\
\operatorname{div} \mathcal{U}_\epsilon = 0 \\
\mathcal{U}_\epsilon = 0 \text{ on } \Gamma_1 \\
\mathcal{U}_\epsilon \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}, \\
\mathcal{Q}_\epsilon = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad \mathcal{Q}_\epsilon = 0 \text{ on } \Gamma_{\mathrm{out}} \\
\mathcal{U}_\epsilon = \epsilon \overline{\overline{\beta}} \frac{\partial \mathcal{U}_\epsilon}{\partial x_2} + O(\epsilon^2) \text{ on } \Gamma_0
\end{cases}
$$

Implicit boundary condition of mixed type

Notations and Methodology



1. Construction of a complete boundary layer corrector: Ω_ϵ
2. Derivation of wall laws: Ω_0

We denote:

- $P = \{y \in \mathbb{R}^2 \text{ s.t. } y_= r(\cos(\theta), \sin(\theta))\}$,
- Ω the "smooth domain", Γ^0 the fictitious interface,
- x the slow space variable , $y = \frac{x}{\epsilon}$ the fast one.

The problem

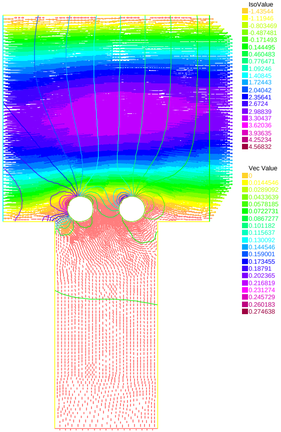
- The flow is laminar

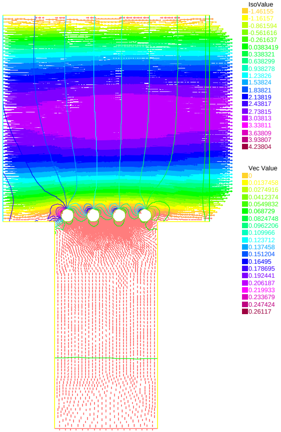
$$
\begin{cases}
-\Delta \mathbf{u}_\epsilon + \nabla p_\epsilon = 0 \text{ in } \Omega_\epsilon \\
\operatorname{div} \mathbf{u}_\epsilon = 0 \\
\mathbf{u}_\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_\epsilon \\
u_{\epsilon,1} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \\
u_{\epsilon,2} = 0 \text{ on } \Gamma_{\mathrm{out},2} \\
p_\epsilon = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad p_\epsilon = p_{\mathrm{out},1} \text{ on } \Gamma_{\mathrm{out},1}, p_\epsilon = p_{\mathrm{out},2} \text{ on } \Gamma_{\mathrm{out},2},
\end{cases}
$$

- Pressure imposed $\neq$ Dirichlet velocity as in

C. Conca.
Étude d'un fluide traversant une paroi perforée. I & II.
J. Math. Pures Appl. (9), 66(1):1–70, 1987.

Expected behaviour





The limit solution when $\epsilon \to 0$

- The Poiseuille flow

$$\begin{cases} -\Delta \mathbf{u}_0 + \nabla p_0 = [\sigma_{\mathbf{u}_0,p_0}] \cdot \mathbf{n}\, \delta_{\Gamma_0} \text{ in } \Omega \\ \operatorname{div} \mathbf{u}_0 = 0 \\ \mathbf{u}_0 = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \\ u_{0,2} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1}, \quad u_{0,1} = 0 \text{ on } \Gamma_{\mathrm{out},2} \\ p_0 = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad p_0 = 0 \text{ on } \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2} \\ \mathbf{u}_0 \neq 0 \text{ on } \Gamma_\epsilon \end{cases}$$

  where $[\sigma_{\mathbf{u}_0,p_0}] \cdot \mathbf{n}$ is the jump across Γ_0
- $(\mathbf{u}_0, p_0)$ is explicit and reads:

$$\begin{cases} \mathbf{u}_0(x) = \frac{p_{\mathrm{in}}}{2}(1 - x_2)x_2 \mathbf{e}_1 1_{\Omega_1}, \quad \forall x \in \Omega \\ p_0(x) = p_{\mathrm{in}}(1 - x_1) 1_{\Omega_1} \end{cases}$$

Theorem 1.11

$$\|\mathbf{u}_\epsilon - \mathbf{u}_0\|_{H^1(\Omega_\epsilon)^2} + \|p_\epsilon - p_0\|_{L^2(\Omega_\epsilon)} \leq k\sqrt{\epsilon}$$

where the constant k does not depend on ϵ.

Higher order approximation ?

- No flow at zeroth order through $\Gamma_{\mathrm{out},2}$!
- Errors treefold
  1. Dirichlet non homogeneous on Γ_ϵ
  2. Jumps at Γ_0 of $\frac{\partial u_{0,1}}{\partial x_2}$
  3. Jumps at Γ_0 of p_0
- Use of two kind boundary layers
  1. verical
  2. horizontal

Dirichlet correction

- Microscopic corrector *à la Mikelić*

$$\begin{cases} -\Delta\beta + \nabla\pi = 0 \text{ in } S \\ \operatorname{div}\beta = 0 \\ \beta = -y_2\mathbf{e}_1 \text{ on } P \\ \beta_2 \to 0, \quad |y_2| \to \infty \end{cases}$$

- Properties

Proposition 3

$\exists!(\beta,\pi)$, π defined up to a constant, s.t.

$$\nabla\beta \in L^2(S)^4, (\beta - \overline{\beta}) \in L^2(S), \quad \pi \in L^2_{\mathrm{loc}}(S)$$

Moreover, one has:

$$\beta(y) \to \overline{\beta}_\pm \mathbf{e}_1, \quad y_2 \to \pm\infty$$

and

$$\begin{cases} \overline{\beta}_2(y_2) = 0, & \forall y_2 \in \mathbb{R} \\ \overline{\beta}_1(y_2) = (\partial_{\mathbf{n}}(y_2\mathbf{e}_1).y_2\mathbf{e}_1)_P - |\nabla\beta|^2_{L^2(S)} + \overline{\beta}(0), & y_2 > y_{2,P}, \\ \overline{\beta}_1(y_2) = \overline{\beta}_1(0), & y_2 < 0 \\ \overline{\pi}(y_2) = 0, & y_2 > y_{2,P} \text{ and } y_2 < 0 \end{cases}$$

where $y_{2,P} := max_{y\in P} y_2$.

Normal derivative horizontal velocity correction

- Microscopic corrector *à la Mikelić*

$$\begin{cases} -\Delta\Upsilon + \nabla\varpi = \delta_\Sigma \mathbf{e}_1 \text{ in } S \\ \operatorname{div}\Upsilon = 0 \\ \Upsilon = 0 \text{ on } P \\ \Upsilon_2 \to 0, \quad |y_2| \to \infty \end{cases}$$

- Properties

Proposition 4

$\exists!(\Upsilon, \varpi)$, ϖ defined up to a constant, s.t.

$$\nabla\Upsilon \in L^2(S)^4, (\Upsilon - \overline{\Upsilon}) \in L^2(S), \quad \varpi \in L^2_{\mathrm{loc}}(S)$$

Moreover, one has:

$$\Upsilon(y) \to \overline{\Upsilon}_\pm \mathbf{e}_1, \quad y_2 \to \pm\infty$$

and

$$\begin{cases} \overline{\Upsilon}_2(y_2) = 0, & \forall y_2 \in \mathbb{R} \\ \overline{\Upsilon}_1(y_2) = \overline{\Upsilon}(0) + \overline{\beta}(0), & y_2 > y_{2,P}, \\ \overline{\Upsilon}_1(y_2) = \overline{\Upsilon}_1(0), & y_2 < 0 \\ \overline{\varpi}(y_2) = 0, & y_2 > y_{2,P} \text{ and } y_2 < 0 \end{cases}$$

where $y_{2,P} := max_{y \in P} y_2$.

Vertical correctors

- Microscopic corrector *à la Conca*

$$\begin{cases} -\Delta\chi + \nabla\eta = 0 \text{ in } S \\ \operatorname{div}\chi = 0 \\ \chi = 0 \text{ on } P \\ \chi_2 \to -1, \quad |y_2| \to \infty \end{cases}$$

- Properties

Proposition 5

$\exists!(\boldsymbol{\chi},\eta)$, η defined up to a constant, s.t.

$$\nabla\boldsymbol{\chi} \in L^2(S)^4, (\boldsymbol{\chi} - \overline{\chi}) \in L^2(S), \quad (\eta - \overline{\eta}) \in L^2_{\mathrm{loc}}(S)$$

Moreover, one has:

$$\boldsymbol{\chi}(y) \to -\overline{\chi}_{\pm}\mathbf{e}_2, \quad y_2 \to \pm\infty$$

and

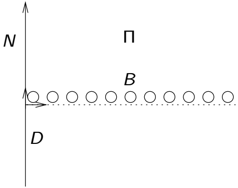
$$\begin{cases} \eta(y) = \overline{\eta}^{\pm}, \quad \forall y_2 \in \mathbb{R}_- \times]y_{2,P}, +\infty[, \\ |\nabla\boldsymbol{\chi}|^2_{L^2(S)} = [\overline{\eta}]^+_- \end{cases}$$

where $y_{2,P} := max_{y\in P} y_2$.

Vertical correctors

- Microscopic corrector

$$\begin{cases} -\Delta w_\beta + \nabla \theta_\beta = 0 \text{ in } \Pi \\ \operatorname{div} w_\beta = 0 \\ w_\beta = 0 \text{ on } D \cup B \\ \left.\begin{array}{l} w_\beta \wedge \mathbf{n} = \beta_2 \\ \theta_\beta = \pi \end{array}\right\} \text{ on } N \end{cases}$$



- the usual weighted Sobolev space :

$$W_\alpha^{m,p}(\Omega) := \left\{ v \in \mathcal{D}'(\Omega) \text{ s.t. } |D^\lambda v|(1+\rho^2)^{\frac{\alpha+|\lambda|-m}{2}} \in L^p(\Omega),\ 0 \le |\lambda| \le m \right\}$$

- Properties

Theorem 1.12

$$\exists!\ (\mathbf{w}, \theta) \in \mathbf{W}_\alpha^{1,2}(\Pi)^2 \times W_\alpha^{0,2}(\Pi) \textit{ if } \alpha < \frac{1}{2}$$

Localizing

1. The corner cut-of Set $\psi_1 := \overline{\psi}(x)$ and $\psi_2 := \overline{\psi}(x-(0,1))$, where $\overline{\psi}$ s.t.

$$\overline{\psi} := \begin{cases} 1 \text{ if } |x| \leq \frac{1}{3} \\ 0 \text{ if } |x| \geq \frac{2}{3} \end{cases}$$

   NB: $\partial_{\mathbf{n}} \overline{\psi} = 0$ on $\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1}$.

2. The "far from the corner" cut-off
   Φ complementary on $\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_2$ s.t.

$$\begin{cases} \psi + \Phi = 1 \\ \partial_{\mathbf{n}} \Phi = 0, \end{cases} \quad \text{on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_2$$

   for instance $\Phi(x) := 1 - \psi(0, x_2)$ for all $x \in \Omega$.

Macro corrector

$$
\begin{cases}
\Delta \mathbf{W} + \nabla Z = 0, \quad \text{in } \Omega_\epsilon \\
\mathrm{div}\, \mathbf{W} = 0 \\
\mathbf{W} \wedge \mathbf{n} = \epsilon \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} (\beta_\epsilon - \overline{\overline{\beta}}) \right\} \wedge \mathbf{n}\, \Phi, \quad \text{and } Z = \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} \beta \pi \right\} \Phi \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma \\
\mathbf{W} = 0 \text{ on } \Gamma_\epsilon \\
\mathbf{W} = \epsilon \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} (\beta_\epsilon - \overline{\beta}) \right\} \Phi \text{ on } \Gamma_1
\end{cases}
$$

Proposition 6

$\exists\,!$ solution $(\mathbf{W}, Z) \in \mathbf{H}^1(\Omega_\epsilon) \times L^2(\Omega_\epsilon)$, moreover:

$$\|\mathbf{W}\|_{\mathbf{H}^1(\Omega_\epsilon)} + \|Z\|_{L^2(\Omega_\epsilon)} \leq k e^{-\frac{\gamma}{\epsilon}}$$

rate γ and constant k do not depend on ϵ.

Macro corrector

$$\begin{cases} \Delta \mathbf{W} + \nabla Z = 0, \quad \text{in } \Omega_\epsilon \\ \operatorname{div} \mathbf{W} = 0 \\ \mathbf{W} \wedge \mathbf{n} = \epsilon \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} (\beta_\epsilon - \overline{\overline{\beta}}) \right\} \wedge \mathbf{n}\, \Phi, \quad \text{and } Z = \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} \beta\pi \right\} \Phi \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{o}} \\ \mathbf{W} = 0 \text{ on } \Gamma_\epsilon \\ \mathbf{W} = \epsilon \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} (\beta_\epsilon - \overline{\beta}) \right\} \Phi \text{ on } \Gamma_1 \end{cases}$$

define

$$\mathcal{W}_\epsilon(x) := \epsilon \left\{ \psi_1(x) \mathbf{w}\left(\frac{x}{\epsilon}\right) + \psi_2((1,0) - x) \mathbf{w}\left(\frac{(1,0)-x}{\epsilon}\right) \right\} + \mathbf{W}(x),$$

$$\mathcal{Z}_\epsilon(x) := \left\{ \psi_1(x) \theta\left(\frac{x}{\epsilon}\right) + \psi_2((1,0) - x) \theta\left(\frac{(1,0)-x}{\epsilon}\right) \right\} + Z(x),$$

First order approximation

$$
\begin{aligned}
\mathcal{V}_\epsilon &:= \mathbf{u}_0 + \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2}(\beta_\epsilon - \overline{\overline{\beta}}) + \left[\frac{\partial u_{0,1}}{\partial x_2}\right](\Upsilon_\epsilon - \overline{\overline{\Upsilon}}) + \frac{[p_0]}{[\overline{\overline{\eta}}]}(\chi_\epsilon - \overline{\overline{\chi}}) + \mathbf{u}_1 \right\} \\
&\quad + \epsilon^2 \left\{ p_{\mathrm{in}}(\varkappa_\epsilon - \overline{\overline{\varkappa}}) + \mathbf{u}_2 \right\} + \mathcal{W}_\epsilon \\
\mathcal{P}_\epsilon &:= p_0 + \left\{ \frac{\partial u_{0,1}}{\partial x_2}\pi_\epsilon + \left[\frac{\partial u_{0,1}}{\partial x_2}\right]\varpi_\epsilon + \frac{[p_0]}{[\overline{\overline{\eta}}]}(\eta_\epsilon - \overline{\overline{\eta}}) + \epsilon p_1 \right\} \\
&\quad + \epsilon p_{\mathrm{in}}(\mu_\epsilon - \overline{\overline{\mu}}) + \epsilon^2 p_2 + \mathcal{Z}_\epsilon
\end{aligned}
$$

- multi-scale version of boundary layer correctors:

$$
\beta_\epsilon(x) := \beta\left(\frac{x}{\epsilon}\right), \quad \boldsymbol{\Upsilon}_\epsilon(x) := \boldsymbol{\Upsilon}\left(\frac{x}{\epsilon}\right), \quad \chi_\epsilon(x) := \chi\left(\frac{x}{\epsilon}\right), \quad \varkappa_\epsilon(x) := \varkappa\left(\frac{x}{\epsilon}\right),
$$

Higher order macroscopic correctors

$$
\begin{cases}
-\Delta \mathbf{u}_1 + \nabla p_1 = 0 \text{ in } \Omega_1 \cup \Omega_2 \\
\operatorname{div} \mathbf{u}_1 = 0 \\
\mathbf{u}_1 = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \\
u_{1,2} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1}, \\
u_{1,1} = 0, \text{ on } \Gamma_{\mathrm{out},2} \\
p_1 = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2} \\
\mathbf{u}_1 = \left\{ \frac{\partial u_{0,1}}{\partial x_2} \overline{\overline{\beta}}^{\pm} + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \overline{\overline{\Upsilon}}^{\pm} \right\} \mathbf{e}_1 + \frac{[p_0]}{[\overline{\overline{\eta}}]} \overline{\overline{\chi}} \mathbf{e}_2 \text{ on } {\Gamma_0}^{\pm}
\end{cases}
$$

$(\mathbf{u}_1, p_1)$ give first order flow-rate trough Γ_0

Main convergence results

Theorem 1.13

Very weak solutions à la Conca

$$\|\mathbf{u}_\epsilon - \mathcal{V}_\epsilon\|_{L^2(\Omega_1 \cup \Omega_2)} + \|p_\epsilon - \mathcal{P}_\epsilon\|_{H^{-1}(\Omega_1 \cup \Omega_2)/\mathbb{R}} \leq k \epsilon^{\frac{3}{2}^-}$$

At this point the approximation $(\mathcal{V}_\epsilon, \mathcal{P}_\epsilon)$ is multi-scale

Averaged macroscopic approximation

Suppress the oscillating boundary layer, keep the rest:

$$\mathcal{U}_\epsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1$$
$$\mathcal{Q}_\epsilon := p_0 + \epsilon p_1$$

$$\begin{cases} -\Delta \mathcal{U}_\epsilon + \nabla \mathcal{Q}_\epsilon = 0 \text{ in } \Omega_1 \cup \Omega_2 \\ \operatorname{div} \mathcal{U}_\epsilon = 0 \\ \mathcal{U}_\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \\ \mathcal{U}_\epsilon \wedge \mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2}, \\ \mathcal{Q}_\epsilon = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad \mathcal{Q}_\epsilon = 0 \text{ on } \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2} \\ \mathcal{U}_\epsilon = \epsilon \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} \overline{\overline{\beta}}^{\pm} + \left[\dfrac{\partial u_{0,1}}{\partial x_2} \right] \overline{\overline{\Upsilon}}^{\pm} \right\} \mathbf{e}_1 + \epsilon \dfrac{[p_0]}{[\overline{\overline{\eta}}]} \overline{\overline{\chi}} \mathbf{e}_2 \text{ on } {\Gamma_0}^{\pm} \end{cases}$$

Averaged macroscopic approximation

Suppress the oscillating boundary layer, keep the rest:

$$\mathcal{U}_\epsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1$$
$$\mathcal{Q}_\epsilon := p_0 + \epsilon p_1$$

$$\begin{cases} -\Delta \mathcal{U}_\epsilon + \nabla \mathcal{Q}_\epsilon = 0 \text{ in } \Omega_1 \cup \Omega_2 \\ \operatorname{div} \mathcal{U}_\epsilon = 0 \\ \mathcal{U}_\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \\ \mathcal{U}_\epsilon \wedge \mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2}, \\ \mathcal{Q}_\epsilon = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad \mathcal{Q}_\epsilon = 0 \text{ on } \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2} \\ \mathcal{U}_\epsilon^+ \cdot \boldsymbol{\tau} = \epsilon(\overline{\overline{\beta}}^+ + \overline{\overline{\Upsilon}}^+) \dfrac{\partial \mathcal{U}_{\epsilon,1}}{\partial x_2}^+, \quad \dfrac{\mathcal{U}_\epsilon^+ \cdot \boldsymbol{\tau}}{\overline{\overline{\beta}}^+ + \overline{\overline{\Upsilon}}^+} = \dfrac{\mathcal{U}_\epsilon^- \cdot \boldsymbol{\tau}}{\overline{\overline{\beta}}^- + \overline{\overline{\Upsilon}}^-} \\ \mathcal{U}_\epsilon^+ \cdot \mathbf{n} = \mathcal{U}_\epsilon^- \cdot \mathbf{n} = \dfrac{\epsilon}{[\overline{\overline{\eta}}]}([\sigma_{\mathcal{U}_\epsilon, \overline{p}_\epsilon}] \cdot \mathbf{n}, \mathbf{n}) \end{cases}$$

Averaged macroscopic approximation

Suppress the oscillating boundary layer, keep the rest:

$$\mathcal{U}_\epsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1$$
$$\mathcal{Q}_\epsilon := p_0 + \epsilon p_1$$

Theorem 1.14

Very weak solutions à la Conca

$$\|\mathbf{u}_\epsilon - \mathcal{U}_\epsilon\|_{L^2(\Omega_1 \cup \Omega_2)} + \|p_\epsilon - \mathcal{Q}_\epsilon\|_{H^{-1}(\Omega_1' \cup \Omega_2)/\mathbb{R}} \leq k\epsilon^{\frac{3}{2}^-}$$

Proof.

Use the triangular inequality

$$\|\mathbf{u}_\epsilon - \mathcal{U}_\epsilon\|_{L^2(\Omega_1 \cup \Omega_2)} \leq \|\mathbf{u}_\epsilon - \mathcal{V}_\epsilon\|_{L^2(\Omega_1 \cup \Omega_2)} + \|\mathcal{V}_\epsilon - \mathcal{U}_\epsilon\|_{L^2(\Omega_1 \cup \Omega_2)}$$

mostly only remains to estimate oscillations

$$\|\mathcal{V}_\epsilon - \mathcal{U}_\epsilon\|_{L^2(\Omega_1 \cup \Omega_2)} \leq \epsilon \left\| \beta - \overline{\overline{\beta}} \right\|_{L^2(\Omega_1 \cup \Omega_2)} + \cdots + O(\epsilon^2) \leq \epsilon^{\frac{3}{2}^-}$$

$\square$

Compute the first order flow rate

- velocity profile normal direction to Γ_0

$$\mathcal{U}_{\epsilon,2}(x) = (u_{0,2} + \epsilon u_{1,2})(x) \equiv -\epsilon \frac{[p_0]}{[\overline{\overline{\eta}}]}(x)$$

- Solve with a computer a cell problem (cheap even in 3D):

$$[\overline{\overline{\eta}}] = \overline{\overline{\eta}}^{+} - \overline{\overline{\eta}}^{-}$$

- First order flow rate

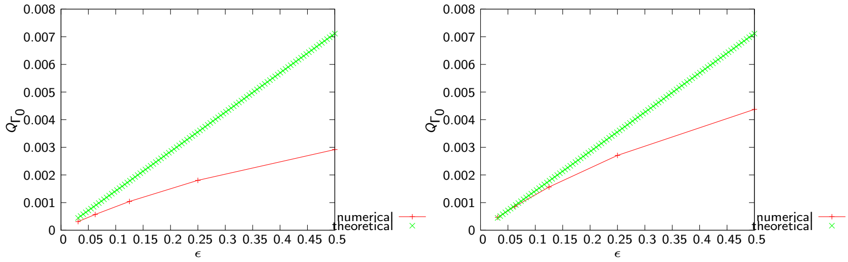
$$Q_{\Gamma_0} = \frac{\epsilon}{[\overline{\overline{\eta}}]} \int_a^b [p_0] \, dx_1$$

Numerical evidence

- Compute the exact problem
- Compute the boundary layers single # cell
  - Extract the constants at infinity

$$[\overline{\eta}] = 52.6961$$

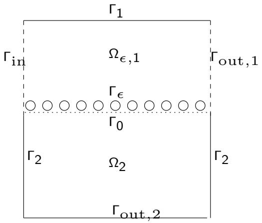
- Compute the flow-rate



Same problem

but . . .

$$\begin{cases} -\Delta \mathbf{u}_\epsilon + \nabla p_\epsilon = 0 \text{ in } \Omega_\epsilon \\ \operatorname{div} \mathbf{u}_\epsilon = 0 \\ \mathbf{u}_\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_\epsilon \\ u_{\epsilon,1} = 0 \text{ on } \Gamma_{\text{in}} \cup \Gamma_{\text{out},1} \\ \mathbf{u}_\epsilon = 0 \text{ on } \Gamma_{\text{out},2} \\ p_\epsilon = p_{\text{in}} \text{ on } \Gamma_{\text{in}}, \quad p_\epsilon = p_{\text{out},1} \text{ on } \Gamma_{\text{out},1}, \end{cases}$$



Pressure imposed at inlet and outlet but not at $\Gamma_{\text{out},2}$

When ϵ goes to 0

- The Poiseuille flow

$$\begin{cases} -\Delta \mathbf{u}_0 + \nabla p_0 = [\sigma_{\mathbf{u}_0,p_0}] \cdot \mathbf{n}\, \delta_{\Gamma_0} \text{ in } \Omega \\ \operatorname{div} \mathbf{u}_0 = 0 \\ \mathbf{u}_0 = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_{\mathrm{out},2} \\ \mathbf{u}_0 \wedge \mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \\ p_0 = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad p_0 = 0 \text{ on } \Gamma_{\mathrm{out},1} \\ \mathbf{u}_0 \neq 0 \text{ on } \Gamma_\epsilon \end{cases}$$

- $(\mathbf{u}_0, p_0)$ is explicit and reads:

$$\begin{cases} \mathbf{u}_0(x) = \frac{p_{\mathrm{in}}}{2}(1-x_2)x_2 \mathbf{e}_1 1_{\Omega_1}, \quad \forall x \in \Omega \\ p_0(x) = p_{\mathrm{in}}(1-x_1)1_{\Omega_1} + p_0^- 1_{\Omega_2}, \quad \forall p_0^- \in \mathbb{R} \end{cases}$$

Theorem 1.15

$$\left\| \mathbf{u}_\epsilon - \mathbf{u}_0 \right\|_{H^1(\Omega_\epsilon)^2} + \left\| p_\epsilon - p_0 \right\|_{L^2(\Omega_{\epsilon,1})} + \left\| p_\epsilon - p_0 \right\|_{L^2(\Omega_2)/\mathbb{R}} \leq k\sqrt{\epsilon}$$

where the constant k does not depend on ϵ.

First order approximation

Again the same trick

$$\mathcal{V}_\epsilon := \mathbf{u}_0 + \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2}(\beta_\epsilon - \overline{\overline{\beta}}) + \left[\frac{\partial u_{0,1}}{\partial x_2}\right](\Upsilon_\epsilon - \overline{\overline{\Upsilon}}) + \frac{[p_0]}{[\overline{\overline{\eta}}]}(\chi_\epsilon - \overline{\overline{\chi}}) + \mathbf{u}_1 \right\}$$
$$+ \epsilon^2 \left\{ p_{\rm in}(\varkappa_\epsilon - \overline{\overline{\varkappa}}) + \mathbf{u}_2 \right\} + \mathcal{W}_\epsilon$$
$$\mathcal{P}_\epsilon := p_0 + \left\{ \frac{\partial u_{0,1}}{\partial x_2}\pi_\epsilon + \left[\frac{\partial u_{0,1}}{\partial x_2}\right]\varpi_\epsilon + \frac{[p_0]}{[\overline{\overline{\eta}}]}(\eta_\epsilon - \overline{\overline{\eta}}) + \epsilon p_1 \right\}$$
$$+ \epsilon p_{\rm in}(\mu_\epsilon - \overline{\overline{\mu}}) + \epsilon^2 p_2 + \mathcal{Z}_\epsilon$$

- multi-scale version of boundary layer correctors:

$$\beta_\epsilon(x) := \beta\left(\frac{x}{\epsilon}\right), \quad \Upsilon_\epsilon(x) := \Upsilon\left(\frac{x}{\epsilon}\right), \quad \chi_\epsilon(x) := \chi\left(\frac{x}{\epsilon}\right), \quad \varkappa_\epsilon(x) := \varkappa\left(\frac{x}{\epsilon}\right),$$

The pressure in the sac

$$\begin{cases} -\Delta \mathbf{u}_1 + \nabla p_1 = 0 \text{ in } \Omega_1 \cup \Omega_2 \\ \operatorname{div} \mathbf{u}_1 = 0 \\ \mathbf{u}_1 = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_{\mathrm{out},2} \\ \mathbf{u}_1 \wedge \mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1}, \\ p_1 = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \\ \mathbf{u}_1 = \left\{ \frac{\partial u_{0,1}}{\partial x_2} \overline{\overline{\beta}}^{\pm} + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \overline{\overline{\Upsilon}}^{\pm} \right\} \mathbf{e}_1 + \frac{[p_0]}{[\overline{\overline{\eta}}]} \overline{\overline{\chi}} \mathbf{e}_2 \text{ on } {\Gamma_0}^{\pm} \end{cases}$$

divergence condition and normal veolicty imposed on Γ_0:

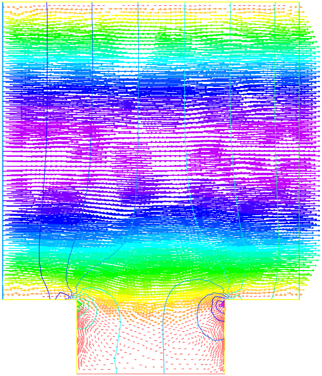
$$\int_{\Omega_2} \operatorname{div} \mathbf{u}_1 dx = \int_{\partial\Omega_2} \mathbf{u}_1 \cdot \mathbf{n} d\sigma = \int_{\Gamma_0} \mathbf{u}_1 \cdot \mathbf{n} d\sigma = 0$$

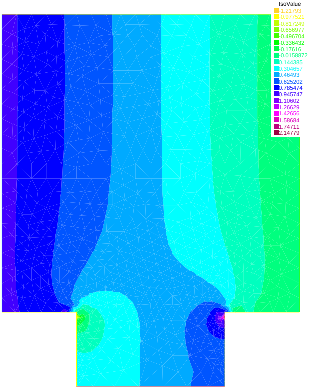
gives in the sac

$$|\Gamma_0| p_0^- = \int_{\Gamma_0} p_0^+(x_1, 0) dx_1$$

Numerics

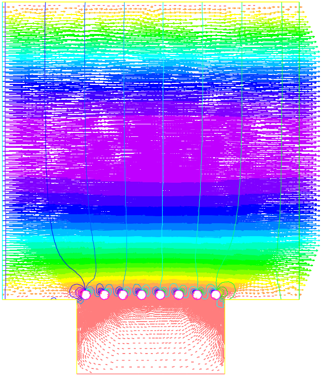
A numerical “pathological” case

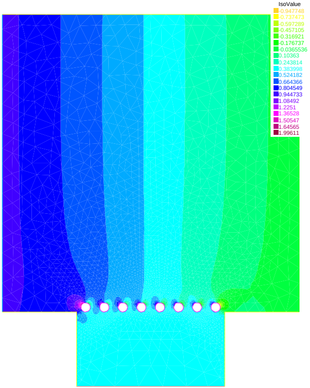




Numerics

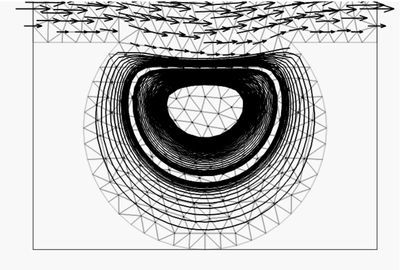
A numerical “stented” case

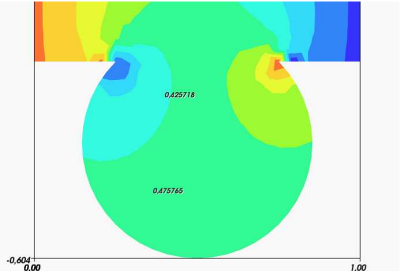




Numerics

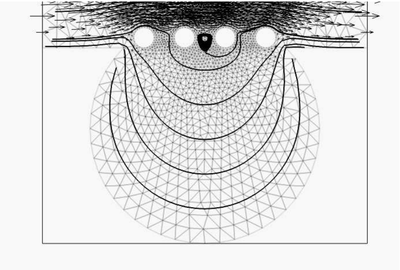
A more realistic geometry: the "pathological" case

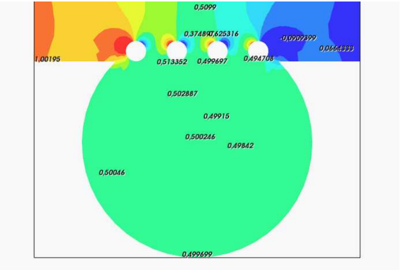




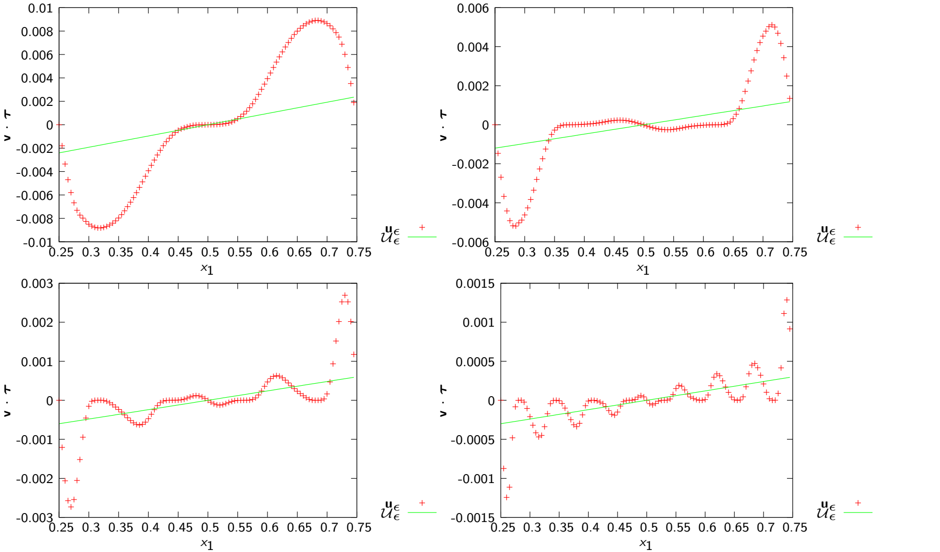
Numerics

A more realistic geometry: the “stented” case





Norma velocity acros Γ_0



Conclusion & Perspectives

Conclusion

- Our approach introduces the vertical correctors
  - Not present in the literature
  - General setting

Perspectives

- Time dependent case: Womersley profile
- Curved boundaries
- Navier-Stokes
- Cell growth