Homework

Exercise 1:

We assume that :

(i) $k \in L^1_{\text{loc}}(\mathbb{R}_+)$
(ii) $\forall \sigma > 0$,

$$\int_{\mathbb{R}_+} \exp(-\sigma t)\,|k(t)|\,dt < \infty$$

(iii) moreover one has

$$\sup_{\mathrm{Re}\, z>0} \left|\hat{k}(z)\right| + \left|1-\hat{k}(z)\right|^{-1} < \infty$$

then show that, if r is the resolvent associated to k

$$r \in L^1(\mathbb{R}_+) \Leftrightarrow k \in L^1(\mathbb{R}_+)$$

Exercise 2:

Consider the model for $(a,t) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$\begin{cases} \partial_t p(a,t) + \partial_a p(a,t) + Bp(a,t) = \nu q(a,t), \\ \partial_t q(a,t) + \partial_a q(a,t) + \nu p(a,t) = 0, \\ p(a=0,t) = 0, \quad q(a=0,t) = 2B\displaystyle\int_{\mathbb{R}_+} p(\tilde{a},t)d\tilde{a} \end{cases} \tag{1}$$

1. Assuming that $B > 0, \nu > 0$, and $B \neq \nu$, show that there exist first eigenelements $(\lambda_0 > 0, P > 0, Q > 0, \varphi_P \geq 0, \varphi_Q \geq 0)$ and compute them explicitly.
2. Give the conserved quantity and the family of generalized relative entropies (GRE) for this system. Show that GRE is decreasing with time.
3. Give the natural weight for the L^1 estimates; Give the L^∞ estimates.

Exercise 3:

We consider the ordinary differential equation

$$\begin{cases} y''(t) + a(t)y'(t) + b(t)y(t) = g(t), \quad t > 0 \\ y'(0) = \alpha \\ y(0) = \beta \end{cases} \tag{2}$$

where the functions a, b and g are $C^\infty(\mathbb{R}_+)$. Find an equivalent formulation with a Volterra equation :

$$y(t) = \int_0^t K(t,s)y(s)ds + f(t)$$

where K and f should be made explicit as functions of a, b and g.