Some structured equations for bio-maths

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Abstract

First, we deal with modeling aspects leading to a certain system of structured integro-differential equations in the context of cell motility. We introduce some of the challenges related to asymptotic features of this system. We give some classical results in the theory of Volterra integral equations. We then present the renewal equation and the generalized entropy method. Next, we introduce more specific tools that turn out to provide explicit computations and new stability estimates. We end with a complete asymptotic analysis of a delayed heat equation and show the convergence of its solutions towards the solution of a heat equation where the time derivative is multiplied by a function accounting in some sense the microscopic structure of the underlying adhesion mechanisms.

Contents

1 Introduction

In these lecture notes, I will present basic concepts in order to handle the mathematical analysis of adhesions terms in certain models for cell-motility.

In this introduction, I will briefly present the biological context and the axisymmetric model derived by C. Schmeiser and D. Oelz [21, 22, 12]. I will focus on specific parts of this approach related to adhesion mechanisms and try to show some mathematical problems related to these. In particular certain formal asymptotics wrt an adimensionalised parameter ε led to friction terms in the Euler Lagrange system associated to a gradient flow minimization procedure. Our final goal is to prove rigorously that these formal computations are actually true.

In a first step I will present a minimal toy model containing similar difficulties. This minimum model turns out to be an equation of Volterra type commonly called the *renewal equation*. I will then introduce mathematical standard tools based on the notion of resolvent that allow to solve it in an abstract way [8, Chap. 2]. Then using this concept of resolvent, we characterize the long-time asymptotics of the solution [8, Chap. 2, 4 and 7].

In order to give an overview of other types of existing methods, we present also the general entropy method [25, Chap. 3]. It relies on Perron-Frobenius arguments and describes in another way long time asymptotic features of the solutions.

Because it's not clear how to extend general enthropy methods to our problems, we then make supplementary hypotheses on Volterra equation's kernel. Namely we assume that it is in some sense dissipative. This allows conservation properties and much more of explicit computations. In particular we show that it's possible to re-formulate the problem thanks to the elongation variable. This new framework allows to obtain a stability results and we're able to fully characterize, under minimal hypothesis on the kernel, the convergence of our solutions to an asymptotic profile. This is not possible to our knowledge with the previous methods.

In order to illustrate all these results in section **??** we present a numerical approximation of our problem. To insure the consistency of our numerical discretization with the continuous problem, we show that, when the discretization parameter Δ tends to zero, solutions provided by the numerical scheme converge to those of the continuous problem. Then we show some numerical results that help to give partial conclusions and we list open problems of interest also for biological reasons.

Because the final convergence problem involves the space variable, and an elliptic operator, in the last section we introduce volterra equation supplemented by a Laplace operator in space. From the modelling point of view, this represents an elastic filament whose reference length is zero. In order to fit to the non-linear framework of the starting model we base our approach on DeGiorgi's minimizing movements and gradient flow techniques [2]. As the Volterra integral operator introduces an infinitely distributed delay, we show how to extend previously mentioned tools to our setting.

At the same time minimizing movements extend the numerical scheme presented in section **??** to this new framework. Energy estimates allow to pass to the limit wrt the discretization parameter Δ and since these are also uniform with respect to ε, they also hold for the continuous solution. This allows to recover the continuous ε-limit as well.

1.1 Filament based lamelipodium model (FBLM)

Cells migrate by protruding at the front and retracting at the rear. Protrusion occurs in thin membrane bound cytoplasmic sheets, 0.2 to 0.3 mμ thick and up to several microns long, termed **lamellipodia**. The major structural components of lamellipodia are actin filaments, which are organised in a more or less two-dimensional diagonal array with the fast growing, plus ends of the actin filaments directed forwards, abutting the membrane [30]. Protrusion is effected by actin polymerisation, whereby actin monomers are inserted at the plus ends of the filaments at the membrane interface and removed at the minus ends, throughout and at the base of the lamellipodium, in a treadmilling regime [24]. Stabilisation of the actin meshwork is achieved by the cross-linking of the filaments by actin-associated proteins, such as filamin[20] as well as protein complexes, such as the Arp2/3 complex [26], although the density and location of such cross-links remains to be established. Since actin polymerisation is involved in diverse motile processes aside from cell motility, including endocytosis and the propulsion of pathogens that invade cytoplasm, the question of how actin filaments are able to push against a membrane has spawned the development of various models.

Comprehensive modelling efforts were initiated in 1996 and fall into two groups. The first group includes continuum models for the mechanical behaviour of cytoplasm : a two phase formulation for cytosol and the actin network [1]; a one dimensional viscoelastic model[7]; a one dimensional model for the actin distribution[19]; and a two dimensional elastic continuum model[27]. The second group makes presumptions about the microscopic organisation of the actin network. The Brownian ratchet model for the polymerisation process introduced by Mogilner and Oster considers actin cross-linking proteins as stabilisers of the lamellipodium meshwork, allowing enough flexibility for actin filaments to bend away from the membrane to accept actin monomers. Other models are based on the current idea, that the actin filaments in lamellipodia form a branched network with the Arp2/3 complex at the branch points.

The filament density decreases from the front to the rear of the lamellipodium, indicative of a graded distribution of filament lengths. According to this structural information, we present a quasi-stationary modelling approach for the simulation of the turnover of the lamellipodium in a circularly symmetric cell, corresponding to real situations such as cytoplasmic fragments of keratocytes. Our approach differs from others in that we describe the lamellipodium in terms of a continuous distribution of filaments of graded length and their linkages. The model depends on four primary parameters : bending elasticity of actin filaments, cross-links between the filaments; the resistance against polymerisation by the membrane; and interactions between the filaments and the substrate via trans-membrane linkages. With a selected set of parameters we compute the dynamics of the network organisation. The simulations reproduce several features also found experimentally: treadmilling, the lateral flow of filament plus ends along the front edge, and persistence of the network organisation after achievement of a steady state.

1.1.1 Modelling hypotheses

A1 At each point in the lamellipodium, actin filaments have one of two directions in the diagonal network, represented by oriented, slightly curved segments with the barbed ends attached to the leading (outer) edge of the lamellipodium. Filaments are inextensible.

A2 The lamellipodium is two dimensional and rotationally symmetric, i.e., at any point in time it has the shape of a circular ring. This compares with the situation of a radially

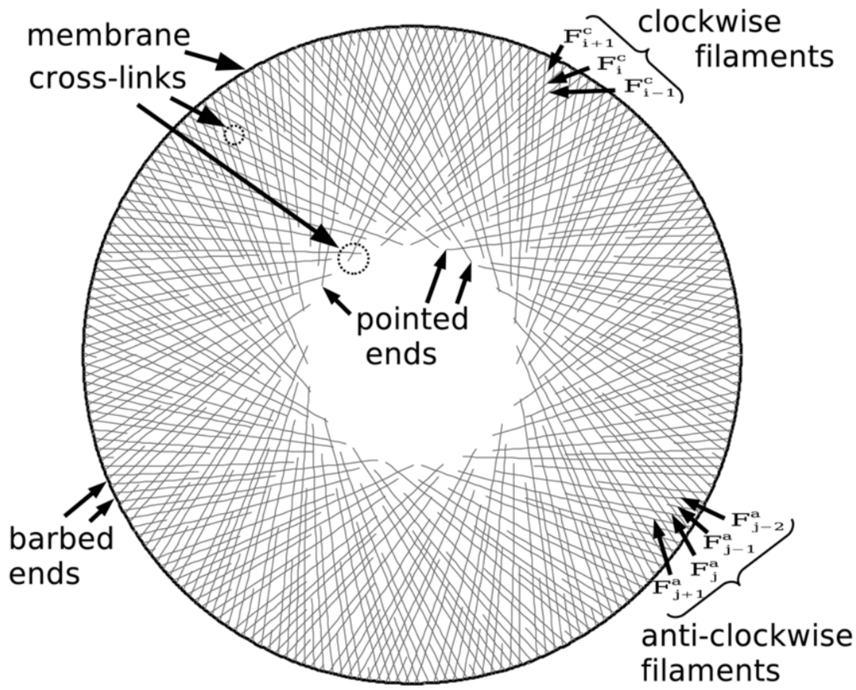


Figure 1: Constituent elements of the model

spread cytoplast from a keratocyte.

A3 Filaments polymerise at the barbed ends with constant polymerisation speed. Depolymerisation at the pointed ends is a stochastic process with prescribed distribution.

As a consequence of A1 and A2 the lamellipodium has the organization depicted in Figure 1.

There are two families of locally parallel filaments. Looking from the centre of the lamellipodium ring, the filaments in the first group bear to the left and the second group to the right; referred to as clockwise and anti-clockwise filaments. As a consequence of rotational symmetry, all filaments can be constructed from one reference filament (which, without loss of generality, we shall take clockwise) with the maximal filament length. All clockwise filaments can then be constructed by rotation of the reference filament and subsequent random cutting at the pointed end; correspondingly, all anti-clock-wise filaments are created by reflection, rotation and cutting.

A central feature of the model is the description of production and decay of cross-links and integrins, consistent with dynamic association/dissociation of linkage molecules with the actin network, leading to the next assumption.

A4 A cross-link is an elastic connection between a clockwise and an anti-clockwise filament. The cross-link has both an elastic and a torsional component. Cross-links form and break stochastically at the crossing between two filaments with at most one cross-link for any pair of filament crossing points at any time.

A5 An adhesion is an elastic link between a filament and a point on the substrate via a transmembrane linkage. Adhesions can form or break spontaneously, breaking being dependent on the degree of link extension.

A6 The position of the filaments in time is determined by a quasistationary balance of elastic forces resulting from bending of the filaments, stretching and twisting the cross-links, stretching the adhesion linkages, and stretching the cell membrane.

The quasistationary assumption means that we neglect elastic oscillations, assuming that the filament network is damped by viscous forces in the cytosol and that the system therefore always operates at minimal potential energy. Thus, the evolution of the network is a consequence of actin polymerization dynamics together with the creation and breaking of cross-links and adhesions. In summary, the model we present has two major ingredients

(1) the making and breaking of cross-links in the filament network and between the filaments and the matrix, based on renewal equations; and
(2) minimisation of the potential energy of the system.

1.1.2 The corresponding mathematical model

In order to obtain a feasible mathematical description we adopt a homogenisation limit, based on the assumption that the density of filaments within the lamellipodium is very high; we let the number of filaments tend to infinity in order to obtain a model based on continuous quantities instead of discrete ones.

With the maximal filament length $L = 1$, an arc length parameterisation of the reference filament at time t is given by $\{x(s,t) : 0 \leq s \leq 1\}$, where $s = 0$ corresponds to the pointed and $s = L$ to the barbed end. We shall need the representation $x(t,s) = |x(t,s)|(\cos\varphi(t,s), \sin\varphi(t,s))$ in polar coordinates with the angle $\varphi(s,t) \in \mathbb{S}^1$. We expect $|x(t,s)|$ to be strictly increasing with respect to s. Then $|x(t,0)|$ is the inner radius of the lamellipodium and $|x(t,L)|$ the radius of its leading edge at time t. The reference filament is assumed to be clockwise *i.e.* $\varphi(s,t)$ is a strictly decreasing function of s. Note that $|\partial_s x(s,t)| = 1$.

Once defined a set of mechanical energy terms related to the previous hypotheses, in [21], the authors use DeGiorgi's *minimizing movements* (this will be described with much details in sections **??** and 3) and obtain the semi-discretized scheme and some stability estimates. These estimates allow to pass to the limit wrt the time discretisation parameter and show that the continuous vector solution (x, λ) satisfies Euler-Lagrange's system of equations reading :

$$\begin{cases} \underbrace{\kappa^B \partial_s^2(\eta \partial_s^2 x)}_{\text{bending}} - \underbrace{\partial_s(\eta\lambda\partial_s x)}_{\text{unextensibility constraint}} + \eta \underbrace{\mu^A D_t x}_{\text{adhesion}} + \\ \quad + \underbrace{\partial_s\left(\eta^2 \mu^T \left(\arccos\left(\partial_s |x|\right) - \varphi_0\right) \partial_s x^\perp\right)}_{\text{twisting}} + \eta^2 \underbrace{\left(\mu^S D_t \varphi\right) x^\perp}_{\text{stretching}} = 0 \\ |\partial_s x| = 1, \end{cases}$$

complemented by a set of natural boundary conditions that we do not detail here. It is a force balance equation. Red terms in this system represent velocity terms scaled by constant factors and that is why they are called friction terms. They are formal limits when ε goes to zero of the following integral operators :

$$\mathcal{L}_\varepsilon[x](s,t) := \frac{1}{\varepsilon} \int_{\mathbb{R}_+} \left(x(s,t) - x(s + \varepsilon v a, t - \varepsilon a)\right) \rho_\varepsilon(s,a,t) da$$

and a similar one for the stretching part. The weight ρ_ε is an unknown of the problem solving a non-local transport problem :

$$\begin{cases} (\varepsilon\partial_t + \partial_a + \zeta(s,a,t,\dots))\,\rho_\varepsilon = 0, & (s,a,t) \in \Omega \times \mathbb{R}_+ \times (0,T) \\ \rho_\varepsilon(s,0,t) = \beta(s,\dots)\left(1 - \int_{\mathbb{R}_+} \rho_\varepsilon(s,a,t)da\right), & (s,a,t) \in \Omega \times \{0\} \times (0,T) \\ \rho_\varepsilon(s,a,0) = \rho_I(s,a), & (s,a,t) \in \Omega \times \mathbb{R}_+ \times \{0\}. \end{cases} \tag{1}$$

where β (resp. ζ) represents linkages' on (resp. off) rate and the non-local term represents a saturation effect accounting for availability of adhesion sites. When the on/off rates are prescribed the ρ_ε system can be solved independently and $(\rho_\varepsilon, x_\varepsilon)$ are *weakly* coupled. If instead β depends on the position x_ε and ζ depends on the elongation

$$u_\varepsilon(s,a,t) := \frac{x_\varepsilon(s,t) - x_\varepsilon(s+\varepsilon va, t-\varepsilon a)}{\varepsilon}$$

then the system is said to be *strongly* coupled. In these lectures no strong coupling should be considered.

Note that a Taylor expansion of $x_\varepsilon(s+\varepsilon va, t-\varepsilon a)$ at the point (s,t) leads to the expansion :

$$u_\varepsilon(s,a,t) = (\partial_t - v\partial_s)\,x_\varepsilon(s,t) + O(\varepsilon)$$

which defines $D_t := (\partial_t - v\partial_s)$ naturally in the equations above. The main project of this course and [15, 16, 18] etc is to give a rigorous meaning to this formal computations. In the next section, we present our strategy in order to tackle this ambitious goal.

1.2 *Simplified* (open) problems

Terms of the first line above provide an equation for the equilibrium of a single filament and represent to some extent the ultimate goal of this research : rigorous justification of memory-like adhesion microscopic description towards friction. Indeed we want to show that starting from the system :

$$\begin{cases} \frac{1}{\varepsilon}\int_{\mathbb{R}_+} (x_\varepsilon(s,t) - x_\varepsilon(s,t-\varepsilon a))\,\rho_\varepsilon(s,a,t)da + x_\varepsilon'''' - (\lambda_\varepsilon x_\varepsilon')' = 0, \\ x_\varepsilon''|_{s=0,1} = 0, \\ x_\varepsilon''' - \lambda_\varepsilon x_\varepsilon'|_{s=0,1} = 0, \\ |x_\varepsilon'| = 1, \quad \forall s \in (0,1) \\ x_\varepsilon(s,t) = x_p(s,t), \quad \forall s \in (0,1), \quad \forall t < 0. \end{cases} \tag{2}$$

the solution x_ε converges towards the system [23]:

$$\begin{cases} \mu_{1,0}(s,t)\partial_t x_0 + x_0'''' - (\lambda_0 x_0')' = 0 \\ x_0''|_{s=0,1} = 0, \\ x_0''' - \lambda_\varepsilon x_0'|_{s=0,1} = 0, \\ |x_0'| = 1, \quad \forall s \in (0,1) \\ x_0(s,t) = x_p(s,0), \quad \forall s \in (0,1), \quad \forall t = 0. \end{cases}$$

where $\rho_0 := \lim_{\varepsilon\to 0} \rho_\varepsilon$ and $\mu_{1,0}(s,t) := \int_{\mathbb{R}_+} a\rho_0(s,a,t)da$. Because even the problem (2) is non-linear and too difficult to handle directly, in a series of works [15, 16, 17], we studied a simplified system where we focused on the integral part and replaced the remaining terms with an imposed external force without any space dependence. This led to study the *weak* coupling between (1) (forgetting the s-dependence) and a simple Volterra equation

$$\begin{cases} \frac{1}{\varepsilon}\int_{\mathbb{R}_+} \left(x_\varepsilon(t) - x_\varepsilon(t-\varepsilon a)\right)\rho_\varepsilon(a,t)da = f(t), & t > 0, \\ x_\varepsilon(t) = x_p(t), & t \leq 0. \end{cases} \tag{3}$$

and we studied the convergence of x_ε towards the solution of the limit problem :

$$\begin{cases} \mu_{1,0}\partial_t x_0(t) = f(t), & t > 0, \\ x_0(0) = x_p(0), & t = 0. \end{cases} \tag{4}$$

In these lectures, in order to simplify even more the problem we assume that the kernel ϱ is time and thus ε independent and is either a given fixed positive function $\varrho \in L^1(\mathbb{R}_+)$ or it solves the limit (when $\varepsilon \to 0$) equation associated to (1) which reads :

$$\begin{cases} \left(\partial_a + \zeta(a)\right)\varrho = 0, & a \in \mathbb{R}_+ \\ \varrho(0) = \beta, & a = 0 \end{cases} \tag{5}$$

For $\varepsilon = 1$, one understands easily that (3) can be reformulated as

$$x(t) - \int_0^t x(t-a)k(a)da = \frac{f}{\mu_0} + \int_t^\infty x_p(t-a)k(a)da, \quad k(a) := \frac{\varrho(a)}{\mu_0}, \quad \mu_0 := \int_{\mathbb{R}_+} \varrho(a)da$$

which can be recasted into a standard Volterra equation

$$x(t) - (k \star x)(t) = F(t), \quad F(t) := \frac{f(t)}{\mu_0} + \int_t^\infty x_p(t-a)k(a)da$$

where the convolution is meant in the following sense :

$$(k \star x)(t) = \int_0^t x(t-a)k(a)da.$$

with the particular feature that $\int_{\mathbb{R}_+} k(a)da = 1$, wich means that $k(a)da$ is a unit measure. In this case the latter equation is called *renewal equation* and has been widely studied in the context of age structured populations.

It appears that the ε-scaling above is closely connected in some cases to the long time asymptotic profile of x. This is why in first sections of these lectures we focus on describing the leading profile of x when t grows large. To do so, we have at our disposal several tools :

- the Laplace transform (sec. 2.2),
- the General Entropy Method (sec. 2.3),
- the elongation variable formulation (sec 2.4)

This motivates the contents of the lectures and the order of exposition. In what follows we show mainly that if

$$\mu_1 := \int_{\mathbb{R}_+} a\varrho(a)da < \infty,$$

then, when $t \to \infty$, $x(t) \sim f_\infty t/\mu_1$ where $f_\infty := \lim_{t\to\infty} f(t)$ for instance, which is exactly the profile satisfied for large times by x_0 solving (4) above.

In order to illustrate what happens in the border cases when for instance, the first order moment of ϱ becomes unbounded, we discretise our problem in the spirit of *Gradient Flows* leading to the *Implicit Euler* discretization of the Volterra equation. We provide a complete analysis of the convergence when the discretisation parameter $\Delta = \Delta t = \Delta a$ tends to zero. We then conclude by listing interesting open problems suggested by the numerics.

Extending the *Gradient Flow* method, in the last part (section 3), and in order to get closer to the original problem, we introduce the space variable and a Laplace operator and focus on the system :

$$\begin{cases} \mathcal{L}_\varepsilon[x_\varepsilon](s,t) - \Delta x_\varepsilon(s,t) = f(s,t) \\ \partial_\nu x_\varepsilon \left|_{s=0,1}\right. = 0, \\ x_\varepsilon(s,t) = x_p(s,t), \quad \forall s \in (0,1), \quad \forall t < 0. \end{cases} \tag{6}$$

Using DeGiorgi's *minimizing movements* [2], we $\frac{1}{2}$-discretize the energy minimization procedure associated with this problem in age and time (but not in space) and show that when Δ goes to zero, the discrete solution converges towards x_ε solving (6). This is possible thanks to specific energy estimates, more complicated to obtain than for the standard heat equation for instance. Then show the convergence of x_ε to x_0 solving the friction-heat equation :

$$\begin{cases} \mu_1(s)\partial_t x_0 - \Delta x_0(s,t) = f(s,t) \\ \partial_\nu x_0 \left|_{s=0,1}\right. = 0, \\ x_0(s,t) = x_p(s,0), \quad \forall s \in (0,1), \quad \forall t = 0. \end{cases} \tag{7}$$

This is possible thanks to stability estimates already obtained for the discrete scheme which are stable both wrt ε and Δ and a stronger supplementary estimate fully based on the reformulation of the problem in the *elongation* variable.

2 The renewal equation

If we turn back to (3), for $\varepsilon = 1$ it can be rewritten as :

$$\mu_0 x(t) - \int_0^t x(t-a)\varrho(a)da = f(t) + \int_t^\infty x_p(t-a)\varrho(a)da$$

which, dividing by μ_0 and redefining f accordingly, transforms into :

$$x(t) - \int_0^t x(t-a)k(a)da = f(t), \quad k(a) := \frac{\varrho(a)}{\mu_0}$$

the latter function having total mass 1. By *renewal equation* we intend precisely that the total mass of k is one, *i.e.*

$$\int_{\mathbb{R}_+} k(a)da = 1.$$

Before treating this special case, we aim to give a general framework allowing to handle Volterra equations of this kind.

2.1 Constructing the resolvent

If k and x are measurable functions we denote

$$(k \star x)(t) := \int_0^t k(a)x(t-a)da.$$

We consider the problem find x in a suitable space s.t.

$$x(t) = k \star x(t) + f(t), \tag{8}$$

First we write few words about convolution :

Theorem 1. Let J be $\mathbb{R}_+$, or an interval with left end-point 0 and right end-point T, where $0 < T < \infty$, and let a be measurable on J. Then each of the following conditions implies that $a \in L^1(J)$:
i) $a \star b \in L^\infty(J)$ for every $b \in L^\infty(J)$
ii) $a \star b \in L^1(J)$ for every $b \in L^1(J)$

Proof. We admit the proof since it is technical and far from the scope of this lecture notes. Nevertheless, the interested reader can consult [8, Lemma 3.10.2 and 9.10.4 p.270–274]. ∎

2.1.1 The Laplace transform

Definition 1. The Laplace transform of a function a defined on $\mathbb{R}_+$ is s.t.

$$\hat{a}(z) := \int_{\mathbb{R}_+} a(t)\exp(-zt)dt$$

defined on those $z \in \mathbb{C}$ for which the integral exists as an absolutely convergent Lebesgue's integral. The (bilateral) Laplace transform of a functiona, defined on $\mathbb{R}$, is the function

$$\hat{a}(z) := \int_{\mathbb{R}} a(t)\exp(-zt)dt$$

which is defined for those $z \in \mathbb{C}$ for which the integral exists.

Theorem 2.
(i) If $a \in L^1(\mathbb{R})$, then $\hat{a}(z_0)$ is defined and continuous on the line $\Re z = 0$, $\hat{a}(i\omega) \to 0$ as $|\omega| \to \infty$, and

$$\hat{a}(i\omega) = 0, \quad \forall \omega \in \mathbb{R} \Leftrightarrow \; a(t) = 0 \text{ a.e. } t \in \mathbb{R}$$

(ii) If $a \in L^1_{\text{loc}}(\mathbb{R})$, and $\hat{a}(z_0)$ is defined for some $z_0 \in \mathbb{C}$, then $\hat{a}(z)$ is defined on the vertical line $\Re z = \Re z_0$.
(iii) If $a \in L^1_{\text{loc}}(\mathbb{R})$ and $b \in L^1_{\text{loc}}(\mathbb{R})$, then $\widehat{(a \star b)}(z) = \hat{a}(z)\hat{b}(z)$ for all those $z \in \mathbb{C}$ for which both $\hat{a}(z)$ and $\hat{b}(z)$ are defined.

Theorem 3.

(i) If $a \in L^1(\mathbb{R}_+)$, then

- $\hat{a}(z)$ is defined and continuous on the closed half plane $\Re z \geq 0$, it is analytic in $\Re z > 0$,
- $\hat{a}(z) \to 0$ as $|z| \to \infty$ in the half plane $\Re z \geq 0$.
- Moreover $\hat{a}(z) = 0$ for all z in $\Re z \geq 0$ iff $a(t) = 0$ a.e. $t \in \mathbb{R}_+$.

(ii) If $a \in L^1_{\text{loc}}(\mathbb{R}_+)$, and $\hat{a}(z_0)$ is defined for some $z_0 \in \mathbb{C}$ then $\hat{a}(z)$ is defined in the closed half plane $\Re z \geq \Re z_0$.

(iii) If $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $b \in L^1_{\text{loc}}(\mathbb{R}_+)$, then $\widehat{(a \star b)}(z) = \hat{a}(z)\hat{b}(z)$ for all those $z \in \mathbb{C}$ for which both $\hat{a}(z)$ and $\hat{b}(z)$ are defined.

Proof. To prove the analyticity it suffices to observe that one can differentiateunder the integral sign [28]. We concentrate on the decrease at infinity. If $z = \sigma + i\omega$, then

$$|\hat{a}(z)| \leq \int_{\mathbb{R}_+} \exp(-\sigma t)\, |a(t)|\, dt$$

and, therefore, by Lebesgue's dominated convergence theorem, $\hat{a}(\sigma + i\omega) \to 0$, as $\sigma \to \infty$ uniformly in ω. It follows from Riemann-Lebesgue's Lemma that for each σ the function $\omega \to \hat{a}(\sigma + i\omega) \in BC_0(\mathbb{R})$. The map $\sigma \to \exp(-\sigma t)a(t)$ is continuous from $\mathbb{R}_+$ into $L^1(\mathbb{R}_+)$, and this implies that the map $\sigma \to \hat{a}(\sigma + i\omega)$ is continuous from $\mathbb{R}_+$ into $BC_0(\mathbb{R})$. This implies that $\hat{a}(\sigma + i\omega) \to 0$ as $\omega \to \infty$ uniformly in $\sigma \in [0, \sigma_0]$ for all $\sigma_0 > 0$, which ends the proof. ∎

2.1.2 Existence of a resolvent

Theorem 4. Let $k \in L^1_{\text{loc}}(\mathbb{R}_+)$, then there exists a solution $r \in L^1_{\text{loc}}(\mathbb{R}_+)$ of the *resolvent equation* :

$$r - k \star r = k \tag{9}$$

This solution is unique and depends continuously on k in the topology of $L^1_{\text{loc}}(\mathbb{R}_+)$.

Proof. First let us show that a solution r of (9) on an interval $[0, T]$ has to be unique. Suppose that q satisfies $q - k \star q = k$ and r satisfies $r - r \star k = k$. Then

$$\begin{aligned} r = k + r \star k &= k + r \star (q - k \star q) = k + r \star q - r \star k \star q \\ &= k + (r - r \star k) \star q = k + k \star q = k + q - k = q \end{aligned}$$

thus the solution is unique. Next, let us observe that it suffices to prove that for each $T \in (0, \infty)$, there is a function $r_T \in L^1(0, T)$ s.t.

$$r_T - k \star r_T = r_T - r_T \star k = k$$

on $[0, T]$. If this is true, then by uniqueness, for each $j \in \mathbb{N}$, the restriction r_{j+1} to $[0, j)$ must be equal to r_j. We get a unique solution in $L^1_{\text{loc}}(\mathbb{R}_+)$ by defining $r(t) = r_{j+1}(t)$ for

$t \in [j, j+1)$. In the special case where $\int_0^T k(a)da < 1$ one can use the iteration method. Indeed we set

$$r_m := \sum_{j=1}^{m} k^{\star j}$$

where $k^{\star j}$ is the $(j-1)$-fold convolution of k by itself :

$$k^{\star 2} = \int_0^t k(a)k(t-a)da.$$

A standard result of convolution provides that

$$\left\| k^{\star j} \right\|_{L^1(0,T)} \leq \|k\|_{L^1(0,T)}^j$$

This means that $\{r_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^1(0,T)$. Therefore there exists a function $r \in L^1(0,T)$ to which r_m tends in $L^1(0,T)$ as $m \to \infty$. Each r_m satisfies :

$$r_m - k \star r_{m-1} = r_m - r_{m-1} \star k = k$$

Again as

$$\|k \star (r_m - r)\|_{L^1(0,T)} \leq \|k\|_{L^1(0,T)} \|r_m - r\|_{L^1(0,T)}$$

one has also that $k \star r_m \to k \star r$ and hence r must satisfy (9).

To finish the existence proof, it suffices to show that we may always, without loss of generality, take $\int_0^T |k(t)| \, dt < 1$. By Lebesgue's dominated convergence theorem, the function $\exp(-\sigma t)k(t)$ tends to 0 in $L^1(0,T)$ as σ tends to ∞. So, by choosing σ large enough, we may assume that $a := \exp(-\sigma t)k(t)$ satisfies $\|a\|_{L^1(0,T)} < 1$. There exists $q \in L^1(0,T)$ solving

$$q - a \star q = q - q \star a = a$$

define $r := \exp(\sigma t)q(t)$, then r solves (9). Thus we have found a solution $r \in L^1(0,T)$ of the original equation. The continuous dependence on k is left as an exercise. ■

Exercise 1. Prove the continuous dependence on k

Proof. We set $\underline{k} = k_1 - k_2$ and $\underline{r} = r_1 - r_2$, then one has

$$\underline{r} - k_1 \star \underline{r} = \underline{k} + r_2 \star \underline{k}$$

and in the same way :

$$\underline{r} - k_2 \star \underline{r} = \underline{k} + r_1 \star \underline{k}$$

then taking the convolution wrt r_1 in the first equation above and r_2 in the second one obtains :

$$r_1 \star \underline{r} - r_1 \star k_1 \star \underline{r} = \underline{k} \star (k_1 \star r_2 + r_1)$$

and similarly

$$r_2 \star \underline{r} - r_2 \star k_2 \star \underline{r} = \underline{k} \star (k_2 \star r_1 + r_2)$$

then using (9) one has for instance that :

$$k_1 \star \underline{r} = \underline{k} \star (k_1 \star r_2 + r_1)$$

which finally gives that :

$$\underline{r} = k_1 \star \underline{r} + \underline{k} + r_2 \star \underline{k} = \underline{k} \star (k_1 \star r_2 + r_1) + \underline{k} + r_2 \star \underline{k}$$

then this latter expression provides the bound

$$\|\underline{r}\|_{L^1(0,T)} \leq C(k_1, r_1, r_2) \|\underline{k}\|_{L^1(0,T)}$$

one concludes. ■

Exercise 2. If the function $\exp(-\sigma t)k(t)$ belongs to $L^1(\mathbb{R}_+)$ for some $\sigma \in \mathbb{R}$, then there is a constant c such that the function $\exp(-ct)r(t)$ belongs to $L^1(\mathbb{R}_+)$, where r is the resolvent of k.

Proof. If the function $\tilde{k} := \exp(-\sigma t)k(t) \in L^1(\mathbb{R}_+)$ then there exists $\delta \in \mathbb{R}$ large enough s.t.

$$\left\| \exp(-\delta \cdot) \tilde{k}(\cdot) \right\|_{L^1(\mathbb{R}_+)} < 1$$

and thus as in the proof of Theorem 4, we set $a := \exp(-\delta t)\tilde{k}(t) = \exp(-(\delta + \sigma)t)k(t)$. There exists $q \in L^1(\mathbb{R}_+)$ s.t.

$$q - a \star q = a$$

and setting $r(t) := \exp((\sigma + \delta)t)q(t)$ is solves (9). Moreover as there exists C s.t.

$$\|q\|_{L^1(\mathbb{R}_+)} \leq C$$

this implies that

$$\|\exp(-(\sigma + \delta) \cdot) r(\cdot)\|_{L^1(\mathbb{R}_+)} < C$$

which ends the proof. ■

Exercise 3. Show that if $k(s) \geq 0$ for almost every $s \in (0, T)$, then so is r solving (9).

Proof. Let's assume first that

$$\int_0^T k(s) ds < 1.$$

Then

$$r = \sum_{j \geq 1} k^{\star j}$$

and thus $r \geq 0$ as long as $k(s) \geq 0$. In the general case, chose σ small enough as in the proof of Theorem 4, and set $a_\sigma(t) := \exp(-\sigma t)k(t)$ so that

$$\int_0^T a_\sigma(s) ds < 1.$$

Thus there exists $q_\sigma \geq 0$ s.t.

$$q_\sigma - a_\sigma \star q_\sigma = a_\sigma$$

and define $r := \exp(\sigma t) q_\sigma(t)$ which solves (9) and it is positive. ■

Theorem 5. Let $k \in L^1_{\text{loc}}(\mathbb{R}_+)$. Then for every $f \in L^1_{\text{loc}}(\mathbb{R}_+)$, there exists a unique solution $x \in L^1_{\text{loc}}(\mathbb{R}_+)$ of (8). This solution is given by the variation of constant formula :

$$x(t) = f(t) + (r \star f)(t), \ t \in \mathbb{R}_+ \tag{10}$$

where r is the resolvent of k. If $f \in L^p_{\text{loc}}(\mathbb{R}_+)$ then $x \in L^p_{\text{loc}}(\mathbb{R}_+)$.

Proof. Let $f \in L^1_{\text{loc}}(\mathbb{R}_+)$ and define x as in (10). Then $x \in L^1_{\text{loc}}(\mathbb{R}_+)$, and

$$x - k \star x = x - k \star (f + r \star f) = x - (k + k \star r) \star f = x - r \ \star f = f$$

so that x solves (8).

Conversely, let x be an arbitrary solution of (8) in $L^1_{\text{loc}}(\mathbb{R}_+)$. Then as $f \in L^1_{\text{loc}}(\mathbb{R}_+)$ and

$$x = f + k \star x = f + (r - r \star k) \star x = f + r \star (x - k \star x) = f + r \star f$$

and so x is given by (10). ∎

Theorem 6. Let k be a positive Volterra kernel in $L^1_{\text{loc}}(0,T)$, assume that $x, f \in L^1_{\text{loc}}(0,T)$ and suppose that

$$x(t) \leq (k \star x)(t) + f(t), \quad \text{a.e. } t \in (0,T).$$

Then

$$x(t) \leq y(t), \quad \text{a.e. } t \in (0,T)$$

where y is the solution of the comparison equation $y(t) = (k \star y)(t) + f(t)$.

Proof. If $x(t) \leq (k \star x)(t) + f(t)$ then there exists $g \geq 0$ s.t.

$$x(t) = (k \star x)(t) + f(t) - g(t).$$

and using the resolvent equation this reads :

$$x = (f - g) + r \star (f - g)$$

then one computes :

$$k \star x = (k + k \star r) \star (f - g) = r \star (f - g) = r \star f - r \star g$$

and thus :

$$x(t) \leq (k \star x)(t) + f(t) = (r \star f)(t) - (r \star g)(t) + f(t) \leq (r \star f)(t) + f(t) = y(t)$$

where we used that $r \geq 0$ and $g \geq 0$ imply that $r \star g \geq 0$, which ends the proof. ∎

This result can be generalized to non-convolution type kernels. The reader may consult [8, Propositions 8.1 and Lemma 8.2]. Now we solve (8) :

Remark 1. To get an explicit solution of the resolvent equation, one can use Laplace transforms. Suppose that for some sufficiently large σ, the function $\exp(-\sigma t)k$ belongs to $L^1(\mathbb{R}_+)$. Then $\hat{k}(z)$ exists for $\Re z > \sigma$, and, by Exercise 2, $\hat{r}(z)$ exists in some other half plane $\Re z > c$. Using the previous Theorem 3 (9) may be transformed and becomes :

$$\hat{r}(z) = \frac{\hat{k}(z)}{1 - \hat{k}(z)}, \quad \Re z \geq \max(\sigma, c)$$

An inversion of the Laplace transform (which occasionally can be done in a closed form) gives r.

Suppose that we know something about the asymptotic behaviour of the forcing function f in (8). For example, $f \in L^1(\mathbb{R}_+)$. What conditions do we need on the kernel k in order to conclude that the solution x of (8) has the same asymptotic behaviour as f?

Exercise 4. Prove that if $f \in L^1(\mathbb{R}_+)$ or $f \in L^\infty(\mathbb{R}_+)$ then $x \in L^1(\mathbb{R}_+)$ is equivalent to $r \in L^1$.

Now one can ask the question : when is it true that $r \in L^1(\mathbb{R}_+)$? As we shall see below, if $k \in L^1(\mathbb{R}_+)$, then one can give a complete answer to this question in terms of a Laplace transform condition.

Theorem 7. [Half Line Paley-Wiener] Let $k \in L^1(\mathbb{R}_+)$, then the resolvent r of k satisfies :

$$r \in L^1(\mathbb{R}_+)$$

if and only if

$$\hat{k}(z) \neq 1, \quad \forall z \in \mathbb{C}\ ;\ \Re z \geq 0.$$

Proof. The Laplace condition is clearly necessary. Indeed, if $r \in L^1(\mathbb{R}_+)$ and $r - k \star r = k$, then we are allowed to transform (9) to get :

$$\hat{r} - \hat{k}\hat{r} = \hat{k}, \quad \forall z \in \mathbb{C}\ ;\ \Re z \geq 0.$$

or equivalently

$$(1 - \hat{k}(z))(1 + \hat{r}(z)) = 1, \quad \Re z \geq 0$$

which clearly implies that $\hat{k}(z) \neq 1$, $\Re z \geq 0$ because $\hat{r}$ is bounded on $\Re z \geq 0$ since $r \in L^1(\mathbb{R}_+)$. Indeed, as $k \in L^1(\mathbb{R}_+)$ $\hat{k}$ is continuous on $\Re z \geq 0$. Assume there exists $z_0 \in \{\Re z \geq 0\}$ s.t. $\hat{k}(z_0) = 1$ then

$$\left|(1 - \hat{k}(z_0))(1 + \hat{r}(z_0))\right| \leq \left(1 + \|r\|_{L^1(\mathbb{R}_+)}\right) \left|1 - \hat{k}(z_0)\right| = 0$$

leading to a contradiction. The rest of the proof is admitted since beyond of these lectures' scope . The interested reader can refer to the proofs of [8, Theorems 4.1 and 4.3]. ∎

2.1.3 Application to the case when total mass is less than 1

This gives :

Lemma 1. Assume that $\int_{\mathbb{R}_+} k(a)da < 1$, $\hat{k}(z)$ is well defined on $\Re z \geq 0$ and

$$\forall z \; ; \; \Re z \geq 0, \quad k(z) \neq 1$$

Proof. Indeed, by Theorem 3, as $k \in L^1(\mathbb{R}_+)$, $\hat{k}(z)$ is defined and continuous on the closed half-plane $\Re z \geq 0$, moreover

$$\left|\hat{k}(z)\right| = \left|\int_{\mathbb{R}_+} k(a) \exp(-za) da\right| \leq \|k\|_{L^1(\mathbb{R}_+)} < 1$$

and the claim follows. ∎

Theorem 8. If $\|k\|_{L^1(\mathbb{R}_+)} < 1$, one has in a straightforward manner that : there exist $r \in L^1(\mathbb{R}_+)$ solving (9) s.t.

$$\|r\|_{L^1(\mathbb{R}_+)} \leq \frac{1}{1 - \|k\|_{L^1(\mathbb{R}_+)}}.$$

Moreover these estimates show also that if $f \in L^1(\mathbb{R}_+)$ then so is x solving (8). Same results hold for any $L^p(\mathbb{R}_+)$ norms for $p \in (1, \infty]$.

Corollary 1. Assume that $f \in L^1(\mathbb{R}_+)$, then

$$\|x\|_{L^1(\mathbb{R}_+)} \leq \|f\|_{L^1(\mathbb{R}_+)}(1 + \|r\|_{L^1(\mathbb{R}_+)})$$

Obviously this implies that

$$\int_t^{t+h} \left|x(\tilde{t})\right| d\tilde{t} \to 0$$

when t grows large.

2.2 The resolvent for the renewal equation

If we turn back to (3), for $\varepsilon = 1$ it can be rewritten as :

$$\mu_0 x(t) - \int_0^t x(t-a)\varrho(a)da = f(t) + \int_t^\infty x_p(t-a)\varrho(a)da$$

which, dividing by μ_0 and redefining f accordingly, transforms into :

$$x(t) - \int_0^t x(t-a)k(a)da = x - k \star x = f(t), \quad k(a) := \frac{\varrho(a)}{\mu_0}$$

the latter function having total mass 1. By *renewal equation* we intend precisely that the total mass of k is one, *i.e.*

$$\int_{\mathbb{R}_+} k(a)da = 1.$$

In a first step we require a quite strong decrease of k at infinity, latter we refine our analysis in order to relax as much as possible this constraint.

2.2.1 The kernel belongs to some exponentially weighted Lebesgue's space

Definition 2. A positive function w on $\mathbb{R}$ is called sub-multiplicative, if $w(0) = 1$ and

$$w(s+t) \leq w(s)w(t)$$

for all s, t in $\mathbb{R}$.
We define the weighted L^p-spaces $L^p(0,T)$ for $p \in [1, \infty]$ to be the set of functions f on $(0,T)$ that satisfy $wf \in L^p(0,T)$, with norm $\|f\|_{L^p(0,T;w)} = \|wf\|_{L^p(0,T)}$.

Examples of sub-multiplicative weights are

$$\begin{cases} w_1(t) := (1+|t|)^\delta, & t \in \mathbb{R},\ \delta \geq 0 \\ w_2(t) := (1+\log(1+|t|))^\gamma w_1(t), & t \in \mathbb{R},\ \gamma \geq 0 \\ w_3(t) := \exp(|t|^\alpha) w_2(t), & t \in \mathbb{R},\ 0 \leq \alpha < 1 \\ w_4(t) := \exp(\beta t) w_3(t), & t \in \mathbb{R},\ \beta \in \mathbb{R} \end{cases}$$

Proposition 1. Let w be a locally bounded sub-multiplicative, then

$$\alpha_w := -\inf_{t \to -\infty} \frac{\ln(w(t))}{t}, \quad \omega_w := -\inf_{t \to \infty} \frac{\ln(w(t))}{t}$$

exists and one has :

$$w(t) \geq \max\left(\exp(-\alpha_w t), \exp(-\omega_w t)\right), \quad \forall t \in \mathbb{R}$$

We recall here the Paley-Wiener Lemma :

Lemma 2. Let $a \in L^1(\mathbb{R}; \mathbb{C})$, let ϕ be a function that is bounded and continuous in the closed half plane $\Re z \geq 0$ and analytic in the open half plane $\Re z > 0$, and suppose that $\hat{a}(z) = \varphi(z)$ for all $\Re z = 0$. Then $a(t) = 0$ for almost all $t < 0$, and $\hat{a}(z) = \phi(z)$ for all $z \in \mathbb{C}$ with $\Re z > 0$.

Proof. We set

$$a_1(t) := \begin{cases} a(t), & t < 0, \\ 0, & t \geq 0 \end{cases}, \quad a_2(t) := \begin{cases} 0, & t < 0, \\ a(t), & t \geq 0 \end{cases}$$

By hypothesis, one has

$$\hat{a}(z) = \hat{a}_1(z) + \hat{a}_2(z) = \varphi(z), \quad \Re z = 0.$$

$\varphi(z)$ and $\hat{a}_1(z)$ are continuous on $\Re z \geq 0$ and analytic in $\Re z > 0$. Similarly as $\hat{a}_1(-z)$ is the ordinary LT of $a(-t)$ we see that $\hat{a}_1$ is continuous on $\Re z \leq 0$ and analytic in $\Re z >< 0$. It follows that the function

$$G(z) := \begin{cases} \hat{a}_1(z), & \Re z \leq 0 \\ \varphi(z) - \hat{a}_2(z), & \Re z > 0 \end{cases}$$

is entire [28, Theorem 16.8, p 323]. As φ and $\hat{a}_2$ are bounded on $\Re z \geq 0$, and similarly because $a_1 \in L^1(\mathbb{R}_-)$ $\hat{a}_1$ is bounded on $\Re z \leq 0$, then G is bounded. By Liouville's Theorem, G is a constant. Since $G(i\omega) \to 0$ when $|\omega| \to \infty$ G is identically zero. Then $\hat{a}_1 = 0$ and by uniqueness of the LT, $a_1(t) = 0$, for a.e. $t < 0$ thus $a(t) = 0$ a.e. $t > 0$. ■

Assume that φ is a sub-multiplicative weight, we define $\mathcal{M}(\mathbb{R}_+; \varphi; \mathbb{C})$ to be the set of measures μ s.t. $\varphi\mu(dt) \in \mathcal{M}(\mathbb{R}_+; \mathbb{C})$, where $\mathcal{M}(\mathbb{R}; \mathbb{C})$ is the set of finite complete Borel Measures [28]. Here we state [8, Theorem 7.2.4].

Theorem 9. Let φ be a sub-multiplicative weight function on $\mathbb{R}$ for which $\alpha_\varphi = \omega_\varphi$, let $k \in \mathcal{M}(\mathbb{R}_+; \varphi; \mathbb{C})$ have no singular part and suppose that

$$\inf_{\Re z = \omega_\varphi} \left| 1 - \hat{k} \right| > 0,$$

and

$$\liminf_{\substack{|z| \to \infty \\ \Re z \geq \omega_\varphi}} \left| 1 - \hat{k}(z) \right| > 0$$

Then the resolvent r of k is of the form :

$$k(dt) = \sum_{\ell=1}^{m} \sum_{j=0}^{p_\ell - 1} \Omega_{\ell,j} t^j \exp(z_\ell t) dt + \xi(dt)$$

where each exponent z_ℓ is complex with $\Re z_\ell > \omega_\varphi$, each $\Omega_{\ell,j}$ is a complex coefficient and $\xi \in \mathcal{M}(\mathbb{R}_+; \varphi; \mathbb{C})$.
More specially, the exponents z_ℓ are the zeros of the characterstic equation $\hat{k}(z) = 1$ in $\Re z > \omega_\varphi$ and the coefficients $\Omega_{\ell,j}$ are determined by the principal part of $\hat{k}(1 - \hat{k})^{-1}$ at z_ℓ in the sense that, in some neighborhood of z_ℓ,

$$\frac{\hat{k}(z)}{1 - \hat{k}(z)} = \sum_{\ell=1}^{m} \sum_{j=0}^{p_\ell - 1} \frac{j! \Omega_{\ell,j}}{(z - z_\ell)^{j+1}} + \xi_\ell(z),$$

where the remainder ξ_ℓ is analytic at z_ℓ.

Theorem 10. We assume here that k is s.t.

- $k(a) \geq 0$,
- $\int_{\mathbb{R}_+} k(a)da = 1$,
- there exists $\sigma > 0$ s.t.

$$k(\cdot)\exp(\sigma\cdot) \in L^1(\mathbb{R}_+)$$

then there exists $h \in L^1(\mathbb{R}_+, \exp(\sigma\cdot))$ s.t. the resolvent r reads

$$r(a) = \frac{1}{\int_{\mathbb{R}_+} \tilde{a}k(\tilde{a})d\tilde{a}} + h(a), \quad \text{a.e. } a \in \mathbb{R}_+$$

Proof. If k is positive and $\int k(a)da = 1$ then its Laplace transform exists and is analytic in $\Re z > 0$. Moreover, $k(\cdot)\exp(\sigma\cdot) \in L^1(\mathbb{R}_+)$ implies that

$$\left|\hat{k}(-\sigma)\right| = \int_{\mathbb{R}_+} k(a)\exp(\sigma a)da < \infty$$

which by Theorem 3 implies that $\hat{k}$ is defined for $\Omega_\sigma := \{\Re z \geq -\sigma\}$. Moreover $\hat{k}$ is analytic in $\Re z > \sigma$. Indeed, one has that

a) a.e. $t \in \mathbb{R}_+$, the map $z \in \Omega_\sigma \to \exp(-zt)k(t)$ is differentiable on Ω_σ and the derivative is $-t\exp(-zt)k(t)$;
b) for every $z \in \Omega_\sigma$, the map $t \in \mathbb{R}_+ \to \exp(-zt)k(t)$ is integrable on $\mathbb{R}_+$ (this is the consequence of the previous result).
c) one has that for all $\sigma' < \sigma$ and all $z \in \Omega_{\sigma'}$ there exists $t_{\sigma'}$ s.t. ,

$$|t\exp(-zt)|\, k(t) \leq \max(\exp(\sigma t), t_{\sigma'}\exp(\sigma' t_{\sigma'}))k(t) \in L^1(\mathbb{R}_+)$$

Indeed,

$$t\exp(-zt) < t\exp(\sigma' t)$$

and then there exists $t_{\sigma'}$ s.t.

$$\forall t > t_{\sigma'}, \quad t\exp(\sigma' t) \leq \exp(\sigma t)$$

so that

$$t\exp(-zt) \leq t\exp(\sigma' t) \leq \begin{cases} \exp(\sigma t) & \text{if } t > t_{\sigma'} \\ t'_\sigma \exp(t_{\sigma'}) & \text{otherwise.} \end{cases}$$

using the Theorem of derivation of an integral wrt a paramter, one has that

$$\hat{k}'(z) = -\int_{\mathbb{R}_+} \exp(-zt)tk(t)dt, \quad \forall \Re z > \sigma.$$

When z is real and $z \in (-\sigma, \infty)$, $\hat{k}$ is a positive strictly decreasing function :

$$\hat{k}'(z) = -\int_{\mathbb{R}_+} \exp(-zt)tk(t)dt$$

Indeed, the derivative is defined and strictly negative since there exists a set $\omega \subset (0, \infty)$ of positive measure for which $k(a) > 0$ and thus

$$\hat{k}'(z) < 0, \quad \forall z \in (-\sigma, \infty).$$

Since $k \in L^1(\mathbb{R}_+)$ it is clear that $\lim_{z\to\infty} \hat{k}(z) = 0$, and since $\hat{k}(0) = 1$ and $k'(0) < 0$, this shows that there exists a unique real and simple root of the characteristic equation $\hat{k} = 1$.

If z_1 is an eigenvalue, then so is $\overline{z_1}$, its conjugate. This leads to write that :

$$\hat{k}(z_1) + \hat{k}(\overline{z_1}) = 2$$

which translates into :

$$\int_{\mathbb{R}_+} (\exp(-z_1 a) + \exp(-\overline{z_1} a))\, k(a) da = 2.$$

This can be rewritten as :

$$\int_{\mathbb{R}_+} \exp(-\Re z_1 a) \cos(\Im z_1 a) k(a) da = 1$$

which can be estimated as

$$\hat{k}(\Re z_1) \geq \int_{\mathbb{R}_+} \exp(-\Re z_1 a) \cos(\Im z_1 a) k(a) da = 1 = \hat{k}(0).$$

Because $\hat{k}$ is a decreasing function on $[-\sigma, \infty)$, this means that $\Re z_1 \leq 0$. Suppose that $\Re z_1 = 0$ and $\Im z_1 \neq 0$, then one arrives at the equation :

$$\int_{\mathbb{R}_+} (1 - \cos(\Im z_1 a)) k(a) da = 0$$

and since the integrand is non-negative, this implies that for almost every $a \in \operatorname{supp} k$ one has :

$$(1 - \cos(\Im z_1 a)) k(a) = 0,$$

since there is at least a non zero measure set K on which $k(a) > 0$ this leads to $(1 - \cos(\Im z_1 a)) = 0$ for a.e. $a \in K$ thus $\Im z_1 = 0$ which is a contradiction. This shows that necessarily the strict inequality holds.

Since $\hat{k}$ is analytic in $\Re z > -\sigma$, the zeros are isolated, there is no accumulation point in $\Re z > -\sigma$. Thus there exists $0 < \delta < \sigma$ s.t. in the strip $\Re z \in (-\delta, \infty)$, $z = 0$ is the only root of the characteristic equation $\hat{k}(z) = 1$.

We are in the position to apply Theorem 9, which states that :

$$r(dt) = \operatorname{Rez}\left(\frac{\hat{k}(z)}{1 - \hat{k}(z)}, 0\right) dt + \xi(dt)$$

where $\xi \in \mathcal{M}(\mathbb{R}_+; \exp(\delta t); \mathbb{C})$. Then a simple computation shows that :

$$\operatorname{Rez}\left(\frac{\hat{k}(z)}{1 - \hat{k}(z)}, 0\right) = \lim_{z\to 0} \hat{k}(z) \frac{z}{1 - \hat{k}(z)} = \lim_{z\to 0} \hat{k}(z) \lim_{z\to 0} \frac{1}{\frac{1-\hat{k}(z)}{z}} = \frac{1}{-\hat{k}'(0)} = \frac{1}{\int_{\mathbb{R}_+} a k(a) da}.$$

As $k \in L^1(\mathbb{R}_+)$, the resolvent is in $L^1_{\text{loc}}(\mathbb{R}_+)$. The constant $1/\int_{\mathbb{R}_+} a k(a) da$ is also in L^1_{loc} which implies that ξ is also in L^1_{loc}.

If a measure $\xi \in \mathcal{M}(\mathbb{R}_+; \mathbb{R})$ is finite and is also in $L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R})$ then there exists $h \in L^1(\mathbb{R}_+)$ s.t. $\xi(dt) = h dt$. It is a consequence of the Lebesgue-Radon-Nikodym decomposition theorem [28, Theorem 6.10, p.121] and the fact that the space of $L^1_{\text{loc}}(\mathbb{R}_+)$ functions coincides with absolutely continuous measures wrt Lebesgue's measure [29, p. 18]. Indeed, Lebesgue's measure λ is σ-additive and positive, ξ is a finite measure s.t $\xi \ll \lambda$ (absolutelly continuous wrt Lebesgue's measure). Since it is also finite, there exists a unique $h \in L^1(\mathbb{R}_+)$ s.t.

$$\xi(E) = \int_E h dt, \quad \forall E \in \mathfrak{B}.$$

where $\mathfrak{B}$ denote the Borelian σ-algebra defined on $\mathbb{R}_+$. So ξ can be identified with h. ■

2.2.2 The total mass is one and no exponential weight is at hand

In this section we extend the previous result in a weaker case for which only integrability and the boundedness of the first moment are required.

Definition 3. A function ϕ is said to belong locally to $\hat{L}^1(w)$

- at a point $z_0 \in \mathbb{C}$, with $\alpha_w \leq \Re z_0 \leq \omega_w$, if there exists functions $\mu_1, \cdots, \mu_k$ in $L^1(\mathbb{R}_+; w)$ and a function $\psi(z, \xi_1, \ldots, \xi_k)$, analytic at $(z_0, \hat{\mu}_1(z_0), \ldots, \hat{\mu}_k(z_0))$, s.t. for some ε,
$$\phi(z) = \psi(z, \hat{\mu}_1(z), \ldots, \hat{\mu}_k(z))),$$
when $|z - z_0| < \varepsilon$ and $\alpha_w \leq \Re z \leq \omega_w$.
- at infinity if there exists functions $\mu_1, \cdots, \mu_k$ in $L^1(\mathbb{R}_+; w)$ and a function $\psi(z, \xi_1, \ldots, \xi_k)$ analytic in $(0, 0, \ldots, 0)$ s.t. for some $\varepsilon > 0$,
$$\phi(z) = \psi(1/z, \hat{\mu}_1(z), \ldots, \hat{\mu}_k(z))),$$
when $|z| > \varepsilon$ and $\alpha_w \leq \Re z \leq \omega_w$.

If ϕ belongs locally to $\hat{L}^1(w)$ at every point z_0 with $\alpha_w \leq \Re z \leq \omega_w$ and at infinity, then we say that ϕ belongs locally to $\hat{L}^1(w)$.

The following theorem motivates the use of the words *belongs locally* $L^1(\varphi)$ in Definition 3. For the proof, see [6, Therorem 1, p. 82] and [10, Prop. 2.3, p.755.].

Theorem 11. If w is a locally bounded sub-multiplicative weight on $\mathbb{R}$, and let $\phi \in \hat{L}^1(w)$ and satisfy $\phi(\infty) = 0$. Then there exists a unique function $a \in L^1(\mathbb{R}_+; w)$ s.t. $\phi(z) = \hat{a}(z)$ for $\alpha_w \leq \Re z \leq \omega_w$.

Theorem 12. Here consider problem (8) with the specific condition that $k \geq 0$ and $\|k\|_{L^1(\mathbb{R}_+)} = 1$. We assume moreover that the first moment of k is bounded *i.e.* :

$$\int_{\mathbb{R}_+} ak(a)da < +\infty$$

there exists a function $\nu \in L^1(\mathbb{R}_+)$, s.t.

$$r(t) = \nu(t) + \int_0^t \nu(s)ds$$

where r is the positive resolvent associated to k and one has :

$$\int_{\mathbb{R}_+} \nu(s)ds = \frac{1}{\int_{\mathbb{R}_+} k(a)ada}.$$

Moreover, if $\int_{\mathbb{R}_+} a^2 k(a)da < \infty$, then there exists $\gamma := \nu - \int_t^\infty \nu(a)da$ and $\gamma \in L^1(\mathbb{R}_+)$ s.t.

$$r(t) = \gamma(t) + \frac{1}{\int_{\mathbb{R}_+} k(a)ada} \tag{11}$$

Proof. By construction we know that the resolvent $r \in L^1_{\text{loc}}(\mathbb{R}_+)$ is positive since

$$r := \sum_{j=1}^{\infty} k^{\star j}.$$

We set $b(t) := \int_t^\infty k(a)da$, since this is an integrable function its Laplace transform is analytic in $\Re z > 0$, continuous in $\Re z \geq 0$ and reads :

$$\hat{b}(z) = \frac{1 - \hat{k}(z)}{z}, \quad \Re z > 0.$$

Moreover a simple computation shows that

$$\hat{b}(0) = \int_{\mathbb{R}_+} b(t)dt = \frac{\mu_1}{\mu_0}.$$

We set

$$\phi(z) := \frac{z}{(1+z)} \frac{\hat{k}(z)}{(1-\hat{k}(z))} = \frac{z}{(1-\hat{k}(z))} \frac{\hat{k}(z)}{(1+z)} = \frac{1}{\hat{b}(z)} \frac{\hat{k}(z)}{(1+z)}.$$

If we denote $\psi(z,\xi) := \frac{z}{1+z}\frac{\xi}{1-\xi}$, it is thus analytic in $(\mathbb{C} \setminus \{-1\}) \times (\mathbb{C} \setminus \{1\})$. Simple computations and previous results show that,

$$\forall z_0 \in \mathbb{C} \setminus \{0\}, \text{ s.t. } \Re z_0 \geq 0, \quad \hat{k}(z_0) \neq 1,$$

(it is sufficient to distinguish between the case $\Re z > 0$ and $z \in i\mathbb{R}^*$). Thus for all $z_0 \in \Re z_0 \geq 0$ and $z_0 \neq 0$,

$$(z_0, \hat{k}(z_0)) \in \{\Re z \geq 0, \ z \neq 0\} \times (\mathbb{C} \setminus \{1\}) \subset (\mathbb{C} \setminus \{-1\}) \times (\mathbb{C} \setminus \{1\})$$

So that $\psi(z,\xi)$ is analytic at $(z_0, \hat{k}(z_0))$. Moreover one has that

$$\phi(z) = \psi(z, \hat{k}(z)), \quad \forall z \in \Re z \geq 0.$$

This shows that ϕ is locally in $\tilde{L}^1(1)$ at points $z_0 \neq 0$ in $\Re z \geq 0$.

We focus now at $z_0 = 0$. We set

$$\tilde{\psi}(z, \xi_1, \xi_2) := \frac{1}{\xi_1} \frac{\xi_2}{(1+z)}.$$

Obviously one has :

$$\phi(z) = \tilde{\psi}(z, \hat{b}(z), \hat{k}(z)), \quad \forall z \in \Re z \geq 0.$$

Then one notices that $\tilde{\psi}(z, \xi_1, \xi_2)$ is analytic in the neighborhood of

$$(0, \hat{b}(0), \hat{k}(0)) = \left(0, \int_{\mathbb{R}_+} k(a) a da, 1\right).$$

Thus ϕ is in $\tilde{L}^1(1)$ at the point $z_0 = 0$.

It remains to prove that actually ϕ is in $\tilde{L}^1(1)$ at infinity. To this aim, we set

$$\varphi(z, \xi) := \frac{1}{(1+z)} \frac{\xi}{1-\xi}$$

that should be analytic in $(0, 0)$. Indeed,

$$\varphi(w_1, w_2) = \left(\sum_{n=0}^{\infty} (-1)^n w_1^n\right) \left(w_2 \sum_{p=0}^{\infty} w_2^p\right) = \sum_{n,p=0}^{\infty} (-1)^n w_1^n w_2^{p+1}$$

which converges for $(w_1, w_2) \in D(0,1)^2$ ($D(0,1)$ is the open unit disc in $\mathbb{C}$). Moreover for all $M \geq 0$, for all $|z| > M$, one has

$$\phi(z) = \varphi(1/z, \hat{k}(z)), \quad \forall z \text{ s.t. } \Re z \geq 0.$$

The previous considerations show that ϕ defined above is in $\hat{L}^1(1)$, in the sense of Definition 3 and Theorem 11 states that there exists $\nu \in L^1(\mathbb{R})$ s.t.

$$\hat{\nu}(z) = \phi(z), \quad \Re z = 0.$$

By Lemma 2 [8, the Paley-Wiener Lemma 5.1 p.50], $\operatorname{supp} \nu \subset \mathbb{R}_+$ and $\hat{\nu}(z) = \phi(z)$ coincide also on $\Re z > 0$. Now by linearity of Laplace's transform one has trivially :

$$\widehat{\left(\nu + \int_0^t \nu(s) ds\right)} = \hat{\nu} + \widehat{\left(\int_0^t \nu(s) ds\right)}$$

and using [4, Theorem 8.1 p.36], $\nu \in L^1(\mathbb{R}_+)$ implies that

$$\widehat{\left(\int_0^t \nu(s) ds\right)}(z) = \frac{1}{z} \hat{\nu}(z), \quad \forall \Re z > 0.$$

By uniqueness of the Laplace transform [4, Lerch's Theorem 5.1, p.21] , if

$$\widehat{r}(z) = \left(1 + \frac{1}{z}\right) \hat{\nu}(z) = \widehat{\left(\nu + \int_0^t \nu(s) ds\right)}(z), \quad \forall \Re z > 0$$

one concludes that $r(a) = \nu(a) + \int_0^a \nu(s)ds$ almost everywhere in $\mathbb{R}_+$. Moreover, one writes :

$$\int_{\mathbb{R}_+} \nu(s)ds = \hat{\nu}(0) = \frac{1}{\hat{b}(0)} = \frac{1}{\int_{\mathbb{R}_+} sk(s)ds}.$$

Next, if the second order moment is bounded, one has that $b \in L^1(\mathbb{R}_+, (1+a))$ and thus also $\nu \in L^1(\mathbb{R}_+, (1+a))$ (thanks again to Theorem 11) which then ends the proof. ■

Corollary 2. If moreover $k \in L^2(\mathbb{R}_+)$, then $\gamma \in L^2(\mathbb{R}_+)$ in the previous claim.

Proof. Using the L^2-isometry of the Fourier transform, one has that if $\phi(i\cdot) \in L^2(\mathbb{R})$ then its inverse fransform is also in L^2. Assume first that $|\xi| > \varepsilon$,

$$|\phi(i\xi)| \leq C \left| \frac{\hat{k}(i\xi)}{1 - \hat{k}(i\xi)} \right|$$

as $k \in L^1(\mathbb{R}_+)$, its Fourier transform is in $C_0(\mathbb{R})$ and thus $\left|1 - \hat{k}(i\xi)\right| \to 1$ when $|\xi| \to \infty$, for all $\delta \in (0,1)$, there exists ξ_0 s.t. $\forall \xi$ s.t. $|\xi| > \xi_0$

$$\left|1 - \hat{k}\right| > 1 - \delta.$$

This provides that

$$\|\phi(i\cdot)\|_{L^2(\mathbb{R} \setminus B(0,\xi_0))} < \infty$$

On the other hand, $\phi(i\xi)$ is continuous on $B(0,\xi_0) \setminus \ \{0\}$ and reaches a finite value in zero $1/\hat{b}(0)$, it is then continuous on $B(0,\xi_0)$ and thus bounded. which shows that $\|\phi(i\cdot)\|_{L^2(\mathbb{R})} < \infty$. Thus its inverse Fourier transform is also an $L^2(\mathbb{R})$ function. One denotes $\upsilon \in L^2(\mathbb{R})$ s.t.

$$\upsilon := \mathcal{F}^{-1}(\phi)$$

Moreover one has that

$$\mathcal{F}(\upsilon)(\xi) = \phi(i\xi) = \hat{\nu}(i\xi) = \mathcal{F}(\nu)(\xi), \text{ a.e. } \xi \in \mathbb{R}$$

the latter equality holding since the Fourier transform of a $L^1(\mathbb{R}_+)$ function extended by zero on $\mathbb{R}_-$ coincides with the definition of the Laplace transform pointwisely. Then in the sense of tempered distributions $\mathcal{S}'(\mathbb{R})$ one has

$$\mathcal{F}(\upsilon - \nu) = 0,$$

which implies that $\upsilon - \nu$ in $\mathcal{S}'(\mathbb{R})$ as $\mathcal{F}$ is an isometry on $\mathcal{S}'(\mathbb{R})$. And thus the equality holds a.e. pointwisely. ■

One may ask what happens if for instance

$$\lim_{R \to \infty} \int_0^R ak(a)da = +\infty.$$

The question is of importance since it seems from above results that in this case the particle should *stop* instead of going further as time grows large.

2.3 Generalized entropy method

The transport form of the renewal equation is as follows :

$$\begin{cases} (\partial_t + \partial_a)n = 0 & (t,a) \in (0,\infty)^2, \\ n(0,t) = \int_{\mathbb{R}_+} k(a)n(a,t)da & t > 0,\ a = 0, \\ n(a,0) = n_0(a) & a > 0,\ t = 0, \end{cases} \tag{12}$$

where we assume that :

Assumptions 1. We assume that

i) $k(a) \geq 0$ a.e. $a \in \mathbb{R}_+$,

ii) $B \in L^1(\mathbb{R}_+)$

iii) either

$$\int_{\mathbb{R}_+} k(a)da > 1 \tag{13}$$

or

$$\int_{\mathbb{R}_+} k(a)da = 1 \text{ and } \int_{\mathbb{R}_+} k(a)ada < \infty \tag{14}$$

If x solves (3), with a null term, then it is solution of (12). Indeed,

$$n(a,t) := \begin{cases} x(t-a) & \text{if } t \geq a \\ x_p(t-a) & \text{otherwise} \end{cases}$$

solves the transport problem in $\mathbb{R}_+ \times (0,T)$. If $a = 0$ then $n(0,t) = x(t) = \int_{\mathbb{R}_+} k(a)n(a,t)da$, where $k = \varrho/\mu_0$ which provides the boundary condition. Chosing $t = 0$ in the definition above defines n_0 as $n_0(a) = x_p(-a)$ and one recovers the initial condition in (12). Conversely, if n solves (12), then setting $x(t) = n(0,t)$ solves (3), with $f = 0$. Indeed,

$$\begin{aligned} x(t) = n(0,t) = \int_{\mathbb{R}_+} n(a,t)k(a)da &= \int_0^t n(0,t-a)k(a)da + \int_t^\infty k(a)n_0(a-t)da \\ &= \int_0^t x(t-a)k(a)da + \int_t^\infty k(a)x_p(t-a)da \end{aligned}$$

2.3.1 The eigen-problem

First we consider the eigen elements of the equation. Namely we look for (λ, N) solving :

$$\begin{cases} (\lambda + \partial_a)N = 0, & a > 0, \\ N(0) = \int_{\mathbb{R}_+} k(\tilde{a})N(\tilde{a})d\tilde{a}, & a = 0, \end{cases} \tag{15}$$

If such a solution exists then λ should satisfy the characteristic equation

$$F(\lambda) := \int_{\mathbb{R}_+} k(a)\exp(-\lambda a)da = 1. \tag{16}$$

We recover here that $F(\lambda)$ is indeed $\hat{k}(\lambda)$.

For $\lambda \in \mathbb{R}_+$, $F(\lambda)$ is finite thanks to the definition of k. F is differentiable with respect to λ on $\Re\lambda > 0$ and the derivative reads :

$$F'(\lambda) = -\int_{\mathbb{R}_+} ak(a)\exp(-\lambda a)da < 0, \quad \Re\lambda > 0$$

showing that F is a monotone decreasing function on $\mathbb{R}_+^*$. If the first moment is bounded then the derivative can be extended by continuity to $\lambda = 0$ (hint: use Lebesgue's Theorem). As seen above it is bounded and strictly negative.

As k is integrable, thanks to Theorem 3, one has :

$$\lim_{\lambda \to +\infty} F(\lambda) = 0$$

In order to conclude two possibilities occur :

- if (13) holds, then

$$F(0) > 1, \qquad F'(\lambda) < 0, \quad \forall \lambda > 0, \qquad \lim_{\lambda \to +\infty} F(\lambda) = 0$$

- if (14) is true, then

$$F(0) = 1, \quad F'(\lambda) < 0, \quad \forall \lambda \geq 0, \qquad \lim_{\lambda \to +\infty} F(\lambda) = 0$$

these facts allow to state that, in any case, there exists a unique $\lambda_0 \in \mathbb{R}_+$ s.t. the characteristic equation holds true.

For practical reasons we set $N(0) = 1$. These arguments lead to

Proposition 2. Under Assumptions 2, there exists a unique real non-negative eigenvalue λ_0 of (16)

We make an important observation :

Proposition 3. If there exists another λ s.t. $\Re\lambda \geq 0$ and

$$F(\lambda) = 1$$

then necessarily one has :

$$\Re\lambda < \lambda_0$$

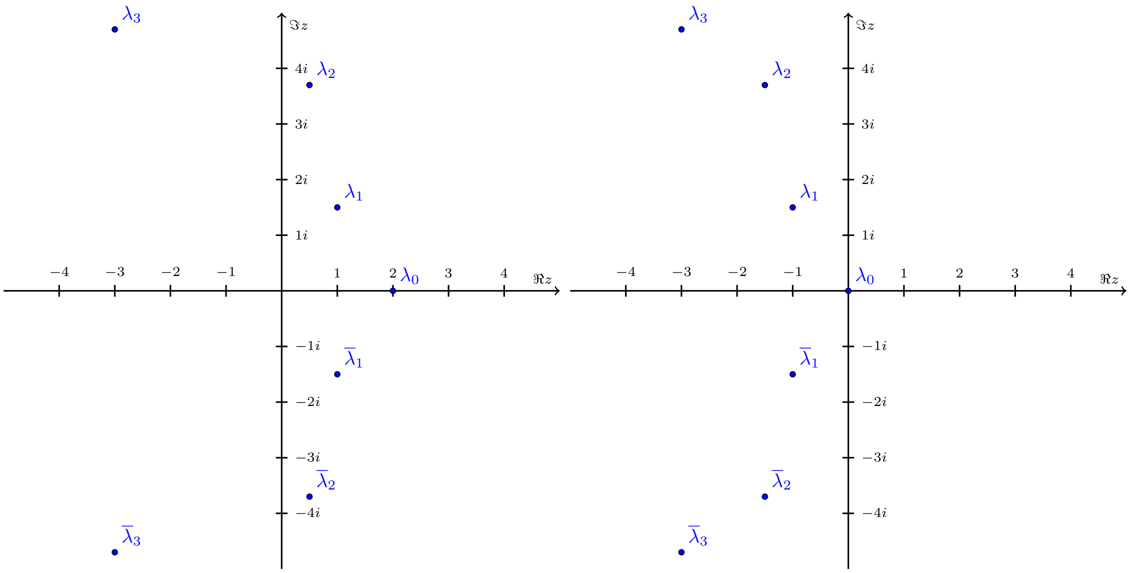
Moreover solutions of the characteristic equation are isolated in $\Re z > 0$ if $\int_{\mathbb{R}_+} k da > 1$, or in the case (14), there is only one eigenvalue in $\Re z \geq 0$ which is thus isolated.

Proof. The proof is left as an exercise following the same steps as in Theorem 10. ∎

The conclusions of the Proposition 3 are illustrated in Fig. 2.

Next, we look now for the dual problem associated with (15). It reads :

$$\begin{cases} (-\lambda + \partial_a)\psi(a) = -\psi(0)k(a) \\ \int_{\mathbb{R}_+} \psi(a)N(a)da = 1. \end{cases} \tag{17}$$



(a) $\int k da > 1$ (b) $\int k da = 1$

Figure 2: Pointwise spectrum in the complex plane

A simple computation shows that

$$
\begin{aligned}
\psi(a) =& \psi(0) \exp(\lambda_0 a) \left(1 - \int_0^a k(s) \exp(-\lambda_0 s) ds\right) \\
=& \psi(0) \exp(\lambda_0 a) \int_a^\infty k(s) \exp(-\lambda_0 s) ds
\end{aligned}
$$

where the latter equality comes from the characteristic equation. One should notice that ψ so defined is a non-negative function for all ages $a \in \mathbb{R}_+$. This is important in what follows.

Proposition 4. Under definition (17), one has the inclusion

$$\operatorname{ess\,supp} k \subset \operatorname{supp} \psi.$$

Proof. We recall that for the continuous function ψ, the support $\operatorname{supp} \psi$ is defined as the closure of the set where $\psi \neq 0$. Suppose that $a_0 \notin \operatorname{supp} \psi$ this implies that there exists η s.t. the open ball $B(a_0, \eta)$ s.t.

$$\psi(a) = 0, \quad \forall a \in B(a_0, \eta)$$

using the explicit definition of ψ this translates into :

$$0 = \psi(0) \exp(\lambda_0 a) \int_a^\infty k(s) \exp(-\lambda_0 s) ds, \quad \forall a \in B(a_0, \eta)$$

which implies because of the non-negativity of the integrand :

$$k(s) \exp(-\lambda_0 s) = 0, \qquad \text{a.e. } s \in (a_0 - \eta, \infty)$$

which in turn means that $k(s) = 0$ for a.e. $s \in (a_0 - \eta, \infty)$, *i.e.* $a_0 \notin \operatorname{ess\,supp} k$. ■

Remark 2. The converse is not true : take $k(a_0) = 0$ for some $a_0 > 0$ and assume that there exist $K \subset (a_0, \infty)$ s.t. $k(a) > 0$ for a.e. $a \in K$ then there exist $a_0 \in \operatorname{supp} \psi$ that does not belong to $\operatorname{supp} k$.

The normalization condition

$$\int_{\mathbb{R}_+} \psi(a) N(a) da = 1$$

then fixes $\psi(0)$ since :

$$\begin{aligned}\int_{\mathbb{R}_+} \psi(a) N(a) da &= \psi(0) \int_{\mathbb{R}_+} \int_a^\infty k(s) \exp(-\lambda_0 s) ds \\ &= \psi(0) \int_{\mathbb{R}_+} k(s) s \exp(-\lambda_0 s) ds = 1\end{aligned}$$

providing that

$$\psi(0) := \frac{1}{\int_{\mathbb{R}_+} k(s) s \exp(-\lambda_0 s) ds}$$

Remark 3. In the case of assumption (13), the integral

$$\int_{\mathbb{R}_+} k(s) s \exp(-\lambda_0 s) ds$$

makes sense since $\lambda_0 > 0$ and thus

$$\int_{\mathbb{R}_+} k(s) s \exp(-\lambda_0 s) ds \leq \frac{1}{\lambda_0 \exp(1)} \|B\|_{L^1(\mathbb{R}_+)}.$$

while in the case of assumption (14), one simply has : $\lambda_0 = 0$ and

$$\int_{\mathbb{R}_+} k(s) s \exp(-\lambda_0 s) ds = \int_{\mathbb{R}_+} k(a) a da < \infty.$$

by hypothesis.

This leads to

Lemma 3. Under Assumptions 1, there exists a unique tuple $(\lambda_0, N, \psi) \in \mathbb{R}_+ \times L^\infty(\mathbb{R}_+)^2$ solving the eigen-problem associated to (12).

2.3.2 Existence and uniqueness

Since there is a positive growing mode in time we rescale (12) so to deal with bounded quantities. Indeed we define

$$\tilde{n}(a,t) := n(a,t) \exp(-\lambda_0 a)$$

which satisfies :

$$\begin{cases} (\partial_t + \partial_a + \lambda_0)\tilde{n} = 0 & (t,a) \in (0,\infty)^2, \\ \tilde{n}(0,t) = \int_{\mathbb{R}_+} k(a) n(a,t) da & t > 0,\ a = 0, \\ \tilde{n}(a,0) = n_0(a) & a > 0,\ t = 0, \end{cases} \tag{18}$$

Definition 4. We define a *mild solution* u of the problem

$$\begin{cases} (\partial_t + \partial_a + \zeta(a,t))u = 0 & (a,t) \in (0,\infty) \times (0,T) \\ u(0,t) = u_b(t) & a = 0, t > 0 \\ u(a,0) = u_I(a) & a > 0, t = 0. \end{cases} \tag{19}$$

with data :

$$u_b \in L^\infty(0,T), \quad u_I \in L^\infty(\mathbb{R}_+), \quad \zeta \in L^\infty(\mathbb{R}_+ \times (0,T)), \quad \zeta \geq 0$$

the solution defined by the Duhamel's principle :

$$u(a,t) := \begin{cases} u_b(t-a) \exp\left(-\int_{-a}^{0} \zeta(a+\tau, t+\tau) d\tau\right) & \text{if } t \geq a \\ u_I(a-t) \exp\left(-\int_{-t}^{0} \zeta(a+\tau, t+\tau) d\tau\right) & \text{if } t \leq a \end{cases} \tag{20}$$

Definition 5. We say that u is a weak solution of problem (19) if

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}_+} u(a,t) \left(\partial_t + \partial_a - \zeta\right) \varphi(a,t) da dt \\ &\qquad = \left[\int_{\mathbb{R}_+} u(a,t)\varphi(a,t) da\right]_{t=0}^{t=T} - \int_0^T u(0,t)\varphi(0,t) dt \end{aligned} \tag{21}$$

for every test function $\varphi \in \mathcal{D}([0,\infty) \times [0,T])$.

Theorem 13. The mild solution of (19) is a weak solution. Moreover, we also have *in the sense of characteristics* :

$$\begin{cases} (\partial_t + \partial_a + \zeta(a,t)) |u| = 0, & (a,t) \in (0,\infty) \times (0,T) \\ |u|(0,t) = |u_b(t)| & a = 0, t > 0 \\ |u|(a,0) = |u_I(a)| & a > 0, t = 0. \end{cases} \tag{22}$$

and if u_i are *mild* solutions of (19) with data u_b^i, u_I^i, ζ_i as above for $i \in \{1,2\}$ then the product $u_1 u_2$ solves (19) in the *mild* sense with data $u_b^1 u_b^2, u_I^1 u_I^2, \zeta_1 + \zeta_2$.

Proof. We set

$$J := \int_{\mathbb{R}_+} \int_0^T u(a,t) \left(\partial_t \varphi + \partial_a \varphi\right)\ dt\ da\ .$$

Performing the change of variables $x = (a - t/)/2, y = (a + t)/2$, one transforms $\mathbb{R}_+ \times (0, T)$ into $\Omega = \{(x, y)\} = \Omega_1 \cup \Omega_2$ where $\Omega_1 := \{(x, y) \in \mathbb{R}^2 \ ; \ x \in] - T/2, 0[\text{ and } y \in] - x, x + T[\}$ and $\Omega_2 := \{(x, y) \in \mathbb{R}^2 \ ; \ x \in]0; \infty[\text{ and } y \in]x, x + T[\}$. Setting $\tilde{\varphi}(x, y) := \varphi(a, t)$ one has then that

$$\partial_t \varphi + \partial_a \varphi = \partial_y \tilde{\varphi} \ ,$$

and

$$J = 2 \int_{\Omega_1} u \partial_y \tilde{\varphi} \ dy \ dx + 2 \int_{\Omega_2} u \partial_y \tilde{\varphi} \ dy \ dx =: J_1 + J_2 \ .$$

We treat each term separately because they correspond to the two cases of Duhamel's formula.

$$J_1 = 2 \int_{-\frac{T}{2}}^{0} \int_{-x}^{x+T} u(0, -x) g(x, y) \ \partial_y \tilde{\varphi}(x, y) \ dy \ dx \ .$$

The function $g(x, y) := \exp\left(- \int_0^{y+x} \zeta(s, s - 2x) ds \right)$ belongs to $H^1_y((-x, x + T))$ and it holds that $\tilde{\varphi}$ is $C^\infty \subset H^1_y$. Hence the integration by parts is well defined,

$$\begin{aligned}
J_1 &= 2 \int_{-\frac{T}{2}}^{0} u(0, -x) \left\{ [g(x, y) \tilde{\varphi}]_{y=-x}^{y=x+T} + \int_{-x}^{x+T} \zeta(x + y, y - x) g(x, y) \tilde{\varphi}(x, y) \ dy \right\} dx \\
&= 2 \int_{-\frac{T}{2}}^{0} u(0, -x) \left\{ \tilde{\varphi}(x, x + T) g(x, x + T) - \tilde{\varphi}(x, -x) \right\} dx \\
&\quad + 2 \int_{\Omega_1} \zeta(x + y, y - x) u \tilde{\varphi} \ dy \ dx \\
&= \int_0^T u(a, T) \varphi(a, T) da - \int_0^T u(0, t) \varphi(0, t) dt \\
&\quad + \int_0^T \int_0^t \zeta(a, t) u(a, t) \varphi(a, t) \ da \ dt \ ,
\end{aligned}$$

and similarly one gets the complementary result for J_2, which ends the proof. We reparametrize (20) by $\tilde{u}(x, y) = u(a, t)$ and obtain

$$\tilde{u}(x, y) = \begin{cases} u_b(-2x) \exp\left(- \int_0^{x+y} \zeta(\tau, \tau - 2x) d\tau \right) & (x, y) \in \Omega_1 \\ u_0(2x) \exp\left(- \int_0^{y-x} \zeta(\tau + 2x, \tau) d\tau \right) & (x, y) \in \Omega_2 \end{cases}$$

and then it is easy to realize that

$$(\partial_y + \zeta) \, \tilde{u} = 0$$

in the domain $\Omega_1 \cup \Omega_2$ parametrized by the variables (x, y). Solving this equation in the y variable and thanks to the assumptions, it is easy to show that $\tilde{u}$ is indeed Lipschitz with respect to y for every fixed x. Thus, one can write in the weak sense that $\partial_y |\tilde{u}| = \operatorname{sgn}(\tilde{u}) \partial_y \tilde{u}$ for every fixed x. Thus $\partial_y |\tilde{u}| + \zeta |\tilde{u}| = 0$ holds a.e. with respect to y for every fixed x. We then integrate and transform back to obtain the system which is the analogous to (20). The last claim involving the product $u_1 u_2$ follows from Duhamel's formula. ■

Exercise 5. Prove the previous theorem when adding a rhs $f \in L^\infty(\mathbb{R}_+ \times (0,T))$ to (19). In particular prove that

$$\begin{cases} (\partial_t + \partial_a + \zeta(a,t))\,|u| = \operatorname{sgn}(u) f(a,t), & (a,t) \in (0,\infty) \times (0,T) \\ |u|\,(0,t) = |u_b(t)| & a = 0, t > 0 \\ |u|\,(a,0) = |u_I(a)| & a > 0, t = 0. \end{cases} \tag{23}$$

and that the product $u_1 u_2$ solves :

$$\begin{cases} (\partial_t + \partial_a + \zeta_1(a,t) + \zeta_2(a,t)) u_1 u_2 = u_1 f_2 + u_2 f_1, & (a,t) \in (0,\infty) \times (0,T) \\ u_1 u_2(0,t) = u_b^1 u_b^2(t) & a = 0, t > 0 \\ u_1 u_2(a,0) = u_I^1 u_I^2(a) & a > 0, t = 0. \end{cases} \tag{24}$$

Hint : First set $\zeta_i = 0$ in order to simplify the expressions. This gives :

$$\int_{-a}^{0} f_2(a+s,t+s) u_1(a+s,t+s) ds = \int_{-a}^{0} f_2(a+s,t+s) u_1(0,t-a) ds \\ + \int_{-a}^{0} f_2(a+s,t+s) \int_{-a}^{s} f_1(a+\tau,t+\tau) d\tau ds$$

and

$$\int_{-a}^{0} f_1(a+s,t+s) u_2(a+s,t+s) ds = \int_{-a}^{0} f_1(a+s,t+s) u_2(0,t-a) ds \\ + \int_{-a}^{0} f_1(a+s,t+s) \int_{-a}^{s} f_2(a+\tau,t+\tau) d\tau ds \\ = \int_{-a}^{0} f_1(a+s,t+s) u_2(0,t-a) ds + \int_{-a}^{0} f_2(a+\tau,t+\tau) \int_{s}^{0} f_1(a+s,t+s) d\tau ds$$

where one swiches the order of integration in the last term in the rhs. Then summing both the expressions ends the part related to the product.

Thanks to these considerations we can claim the following result :

Theorem 14. Under the previous hypotheses and if moreover one has :

$$n_0 \in L^1(\mathbb{R}_+, \psi(a))$$

there is a unique solution in the weak sense of Definition 5 s.t. $\tilde{n} \in C(\mathbb{R}_+ \times L^1(\mathbb{R}_+; \psi(a)da))$ to (18) and we have :

i) if there exists C_0 s.t.

$$|n_0(a)| \leq C_0 N(a), \quad \text{a.e. } a \geq 0,$$

then the maximum principle holds :

$$|\tilde{n}(a,t)| \leq C_0 N(a), \quad \text{a.e. } a \geq 0, \quad \forall t > 0,$$

ii) the comparison principle holds as well *i.e.*

$$n_{0,1}(a) \leq n_{0,2}(a) \Rightarrow \tilde{n}_1(a,t) \leq \tilde{n}_2(a,t), \quad \forall t > 0,$$

iii) the conservation law and the $L^1(\mathbb{R}_+, \psi(a))$ contraction principle,

$$\int_{\mathbb{R}_+} \tilde{n}(a,t)\psi(a)da = \int_{\mathbb{R}_+} n_0(a)\psi(a)da,$$
$$\int_{\mathbb{R}_+} |\tilde{n}(a,t)|\, \psi(a)dx \leq \int_{\mathbb{R}_+} |n_0(a)|\, \psi(a)dx$$

Proof. First we prove existence by a fixed point argument in $X = C([0,T]; L^1(\mathbb{R}))$, with the topology $\|n\|_X := \sup_{0\leq t\leq T} \|n(\cdot,t)\|_{L^1_a(\mathbb{R}_+)}$. We define the following operator $n := \mathcal{T}[m]$ as the solution of the problem :

$$\begin{cases} (\partial_t + \partial_a + \lambda_0)n = 0 & (t,a) \in (0,\infty)^2, \\ n(0,t) = \int_{\mathbb{R}_+} k(a)m(a,t)da & t > 0,\ a = 0, \\ n(a,0) = n_0(a) & a > 0,\ t = 0. \end{cases}$$

Namely we set n as the mild solution defined through Duhamel's formula :

$$n(a,t) := \begin{cases} \int_{\mathbb{R}_+} k(\tilde{a})m(\tilde{a}, t-a)d\tilde{a} \exp(-\lambda_0 a) & \text{if } t > a \\ n_0(a-t)\exp(-\lambda_0 t) & \text{otherwise} \end{cases}$$

First we check that $\mathcal{T}$ is endomorphic in X *i.e.* we want to show that if w is s.t.

$$\forall \varepsilon > 0,\ \forall t > 0,\ \exists \eta_{\varepsilon,t} > 0\ ;\ \forall s\ ;\ |s-t| < \eta_{\varepsilon,t} \implies \|w(\cdot,t) - w(\cdot,s)\|_{L^1(\mathbb{R}_+)} < \varepsilon,$$

we have the same property for n. Lets assume that $s < t$ (the opposite case works the same),

$$\begin{aligned} \int_0^s |n(a,t) - n(a,s)|\, da &\leq \int_0^s \int_{\mathbb{R}_+} k(\tilde{a})|m(\tilde{a}, t-a) - m(\tilde{a}, s-a)|da \\ &\leq \|B\|_{L^\infty(\mathbb{R}_+)} \int_0^s \|m(\cdot, t-a) - m(\cdot, s-a)\|_{L^1(\mathbb{R}_+)} da \\ &\leq \|B\|_{L^\infty} s \sup_{a\in(0,s)} \|m(\cdot, t-a) - m(\cdot, s-a)\|_{L^1(\mathbb{R}_+)} \end{aligned}$$

as a continuous function is uniformly continuous on compact sets,

$$\begin{aligned}&\forall \varepsilon > 0, \forall t > 0,\ \exists \eta_{\varepsilon,t} > 0\ ;\ \forall s\ ;\\ &\qquad |s-t| < \eta_{\varepsilon,t} \implies \|w(\cdot,t-a) - w(\cdot,s-a)\|_{L^1(\mathbb{R}_+)} < \varepsilon, \quad \forall a \in (0,s).\end{aligned}$$

Then one has the term

$$\int_s^t |n(a,t) - n(a,s)|\, da \leq \|n\|_{L^\infty(\mathbb{R}_+\times(0,T))}(t-s)$$

which can be made arbitrarily small. And finally :

$$\begin{aligned}\int_t^\infty |n(a,t) - n(a,s)|\, da \leq \int_{\mathbb{R}_+} |n_0(\tilde{a}+t-s) - n_0(\tilde{a})| d\tilde{a}\\ + \|n_0\|_{L^1(\mathbb{R}_+)} |\exp(-\lambda_0 t) - \exp(-\lambda_0 s)|\end{aligned}$$

By the continuity of the translation operator in $L^1(\mathbb{R}_+)$ the first term in the rhs can be made arbitrarily small and because the exponential function is continuous as well, the second term is controlled also. Thus $\mathcal{T}$ maps X into itself as soon as n is bounded.

The boundedness follows from the simple estimates : if $a \leq t$

$$|n(a,t)| \leq \|B\|_{L^\infty} \|w\|_X,$$

whereas if $a > t$, then

$$|n(a,t)| \leq \|n_0\|_{L^\infty(\mathbb{R}_+)}.$$

In order to show contraction, we denote $n_i := \mathcal{T}(w_i)$ for $i \in \{1,2\}$ and

$$\begin{aligned}\int_{\mathbb{R}_+} |n_2(a,t) - n_1(a,t)| da = \int_0^t |n_2(a,t) - n_1(a,t)| da \leq t\|B\|_{L^\infty} \|w_2 - w_1\|_X\\ < \|w_2 - w_1\|_X\end{aligned}$$

provided that $t\|B\|_{L^\infty} < 1$. So for a final time T s.t. $T\|B\|_{L^\infty} < 1$, by the Banach fixed point Theorem, there exists a unique fixed point $\tilde{n} \in X$ s.t.

$$\tilde{n}(a,t) := \begin{cases} \int_{\mathbb{R}_+} k(\tilde{a})\tilde{n}(\tilde{a}, t-a) d\tilde{a} \exp(-\lambda_0 a) & \text{if } t > a \\ n_0(a-t)\exp(-\lambda_0 t) & \text{otherwise}\end{cases} \tag{25}$$

As usual, we can iterate the operator on $[T, 2T]$, $[2T, 3T]$, ..., since the condition on T does not depend on the iteration. According to Theorem 13, if $\tilde{n}$ solves (25) then it is also a weak solution *i.e.* it solves :

$$\begin{aligned}\int_{\mathbb{R}_+\times(0,T)} \tilde{n}(a,t)\left(\partial_t + \partial_a - \lambda_0\right)\varphi(a,t) da dt - \left[\int_{\mathbb{R}_+} \tilde{n}(a,s)\varphi(a,s) da\right]_{s=0}^{s=T}\\ + \int_0^T \tilde{n}(0,t)\varphi(0,t) dt = 0\end{aligned} \tag{26}$$

for all $\varphi \in C^\infty(\mathbb{R}_+^2) \cap L^\infty(\mathbb{R}_+^2)$.

The comparison principle (ii) follows from the construction of the Banach-Picard fixed point. To show this, consider two initial data $n_{0,1}$, $n_{0,2}$ and denote by $\mathcal{T}_1$, $\mathcal{T}_2$ the corresponding operators of step 1. If $n_{0,1} \leq n_{0,2}$, then for all m we have $\mathcal{T}_1[m] \leq \mathcal{T}_2[m]$ and thus the fixed point itself (recall the Picard iteration process) satisfies $n_1 \leq n_2$. As a direct consequence the maximum principle holds true because $\pm C_0 N(x)$ are solutions and can be used in (ii).

Consider now $n_0 \in L^1(\mathbb{R}_+, \psi(a)da)$ by density (recall that ψ is bounded), we can find a sequence $n_0^k \in L^1(\mathbb{R}_+; da)$ such that $n_0^k \to n_0$ in $L^1(\mathbb{R}_+; \psi(a)da)$ when $k \to \infty$. We denote by $\tilde{n}^k(a,t)$ the corresponding solution to (18). Combining (18) with the dual equation (17), we compute for the solution $\hat{n} = \tilde{n}^k - \tilde{n}^p$,

$$(\partial_t + \partial_a)(\hat{n}(a,t)\psi(a)) = -\psi(0)k(a)\hat{n}(a,t). \tag{27}$$

This implies (cf. Theorem 13 and Exercise 5) the identity

$$(\partial_t + \partial_a)\left(|\hat{n}|\,\psi\right) = -\psi(0)k(a)\,|\hat{n}|$$

after integrating in a we deduce that

$$\frac{d}{dt}\int_{\mathbb{R}_+} |\hat{n}|\,\psi da = -\psi(0)\int_{\mathbb{R}_+} k(a)\,|\hat{n}|\,da + \psi(0)\left|\int_{\mathbb{R}_+} k(a)\hat{n}(a,t)da\right| \leq 0$$

which after integration in time gives :

$$\int_{\mathbb{R}_+} |\hat{n}(a,t)|\,\psi(a)da \leq \int_{\mathbb{R}_+} \left|n_0^k(a) - n_0^p(a)\right|\psi(a)da \tag{28}$$

and thus $\{\tilde{n}_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence of $C(\mathbb{R}_+; L^1(\mathbb{R}_+, \psi(a)))$. Notice also the uniform bound $|\tilde{n}_k(a,t)| \leq C_0 N(a)$. Therefore it converges strongly in $C(\mathbb{R}_+; L^1(\mathbb{R}_+, \psi(a)da))$ and weakly in $L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ leading to a weak solution in the sense of Definition 5.

It is a unique solution since : (28) shows that two possible solutions coincide on the support of ψ and thus on the support of k. So the difference of the two solutions solve an homogeneous transport problem with zero boundary and initial data, thus it is zero. Integrating (27) wrt a gives :

$$\frac{d}{dt}\int_{\mathbb{R}_+} \tilde{n}(a,t)\psi(a)da = 0,$$

from which we recover the conservation law iii). ■

2.3.3 Regularity of solutions

Later on, we will need some regularity results that we state now.

Theorem 15. Under hypotheses (1), and if the initial data n_0 is Lipschitz *i.e.*

$$|n_0(a)| \leq C_0 N(a), \quad |n_0'(a)| \leq C_1 N(a), \quad \forall a \in \mathbb{R}_+,$$

assume moreover that $n_0(0) = \int_{\mathbb{R}_+} k(a)n_0(a)da$, then the solution satisfies :

$$|\partial_t \tilde{n}(a,t)| \leq (C_1 + \lambda_0)N(a), \quad |\partial_a \tilde{n}(a,t)| \leq (C_1 + \lambda_0(1 + C_0))N(a),$$

for all $(a,t) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Proof. Since we assumed the compatibility condition $n_0(0) = \int_{\mathbb{R}_+} k(a)n_0(a)da$, (25) is continuous through the interface $t = a$. Differentiating (25) wrt time, one recovers easily that it is a weak solution of the problem :

$$\begin{cases} (\partial_t + \partial_a + \lambda_0)\partial_t \tilde{n} = 0 & (a,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ \partial_t \tilde{n}(0,t) = \int_{\mathbb{R}_+} k(\tilde{a})\partial_t \tilde{n}(\tilde{a})d\tilde{a} & a = 0,\ t > 0 \\ \partial_t \tilde{n}(a,0) = -n_0'(a) - \lambda_0 n_0(a), & a > 0,\ t = 0 \end{cases} \tag{29}$$

and thus the maximum principle from Theorem 14 holds also for the time derivative :

$$|\partial_t \tilde{n}(a,t)| \leq (C_1 + \lambda_0)N(a), \quad a \geq 0$$

As now the time derivative is a bounded function, and since equation (18) holds in the sense of distributions it is also true in a pointwise sense, one has that

$$|\partial_a \tilde{n}(a,t)| = |\partial_t \tilde{n} + \lambda_0 \tilde{n}| \leq (C_1 + \lambda_0)N(a) + \lambda_0 C_0 N(a)$$

which gives the result. ■

Remark 4. If we do not assume the compatibility condition $n_0(0) = \int_{\mathbb{R}_+} k(a)n_0(a)da$, then $\tilde{n}$ is differentiable wrt time only in $\Omega_1 \cup \Omega_2$ where $\Omega_1 := \{t > a\}$ and $\Omega_2 := \{t < a\}$. We cannot use the comparison principle since the derivative does not solve (29). Indeed, if $\tilde{n}$ is a weak solution and $\partial_t \tilde{n}$ is defined by derivation of (25) picewisely on $\Omega_1 \cup \Omega_2$, then setting $\varphi = \partial_t \phi$ as a test function in (26) provides after integration by parts that $\partial_t \tilde{n}$ solves :

$$\begin{aligned} &- \int_{\mathbb{R}_+ \times (0,T)} \partial_t \tilde{n}(\partial_t + \partial_a - \lambda_0)\phi da dt - \left[\int_{\mathbb{R}_+} (\partial_a + \lambda_0)\tilde{n}\phi da \right]_{t=0}^{t=T} \\ &+ \int_0^T \partial_t \tilde{n}(0,t)\phi(0,t)dt + \left(n(0,0^+) - n(0^+,0)\right)\phi(0,0) \\ &+ \int_0^T [\tilde{n}(t,t)](\partial_t + \partial_a - \lambda_0)\phi(t,t)dt = 0 \end{aligned}$$

where $[\tilde{n}(t,t)] := \lim_{s\to 0} \tilde{n}(t, t+s) - \tilde{n}(t, t-s)$ and $\tilde{n}(0^+, 0) := \lim_{s\to 0^+} \tilde{n}(s, 0)$ and the same for $\tilde{n}(0, 0^+)$.

2.3.4 Generalized relative entropy

Theorem 16. Under the assumptions of Theorem 14, for all locally Lipschitz convex functions $H : \mathbb{R} \to \mathbb{R}$ one has :

i) for all $t > 0$,

$$\int_{\mathbb{R}_+} \psi(a)N(a)H\left(\frac{\tilde{n}(a,t)}{N(a)}\right) da \leq \int_{\mathbb{R}_+} \psi(a)N(a)H\left(\frac{n_0(a)}{N(a)}\right) da$$

ii) denoting the probability measure $d\mu(a) = k(a)N(a)/N(0)da$,

$$\int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} H\left(\frac{\tilde{n}(a,t)}{N(a)}\right) d\mu(a) - H\left(\int_{\mathbb{R}_+} \frac{\tilde{n}(a,t)}{N(a)} d\mu(a)\right)\right] dt$$
$$\leq \int_{\mathbb{R}_+} \psi(a)N(a)H\left(\frac{n_0(a)}{N(a)}\right) da$$

Proof. We use that

$$(\partial_t + \partial_a)\frac{\tilde{n}(a,t)}{N(a)} = 0,$$

so that by similar arguments as in Theorem 13,

$$(\partial_t + \partial_a)H\left(\frac{\tilde{n}(a,t)}{N(a)}\right) = 0,$$

and finally

$$(\partial_t + \partial_a)\left(\psi(a)N(a)H\left(\frac{\tilde{n}(a,t)}{N(a)}\right)\right) = -\psi(0)k(a)N(a)H\left(\frac{\tilde{n}(a,t)}{N(a)}\right).$$

After integration in $a \in \mathbb{R}_+$ we find

$$\frac{d}{dt}\int_{\mathbb{R}_+} \psi(a)N(a)H\left(\frac{\tilde{n}(a,t)}{N(a)}\right) da$$
$$= -\psi(0)N(0)\int_{\mathbb{R}_+} H\left(\frac{\tilde{n}(a,t)}{N(a)}\right) d\mu(a) + \psi(0)N(0)H\left(\frac{\tilde{n}(0,t)}{N(0)}\right)$$
$$= \psi(0)N(0)\left[-\int_{\mathbb{R}_+} H\left(\frac{\tilde{n}(a,t)}{N(a)}\right) d\mu + H\left(\int_{\mathbb{R}_+} \frac{\tilde{n}(a,t)}{N(a)} d\mu\right)\right] \leq 0$$

for all convex functions H by Jensen's inequality. The statements (i) and (ii) follow from this inequality. ■

2.3.5 Long time asymptotic : entropy method

In practice, one observes the *stable age distribution*, *i.e.*, the long time limit of $\tilde{n}$ which is expected to be proportional to the steady state N given by equation (15). In this section, we prove a general statement without a rate. Assuming more restrictive hypotheses on k, one can obtain exponential convergence.

Theorem 17. Under hypotheses (1), and assuming that $n_0 \in L^1(\mathbb{R}_+, \psi(a)da) \cap L^\infty(\mathbb{R}_+)$ and that $k(a) > 0$ a.e. $a \in \mathbb{R}_+$, the solution given by Theorem 14 satisfies

$$\lim_{t\to\infty} \int_{\mathbb{R}_+} |\tilde{n}(a,t) - m_0 N(a)|\, \psi(a)da = 0 \tag{30}$$

where $m_0 := \int_{\mathbb{R}_+} n_0(a)\psi(a)da$ (a conserved quantity).

Proof. i) Truncating and regularizing $n_0 \in L^1(\mathbb{R}_+; \psi(a)da)$, one can construct a sequence $\{n_{0,k}\}_{k\in\mathbb{N}}$ approximating n_0 in the $L^1(\mathbb{R}_+; \psi(a)da)$ norm s.t. for any fixed k the compatibility constraint holds true and there exists constants (C_0^k, C_1^k) such that we are fully in the hypotheses of Theorem (15). The proof is left as an exercise. Using the comparison principle we have :

$$\int_{\mathbb{R}_+} |\tilde{n}(a,t) - \tilde{n}_\varepsilon(a,t)|\, \psi(a)da \le \int_{\mathbb{R}_+} |n_0(a) - n_{0,\varepsilon}(a)|\, \psi(a)da =: r_\varepsilon \to 0,$$

and also $|m_0 - m_{0,\varepsilon}| \le r_\varepsilon$. Therefore it is enough to prove the result for the regularized initial data. Indeed

$$\int_{\mathbb{R}_+} |\tilde{n}(a,t) - m_0 N(a)|\, \psi(a)da \le 2r_\varepsilon + \int_{\mathbb{R}_+} |\tilde{n}_\varepsilon - m_{0,\varepsilon} N(a)|\, \psi(a)da.$$

ii) In the regularized case, we set $h(a,t) := \tilde{n}(a,t) - m_0 N(a)$, which is also a solution to (18) and satisfies $|h(a,t)| \le CN(a)$ and $\int_{\mathbb{R}_+} h(a,t)\psi(a)da = 0$. We prove that such a solution vanishes over a long time. Notice that, by the GRE property, we have

$$f(t) := \int_{\mathbb{R}_+} |h(a,t)|\, \psi(a)da \to L \ge 0, \text{ as } t \to \infty.$$

Indeed for any increasing sequence of times $\{t_p\}_{p\in\mathbb{N}}$, the quantity

$$q_p := f(t_p) = \int_{\mathbb{R}_+} |h(a,t_p)|\, \psi(a)da$$

is a monotone non-increasing positive sequence which is thus convergent to some limit L. Then for any ε there exists an k s.t. $t \in [t_k, t_{k+1})$ and thus $f(t_{k+1}) \le f(t) \le f(t_k)$ which implies that $|f(t) - L| < \varepsilon$ which then shows the claim. And it remains to show that L = 0.

iii) Now we define a sequence of functions :

$$h_k(a,t) := h(a, t+k),$$

which satisfies $|h_k(a,t)| \le CN(a)$. Using Theorem 16, for any strictly convex C^1 function $H : \mathbb{R} \to \mathbb{R}$, one has :

$$\int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} H\left(\frac{h_k(a,t)}{N(a)}\right) d\mu(a) - H\left(\int_{\mathbb{R}_+} \frac{h_k(a,t)}{N(a)} d\mu\right)\right] dt =: I_k \to 0, \text{ as } k \to \infty$$

Indeed, from the very definitions of I_k and h_k , we have

$$I_k = \int_k^\infty \left[\int_{\mathbb{R}_+} H\left(\frac{h(a,t)}{N(a)}\right) d\mu(a) - H\left(\int_{\mathbb{R}_+} \frac{h(a,t)}{N(a)} d\mu(a)\right)\right] dt$$

which tends to zero by Lebesgue's Theorem since the quantity inside the brackets is positive and integrable. We define

$$J_k := \int_k^{k+T} \int_{\mathbb{R}_+} H\left(\frac{h(a,t)}{N(a)}\right) d\mu(a)dt, \quad M_k := \int_k^{k+T} H\left(\int_{\mathbb{R}_+} \frac{h(a,t)}{N(a)} d\mu(a)\right) dt$$

one has then that

$$0 \leq J_k - M_k \leq I_k \to 0$$

which shows that

$$\lim_{k\to\infty} J_k - M_k = 0. \tag{31}$$

One also notices that

$$\begin{cases} (\partial_t + \partial_a + \lambda_0)h_k = 0, & a > 0,\ t > 0 \\ h_k(0,t) = \int_{\mathbb{R}_+} k(a)h_k(\tilde{a},t)d\tilde{a} & a = 0,\ t > 0 \\ \int_{\mathbb{R}_+} h_k(a,t)\psi(a)da = 0. \end{cases} \tag{32}$$

Next, using the regularity of h (via Theorem (15)), we may extract a sub-sequence (still denoted h_k) such that, for all $T > 0$,

$$\begin{aligned} & h_k \to g \text{ in } C(\mathbb{R}_+ \times [0,T]), \qquad 0 \ \leq |g| \leq CN(a), \\ & \int_{\mathbb{R}_+} k(a)h_k(a,t)da \to \int_{\mathbb{R}_+} k(a)g(a,t)da \text{ in } C([0,T]), \\ & \int_{\mathbb{R}_+} \psi(a)g(a,t)da = 0, \qquad \int_{\mathbb{R}_+} \psi(a)|g(a,t)|da = L. \end{aligned}$$

Now we write :

$$\begin{aligned} \lim_{k\to\infty} M_k &= \lim_{k\to\infty} \int_0^T H\left(\int_{\mathbb{R}_+} \frac{h_k(a,t)}{N(a)} d\mu\right) dt \\ &\overset{\text{Lebesgue}}{=} \int_0^T H\left(\int_{\mathbb{R}_+} \frac{g(a,t)}{N(a)} d\mu\right) dt \overset{\text{thanks to (31)}}{=} \lim_{k\to\infty} J_k \end{aligned}$$

Since $h_k \leq CN$ one has that

$$\int_{\mathbb{R}_+} \left| H\left(\frac{h_k}{N}\right)\right| d\mu \leq \sup_{u\in B(0,C)} |H(u)| \int_{\mathbb{R}_+} d\mu = \sup_{u\in B(0,C)} |H(u)| < \infty$$

the latter inequality being true since H defined on $\mathbb{R}$. This allows to use Lemma 4, which thus provides :

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}_+} H\left(\frac{g(a,t)}{N(a)}\right) d\mu(a)dt \leq \liminf_{k\to\infty} \int_0^T \int_{\mathbb{R}_+} H\left(\frac{h_k(a,t)}{N(a)}\right) d\mu(a)dt \\ & \quad = \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}_+} H\left(\frac{h_k(a,t)}{N(a)}\right) d\mu(a)dt = \lim_{k\to\infty} \int_0^T H\left(\int_{\mathbb{R}_+} \frac{h_k(a,t)}{N(a)} d\mu(a)\right) dt \\ & \quad = \int_0^T H\left(\int_{\mathbb{R}_+} \frac{g(a,t)}{N(a)} d\mu(a)\right) dt \end{aligned}$$

provides that

$$\int_0^T \int_{\mathbb{R}_+} H\left(\frac{g(a,t)}{N(a)}\right) d\mu dt \leq \int_0^T H\left(\int_{\mathbb{R}_+} \frac{g(a,t)}{N(a)} d\mu\right) dt.$$

Since the reverse inequality holds thanks to Jensen, these two terms are in fact equal. Using Lemma 9, one has that

$$\frac{g(a,t)}{N(a)} = C(t), \quad \text{a.e. } a \in \operatorname{supp} k \equiv \mathbb{R}_+, \quad \text{a.e. } t \in (0,T).$$

On the other hand, since g satisfies in the sense of distribution :

$$(\partial_t + \partial_a + \lambda_0) g = 0,$$

one has, also in the distributional sense, that

$$(\partial_t + \partial_a) \frac{g(a,t)}{N(a)} = 0$$

and thus by [29, Section 2.4, Thm 1], $C(t)$ is in fact constant in time. Thus there exists $\alpha \in \mathbb{R}$ s.t. $C(t) = \alpha$ and

$$g(a,t) = \alpha N(a), \quad \text{a.e. } a \in \mathbb{R}_+,$$

and using that

$$\int_{\mathbb{R}_+} g(a,t) \psi(a) da = 0$$

we find that $\alpha = 0$. Now we can conclude that the limit L of the second step vanishes because, passing to the limit when k goes to ∞, we have $L = \int_{\mathbb{R}_+} |g(a,t)| \psi(a) da = 0$. ■

Lemma 4. **Convex functional and convergent sequences** Assume that H is convex and lower semi continuous (l.s.c.). Assume moreover that there exists $s \in \partial H(0)$, assume as well that there exists a sequence f_k s.t. $f_k \to f$ in $L^1(\mathbb{R}_+, d\mu)$ and that $\sup_n \|H(f_n)\|_{L^1(\mathbb{R}_+, d\mu)} < \infty$, then one has

$$\int_{\mathbb{R}_+} H(f) d\mu \leq \liminf_{n\to\infty} \int_{\mathbb{R}_+} H(f_n) d\mu$$

Proof. As there exists $s \in \mathbb{R}$ s.t. $s \in \partial H(0)$ this means that

$$H(w) \geq H(0) + sw = sw, \quad \forall w \in \mathbb{R}$$

which implies that setting $g_n = H(f_n) - s f_n$ is a non-negative sequence, moreover,

$$\sup_n \|g_n\|_{L^1(\mathbb{R}_+, d\mu)} < \infty$$

by hypotheses. Then one has, thanks to the l.s.c. property of H, that $H(f) - sf \leq g := \liminf_{n\to\infty} g_n$ and

$$\int_{\mathbb{R}_+} H(f) - sf d\mu(a) \leq \int_{\mathbb{R}_+} g(a) d\mu(a) \leq \liminf_n \int_{\mathbb{R}_+} g_n d\mu(a)$$

which reads :

$$\int_{\mathbb{R}_+} H(f)d\mu - s\int_{\mathbb{R}_+} fd\mu \leq \liminf_n \int_{\mathbb{R}_+} H(f_n)d\mu - s\int_{\mathbb{R}_+} fd\mu$$

which ends the proof. ■

Corollary 3. Under the previous hypotheses, there exist a sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ s.t.

$$\forall\varepsilon > 0,\ \exists k_0\ ;\ \forall k > k_0 \implies |\tilde{n}(0, t + n_k) - m_0| < \varepsilon, \quad \text{a.e. } t \in (0, 1)$$

Remark 5. One would like to show instead :

$$\lim_{t\to\infty} |\tilde{n}(0, t) - m_0| = 0$$

but it does not seem straightforward.

Proof. By Theorem 17, one has :

$$\forall\varepsilon > 0,\ \exists n_0\ ;\ t > n_0 \implies \int_{\mathbb{R}_+} |\tilde{n}(a, t) - m_0 N(a)|\, \psi(a)da \leq \varepsilon$$

which gives after integration in time that

$$\forall\varepsilon > 0,\ \exists n_0\ ;\ \forall n > n_0 \implies \int_0^1 \int_{\mathbb{R}_+} |\tilde{n}(a, t + n) - m_0 N(a)|\, \psi(a)dadt \leq \varepsilon$$

There exists a sub-sequence $(n_k)_{k\in\mathbb{N}}$ s.t.

$$|\tilde{n}(a, t + n_k) - m_0 N(a)| \to 0, \quad \text{a.e. } (a, t) \in \operatorname{supp}\psi \times (0, 1).$$

Where we used that L^p convergence of a sequence implies that there exists a sub-sequence converging point-wisely. Since both terms of the difference are bounded in age, one can apply Lebesgue's Theorem, showing that for a.e. $t \in (0, 1)$,

$$\int_{\mathbb{R}_+} |\tilde{n}(a, n_k + t) - m_0 N(a)|\, k(a)da \to 0,$$

then a simple estimate ends the proof :

$$\begin{aligned} |\tilde{n}(0, t + n_k) - m_0| &= \left| \int_{\mathbb{R}_+} (\tilde{n}(a, t + n_k) - m_0 N(a))k(a)da \right| \\ &\leq \int_{\mathbb{R}_+} |\tilde{n}(a, t + n_k) - m_0 N(a)|\, k(a)da \to 0, \end{aligned}$$

■

For further applications and theory of General Relative Entropy (GRE), the reader can consult [25, Chap. 3] from where the claims are extracted. Our contribution concerns Theorems 13 and 15 and the extension of the GRE theory to the limit case 14 from Assumtions 1.

2.4 The elongation problem

2.4.1 The kernel is dissipative

In what follows we assume that ϱ, linkages' density, satisfies the following problem :

$$\begin{cases} \partial_a \varrho(a) + \zeta(a)\varrho(a) = 0, & \text{a.e. } a > 0, \\ \varrho(0) = \beta, & a = 0. \end{cases} \tag{33}$$

In order to set up the problem in a certain frame, we make the following assumptions.

Assumptions 2. The data of problem (33) satisfies :

- $\zeta(a) \geq 0$ and $\zeta \in L^\infty(\mathbb{R}_+)$.
- moreover there exists a strictly positive decreasing function $m \in L^1(\mathbb{R}_+)$ s.t. $m' \in L^\infty(\mathbb{R}_+)$ and for a.e. $a > a_0$,
$$\zeta(a) \geq -\frac{m'(a)}{m(a)}$$

Proposition 5. Under the previous assumptions, one has

$$\int_{\mathbb{R}_+} \varrho(a) da < +\infty$$

Proof. Since (33) can be solved explicitly, one has according to the assumptions that : for a.e. $a \in (a_0, \infty)$,

$$\begin{aligned} 0 \leq & \varrho(a) = \beta \exp\left(-\int_0^a \zeta(\tilde{a}) d\tilde{a}\right) = \beta \exp\left(-\int_0^{a_0} \zeta(\tilde{a}) d\tilde{a}\right) \exp\left(-\int_{a_0}^a \zeta(\tilde{a}) d\tilde{a}\right) \\ & \leq \beta \exp\left(-a_0 \inf_{a \in (0,a_0))} \zeta(a)\right) \exp(\ln m(a) - \ln m(a_0)) = C \frac{m(a)}{m(a_0)} \in L^1(a_0, \infty) \end{aligned}$$

■

Remark 6. Taking for instance $\varrho(a) = \frac{1}{(1+a)^\sigma}, \quad \sigma \in (1,2)$ satisfies (33), with $\beta = 1$ and $\zeta(a,t) = \frac{\sigma}{(1+a)}$ and one has $\int_{\mathbb{R}_+} \varrho(a) da = 1/(\sigma - 1)$. Nevertheless,

$$\int_{\mathbb{R}_+} \varrho(a) a da = +\infty.$$

Proposition 6. Assume that hypotheses 2 hold and that ϱ solves (33), then setting $k(a) := \varrho(a) / \int_{\mathbb{R}_+} \varrho(\tilde{a}) d\tilde{a}$ one has

$$(1 + a)\zeta(\cdot)k(\cdot) \in L^1(\mathbb{R}_+)$$

Proof. Integrating by parts :

$$v_R := \int_0^R (1+a)\zeta(a)k(a)da = -\int_0^R (1+a)\partial_a k(a)da = -\left[(1+a)k(a)\right]_{a=0}^R + \int_0^R k(a)da$$
$$= k(0) - (1+R)k(R) + \int_0^R k(a)da \leq k(0) + \int_{\mathbb{R}_+} k(a)da = k(0)+1$$

as $(1+R)k(R) \geq 0$, v_R is a monotone non-decreasing bounded from above function of R thus it converges to some finite limit below $k(0)+1$, which ends the proof. ■

Remark 7. Although this hypothesis is coommonly made in our references [15, 16, 17, 18, 14, 13], we do not want ζ to have a constant positive definite lower bound $\zeta_{\min}$. Indeed if there were $\zeta_{\min} > 0$ s.t.

$$\text{a.e. } a \in \mathbb{R}_+, \quad \zeta(a) \geq \zeta_{\min} > 0$$

then

$$\varrho(a) \leq \beta \exp(-\zeta_{\min} a)$$

and

$$\int_{\mathbb{R}_+} \varrho(a)a^p da \leq \int_{\mathbb{R}_+} \beta \exp(-\zeta_{\min} a)a^p da = \beta\Gamma(p+1)\zeta_{\min}^{-(p+1)} < \infty \quad \forall p \in \mathbb{R}_+$$

2.4.2 The elongation variable

If one sets,

$$u(a,t) := \begin{cases} x(t) - x(t-a) & t > a \\ x(t) - x_p(t-a) & t \leq a \end{cases}$$

then u can be rewritten as :

$$u(a,t) = \begin{cases} \int_{t-a}^t x'(s)ds = \int_{-a}^0 x'(t+s)ds, & t > a \\ \int_0^t x'(s)ds + x(0^+) - x_p(t-a) = \int_{-t}^0 x'(t+s)ds + u_I(a), & t \leq a \end{cases} \tag{34}$$

where we defined

$$u_I(a) = x(0^+) - x_p(a).$$

In (34) we recognize Duhamel's formula of a *mild* solution associated to the system :

$$\begin{cases} (\partial_t + \partial_a)u(a,t) = x'(t), & a > 0,\ t > 0, \\ u(0,t) = 0, & a = 0,\ t > 0, \\ u(a,0) = u_I(a), & a > 0,\ t = 0. \end{cases} \tag{35}$$

Now we compute the problem solved by x'. As x solves :

$$x(t) = \frac{f}{\mu_0} + \int_{\mathbb{R}_+} x(t-a)k(a)da = \frac{f}{\mu_0} + \int_0^t x(t-a)k(a)da + \int_t^\infty x_p(t-a)k(a)da$$

one has differentiating

$$\begin{aligned}
x'(t) &= \frac{f'}{\mu_0} + \int_0^t x'(t-a)k(a)da + k(t)\left(x(0^+) - x_p(0)\right) + \int_t^\infty x_p'(t-a)k(a)da \\
&= \frac{f'}{\mu_0} - \int_0^t \partial_a(x(t-a))k(a)da - \int_t^\infty \partial_a(x_p(t-a))k(a)da + k(t)\left(x(0^+) - x_p(0)\right) \\
&= \frac{f'}{\mu_0} - [x(t-a)k(a)]_{a=0}^t + \int_0^t x(t-a)\partial_a k(a)da \\
&\quad - [x(t-a)k(a)]_{a=t}^\infty + \int_t^\infty x_p(t-a)\partial_a k(a)da + k(t)\left(x(0^+) - x_p(0)\right) \\
&= \frac{f'}{\mu_0} + x(t)k(0) - \int_{\mathbb{R}_+} \zeta k(a)x(t-a)da
\end{aligned}$$

Integrating (33) in age one finds that

$$-k(0) + \int_{\mathbb{R}_+} \zeta(a)k(a)da = 0$$

which then allows to rephrase the previous equation as

$$x'(t) = \frac{f'}{\mu_0} + \int_{\mathbb{R}_+} (x(t) - x(t-a))\zeta(a)k(a)da = \frac{f'}{\mu_0} + \int_{\mathbb{R}_+} u(a,t)\zeta(a)k(a)da \tag{36}$$

Inserting this equality in (35), one obtains a closed problem where the only unknown is u (we can forget x):

$$\boxed{\begin{cases} (\partial_t + \partial_a)u = \dfrac{f'}{\mu_0} + \displaystyle\int_{\mathbb{R}_+} u(a,t)\zeta(a)k(a)da, & a > 0,\ t > 0, \\ u(0,t) = 0, & a = 0,\ t > 0, \\ u(a,0) = u_I(a), & a > 0,\ t = 0. \end{cases}} \tag{37}$$

Once u is computed one can return to x using (36) and recover x writing :

$$x(t) = x(0^+) + \int_0^t \left\{ \frac{f'(s)}{\mu_0} + \int_{\mathbb{R}_+} u(a,s)\zeta(a)k(a)da \right\} ds$$

2.4.3 A new scaling

We are interested in the long time asymptotics. For this sake, we make the following scaling assumption :

$$x_\varepsilon(\tilde{t}) := \varepsilon x\left(\frac{\tilde{t}}{\varepsilon}\right), \quad \forall \tilde{t} \in (0,1).$$

So whenever ε is small, one recovers large times when $\tilde{t} > 0$. In this scaling one has :

$$\partial_t x_\varepsilon(\tilde{t}) = \partial_t x(\tilde{t}/\varepsilon), \quad u_\varepsilon(a,\tilde{t}) := \frac{x_\varepsilon(\tilde{t}) - x_\varepsilon(\tilde{t} - \varepsilon a)}{\varepsilon} = x(\tilde{t}/\varepsilon) - x(\tilde{t}/\varepsilon - a) = u(a, \tilde{t}/\varepsilon).$$

We would like to check what equation u_ε solves.

$$\begin{aligned}
(\varepsilon\partial_t + \partial_a)u_\varepsilon &= x_\varepsilon'(t) - x_\varepsilon'(t - \varepsilon a) + x_\varepsilon'(t - \varepsilon a) = x_\varepsilon'(t) = x'(t/\varepsilon) \\
&= \frac{f'(t/\varepsilon)}{\mu_0} + \int_{\mathbb{R}_+} u(a, t/\varepsilon)k(a)da = \frac{f'(t/\varepsilon)}{\mu_0} + \int_{\mathbb{R}_+} u_\varepsilon(a,t)k(a)da
\end{aligned}$$

while for the initial condition we write :

$$u_\varepsilon(a,0) = \frac{x_\varepsilon(0) - x_\varepsilon(-\varepsilon a)}{\varepsilon} = x(0) - x_p(-a) = u_I(a)$$

So that finally u_ε solves :

$$\boxed{\begin{cases} (\varepsilon \partial_t + \partial_a) u_\varepsilon = \dfrac{f'(t/\varepsilon)}{\mu_0} + \displaystyle\int_{\mathbb{R}_+} u_\varepsilon(a,t)\zeta(a)k(a)da =: g_\varepsilon(t), & a > 0,\ t > 0, \\ u_\varepsilon(0,t) = 0, & a = 0,\ t > 0, \\ u_\varepsilon(a,0) = u_I(a), & a > 0,\ t = 0. \end{cases}} \tag{38}$$

And we make the assumptions on the data f :

Assumptions 3. The load f is
i) bounded : $f \in L^\infty(\mathbb{R}_+)$
ii) locally integrable and the derivative is integrable :

$$f \in L^1_{\text{loc}}(\mathbb{R}_+), \quad f' \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$$

iii) and there exists $f_\infty \in \mathbb{R}$ s.t.

$$\int_{\mathbb{R}_+} |f(t) - f_\infty|\, dt < \infty.$$

In what follows, we aim first to give a meaning to a weak solution of (37) in a proper framework. Then we show how this is equivalent to the renewal problem (3). We define the weighted Banach space :

$$X_T := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}_+ \times (0,T)) \text{ s.t. } u(a,t)/(1+a) \in L^\infty(\mathbb{R}_+ \times 0, T) \right\}.$$

Theorem 18. Under Assumptions 2 and 3, there exists a weak unique solution $u \in X_T$ of problem (38).

Proof. The proof uses Proposition 6 and follows the same lines as in Theorem 14 and is left as an exercise. ■

2.4.4 A stability result

Theorem 19. Under hypotheses 2 and 3, one has for almost every $t \in (0,T)$

$$\int_{\mathbb{R}_+} |u_\varepsilon(a,t)|\, k(a)da \leq \|f'\|_{L^1(\mathbb{R}_+)} + \int_{\mathbb{R}_+} |u_I(a)|\, k(a)da \leq \|f'\|_{L^1(\mathbb{R}_+)} + \left| \frac{f(0)}{\mu_0} \right| + 2\|x_p\|_{L^\infty(\mathbb{R}_-)}$$

Proof. To be rigorous the proof uses the same arguments as in the proof of Theorem 13. So the calculus is here only formal. In the sense of characteristics, one has

$$(\varepsilon \partial_t + \partial_a) |u_\varepsilon| \leq \frac{1}{\mu_0} |f'(t/\varepsilon)| + \int_{\mathbb{R}_+} |u_\varepsilon(a,t)| \zeta(a) k(a) da$$

then considering $|u_\varepsilon| B$ as the unknown it solves :

$$(\varepsilon \partial_t + \partial_a) (|u_\varepsilon(a,t)| k(a)) + \zeta(a) k(a) |u_\varepsilon(a,t)| \leq \frac{1}{\mu_0} |f'(t/\varepsilon)| + \int_{\mathbb{R}_+} |u_\varepsilon(\tilde{a},t)| \zeta(\tilde{a}) k(\tilde{a}) d\tilde{a}$$

Integrating in age this gives :

$$\begin{aligned} \varepsilon \frac{d}{dt} \int_{\mathbb{R}_+} |u_\varepsilon(a,t)| k(a) da + \int_{\mathbb{R}_+} |u_\varepsilon(a,t)| \zeta(a) k(a) da \\ \leq \frac{1}{\mu_0} |f'(t/\varepsilon)| + \int_{\mathbb{R}_+} |u_\varepsilon(\tilde{a},t)| \zeta(\tilde{a}) k(\tilde{a}) d\tilde{a} \end{aligned}$$

since the two dissipation terms cancel, one gets the result noticing that

$$\frac{1}{\varepsilon} \int_0^t \frac{1}{\mu_0} |f'(s/\varepsilon)| ds = \int_0^{\frac{t}{\varepsilon}} \frac{1}{\mu_0} |f'(\tau)| d\tau \leq \|f'\|_{L^1(\mathbb{R}_+)}.$$

Now we detail, the bound on the initial condition :

$$\int_{\mathbb{R}_+} |u_I(a)| k(a) da \leq \left| \frac{f(0)}{\mu_0} \right| + 2 \int_{\mathbb{R}_+} |x_p(-a)| k(a) da \leq \left| \frac{f(0)}{\mu_0} \right| + 2 \|x_p\|_{L^\infty(\mathbb{R}_-)}$$

■

Corollary 4. Under the previous hypotheses, one has continuous dependence on the data. Indeed, if u_i solves (38) with data $(f_i, u_{I,i})$ for $i \in \{1,2\}$, then one has : for a.e. $t \in \mathbb{R}_+$,

$$\int_{\mathbb{R}_+} |u_2(a,t) - u_1(a,t)| k(a) da \leq \frac{1}{\mu_0} \int_0^t |f_2'(\tau) - f_1'(\tau)| d\tau + \int_{\mathbb{R}_+} |u_{I,2}(a) - u_{I,1}(a)| k(a) da.$$

This result shows as well that $u \in C(\mathbb{R}_+; L^1(\mathbb{R}_+; B))$ (which means in the sense of uniform convergence on compact subintervals of $\mathbb{R}_+$ in time).

Proof. Using the previous result, one has :

$$\int_{\mathbb{R}_+} |u(a,t+h) - u(a,t)| k(a) da \leq \int_0^t |f'(\tau+h) - f'(\tau)| d\tau + \int_{\mathbb{R}_+} |u(a,h) - u_I(a)| k(a) da.$$

then one uses the continuity of the translation operator in $L^1(\mathbb{R}_+, B)$ and the $L^\infty(0,T)$ bound establihed on g_ε in the previous results to show that

$$\begin{aligned} \int_{\mathbb{R}_+} |u(a,h) - u_I(a)| k(a) da \leq \left(h \|g_\varepsilon\|_{L^\infty(0,T)} + \|u_I\|_{L^\infty(0,h)} \right) \int_0^h k(a) da \\ + \int_{\mathbb{R}_+} |u_I(a+h) - u_I(a)| k(a) da + h \|g_\varepsilon\|_{L^\infty(0,T)} \end{aligned}$$

and concludes. ■

Remark 8. What really matters here is the fact that the bound is uniform with respect to ε.

Corollary 5. Under the previous assumptions, one has :

$$\int_{\mathbb{R}_+} |u_\varepsilon(a,t)|\,\zeta(a)k(a)da < \infty, \quad \forall t \in (0,T)$$

Proof. The proof is a simple consequence of the previous result and the fact that $\zeta \in L^\infty_a(\mathbb{R}_+)$. ■

Theorem 20. Under the previous assumptions, one has :

$$\|u_\varepsilon\|_{X_T} \leq \frac{1}{\mu_0}\|f'\|_{L^\infty} + \left(1 + \|\zeta\|_{L^\infty(\mathbb{R}_+)}\right)\left(\|f'\|_{L^1(\mathbb{R}_+)} + 2\|x_p\|_{L^\infty(\mathbb{R}_-)}\right).$$

Moreover, there are two possibilities :

- either $\int_{\mathbb{R}_+} ak(a)da = \infty$ and then

$$u_\varepsilon \overset{*}{\rightharpoonup} 0 \text{ in } X_T$$

- or $\int_{\mathbb{R}_+} ak(a)da < \infty$ and then

$$u_\varepsilon \overset{*}{\rightharpoonup} u_0 := \frac{f_\infty}{\mu_0 \int_{\mathbb{R}_+} ak(a)da} a \equiv \frac{f_\infty}{\int_{\mathbb{R}_+} a\varrho(a)da} a \text{ in } X_T$$

Proof. We denote

$$g_\varepsilon(t) := \frac{f'(t/\varepsilon)}{\mu_0} + \int_{\mathbb{R}_+} u_\varepsilon(a,t)\zeta(a)k(a)da.$$

thanks to the previous results, $g_\varepsilon \in L^\infty(0,T)$ uniformly wrt ε. Using (34) adapted to the ε-scaling, gives that

$$u_\varepsilon(a,t) = \begin{cases} \int_{-a}^0 g_\varepsilon(t+\varepsilon s)ds & \text{if } t > \varepsilon a \\ u_I(a - t/\varepsilon) + \int_{-\frac{t}{\varepsilon}}^0 g_\varepsilon(t+\varepsilon s)ds & \text{otherwise.} \end{cases}$$

The latter $L^\infty(0,T)$-bound provides then that, when $t > \varepsilon a$

$$|u_\varepsilon(a,t)| \leq a\|g\|_{L^\infty(0,T)} \leq (1+a)\|g\|_{L^\infty(0,T)}$$

whereas when $t \leq \varepsilon a$ one has

$$|u_\varepsilon(a,t)| \leq \|u_I\|_{L^\infty_a(\mathbb{R}_+)} + \frac{t}{\varepsilon}\|g_\varepsilon\|_{L^\infty(0,T)} \leq \|u_I\|_{L^\infty_a(\mathbb{R}_+)} + a\|g_\varepsilon\|_{L^\infty(0,T)}$$

then dividing the inequality by the weight $(1+a)$ shows that :

$$\|u_\varepsilon\|_{X_T} \leq \|g_\varepsilon\|_{L^\infty(0,T)} + \|u_I\|_{L^\infty_a(\mathbb{R}_+)}$$

gathering estimates already established above ends this part of the proof.

Thanks to this latter bound one has that there exists $u_0 \in X_T$ s.t.

$$u_\varepsilon \overset{*}{\rightharpoonup} u_0 \text{ weak} - * \text{ in } X_T.$$

which means that

$$\int_0^T \int_{\mathbb{R}_+} u_\varepsilon(a,t)\varphi(a,t)da \to \int_0^T \int_{\mathbb{R}_+} u_0(a,t)\varphi(a,t)da$$

for all $\varphi \in L^1(\mathbb{R}_+ \times (0,T); (1+a))$. Indeed since $u_\varepsilon/(1+a) \in L^\infty(\mathbb{R}_+ \times (0,T))$ there exists a limit $\ell(a,t) \in L^\infty(\mathbb{R}_+ \times (0,T))$ s.t. for all $\phi \in L^1(\mathbb{R}_+ \times (0,T))$

$$\int_0^T \int_{\mathbb{R}_+} \frac{u_\varepsilon}{(1+a)}\phi(a,t)dadt \to \int_0^T \int_{\mathbb{R}_+} \ell(a,t)\phi(a,t)dadt$$

if $\varphi \in L^1(\mathbb{R}_+ \times (0,T); (1+a))$ then choosing $\phi(a,t) = (1+a)\varphi$ shows that

$$\int_0^T \int_{\mathbb{R}_+} u_\varepsilon \varphi(a,t)dadt \to \int_0^T \int_{\mathbb{R}_+} \ell(a,t)(1+a)\varphi(a,t)dadt$$

and setting $u_0(a,t) = \ell(a,t)(1+a)$ provides a well-suited limit in X_T in this topology. The weak form associated to (38) reads :

$$\begin{aligned} \int_0^T \int_{\mathbb{R}_+} u_\varepsilon(a,t)\left(-\varepsilon\partial_t - \partial_a\right)\varphi(a,t)da + \left[\int_{\mathbb{R}_+} u_\varepsilon(a,t)\varphi(a,t)da\right]_{t=0}^{t=T} \\ = \int_0^T \left\{\frac{f'(t/\varepsilon)}{\mu_0} + \int_{\mathbb{R}_+} u_\varepsilon(\tilde{a},t)\zeta(\tilde{a})k(\tilde{a})d\tilde{a}\right\} \int_{\mathbb{R}_+} \varphi(a,t)dadt \end{aligned}$$

for all φ s.t. $(1+a)\varphi \in W^{1,1}(\mathbb{R}_+ \times (0,T))$. Taking $\varphi \in \mathcal{D}(\mathbb{R}_+ \times (0,T))$ and then passing to the limit thanks to the previous weak-* convergence result gives that

$$-\int_0^T \int_{\mathbb{R}_+} u_0(a,t)\partial_a\varphi(a,t)da \;\; = \int_0^T \int_{\mathbb{R}_+} u_0(\tilde{a},t)\zeta(\tilde{a})k(\tilde{a})d\tilde{a} \int_{\mathbb{R}_+} \varphi(a,t)dadt$$

where we make the following computation regarding the source term part :

$$\left|\int_0^T \frac{f'(t/\varepsilon)}{\mu_0} \int_{\mathbb{R}_+} \varphi(a,t)dadt\right| = \varepsilon \left|\int_0^{\frac{T}{\varepsilon}} \frac{f'(t)}{\mu_0} \int_{\mathbb{R}_+} \varphi(a,\varepsilon t)dadt\right| \leq C\varepsilon \|f'\|_{L^1} \|\varphi\|_{L^\infty(\mathbb{R}_+ \times (0,T))}$$

showing that it vanishes. For any $\varphi \in \mathcal{D}(\mathbb{R}_+ \times (0,T))$ one has thus the weak formulation :

$$\int_0^T \int_{\mathbb{R}_+} u_0(a,t)(-\partial_a)\varphi(a,t)dadt = \int_0^T \int_{\mathbb{R}_+} g_0(a)\varphi(a,t)dadt.$$

where $g_0(t) := \int_{\mathbb{R}_+} u_0(a,t)\zeta(a)k(a)da$. The latter weak expression translates into :

$$\partial_a u_0(a,t) = g_0(t), \qquad \text{in } \mathcal{D}'(\mathbb{R}_+^* \times (0,T))$$

but since

$$\int_0^T \int_{\mathbb{R}_+} |u_0(a,t)|\, k(a)da\phi(t)dt \leq \liminf_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}_+} |u_\varepsilon(a,t)|\, k(a)da\phi(t)dt < \infty, \quad \forall \phi \in L^1_+(0,T)$$

and because $\zeta \in L^\infty(\mathbb{R}_+)$, g_0 is bounded. Thus $u_0(\cdot,t) \in \mathrm{Lip}(\mathbb{R}_+)$ for a.e. $t \in (0,T)$, returning to the weak formulation satisfied by u_0, one can integrate by parts, which shows that

$$u_0(0,t) = 0.$$

Finally all this allows to write

$$u_0(a,t) = g_0(t)a.$$

We suppose at this point that $\int_{\mathbb{R}_+} ak(a)da = \infty$. Now we write :

$$\int_0^T \left\{ \left(\int_0^R + \int_R^\infty \right) u_\varepsilon(a,t)k(a)da \right\} \varphi(t)dt = \int_0^T \frac{f(t/\varepsilon)}{\mu_0}\varphi(t)dt$$

we denote : $b_{\varepsilon,R}(t) := \int_R^\infty u_\varepsilon(a,t)k(a)da$ it is a bounded function and the bound is uniform wrt ε and R (cf Theorem 19) thus there exists a bounded weak-* limit $b_{0,R}$. Thanks to hypotheses made on f, $f(t/\varepsilon)$ tends to f_∞ as ε goes to zero (if not clear exercise). Thus one writes :

$$\int_0^T \int_0^R g_0(t)ak(a)da\varphi(t)dt = \int_0^T (f_\infty + b_{0,R}(t))\varphi(t)dt$$

this shows that

$$\left| \int_0^T \int_0^R g_0(t)ak(a)da\varphi(t)dt \right| = \int_0^R ak(a)da \left| \int_0^T g_0(t)\varphi(t)dt \right| \leq C$$

where due to the uniform boundedness of $b_{0,R}$ C does not depend on R. then this shows that as $\int_0^R ak(a)da \to \infty$ as R grows large

$$\int_0^T g_0(t)\varphi(t)dt = 0, \quad \forall \varphi \in \mathcal{D}((0,T))$$

and thus $u_0 = 0$ for almost every $(a,t) \in \mathbb{R}_+ \times (0,T)$.

At the contrary if $\int_{\mathbb{R}_+} ak(a)da < \infty$ then one has simply that

$$\int_0^T \int_{\mathbb{R}_+} g_0 ak(a)da\varphi(t)dt = \int_0^T \frac{f_\infty}{\mu_0}\varphi(t)dt$$

which by similar arguments as above shows that

$$g_0(t) = \frac{f_\infty}{\mu_0 \int_{\mathbb{R}_+} ak(a)da}$$

and the conclusion follows. ■

Corollary 6. Under the previous hypotheses, x_ε converges to x_0 in $C([0,T])$ *i.e.*

$$\sup_{t\in(0,T)} |x_\varepsilon(t) - x_0(t)| \leq o_\varepsilon(1)$$

where x_0 solves :

$$\mu_1 \partial_t x_0 = f, \quad x_0(0) = x_p(0)$$

Proof. As we have seen above,

$$x_\varepsilon(t) = x_\varepsilon(0) + \int_0^t \frac{f(\tau/\varepsilon)}{\mu_0} d\tau + \int_0^t \int_{\mathbb{R}_+} \zeta(a)k(a)u_\varepsilon(a,\tau) da d\tau$$

which leads to write :

$$[x_\varepsilon(t) - x_0(t)] \leq |x_\varepsilon(0) - x_p(0)| + \varepsilon \|f\|_{L^1(\mathbb{R}_+)}/\mu_0 + \left| \int_0^t \int_{\mathbb{R}_+} \zeta(a)\varrho(a) \left(u_\varepsilon(a,\tau) - u_0(a,\tau)\right) da d\tau \right|$$

Noticing that

$$x_\varepsilon(0) - x_p(0) = \varepsilon \frac{f(0)}{\mu_0} + \int_{\mathbb{R}_+} \left(x_p(-\varepsilon a) - x_p(0)\right) k(a) da$$

Moreover,

$$\int_{\mathbb{R}_+} \zeta(a)k(a)u_0(a,t) da = \int_{\mathbb{R}_+} \zeta(a)k(a)a \frac{f(\tau)}{\mu_1} da = \frac{f(t)}{\mu_1}, \quad \text{a.e. } t \in (0,T)$$

since $\int_{\mathbb{R}_+} \zeta(a)ak(a)da = 1$ (using (33)). ■

Theorem 21. Under the previous hypotheses and if moreover $\int_{\mathbb{R}_+} ak(a)da < \infty$, then

$$\lim_{t\to\infty} \left| \frac{z(t)}{t} - \frac{f_\infty}{\mu_0 \int_{\mathbb{R}_+} ak(a)da} \right| = 0$$

Proof. Recalling the change of unknowns $x_\varepsilon(t) = \varepsilon z(t/\varepsilon)$ where z solves the renewal equation $z - z \star k = f/\mu + \int_t^\infty k(a)x_p(t-a)da$ and thanks to the previous result one simply writes :

$$\lim_{\varepsilon\to 0} |\varepsilon z(t/\varepsilon) - x_0(t)| = 0, \quad \forall t \in (0,1]$$

Choosing $t = 1$ in the previous expression this translates into

$$\forall \delta > 0, \ \exists t_\delta > 0 \ : \ \forall t > t_\delta \implies |z(t)/t - x_0(1)| < \delta$$

from which the claim follows. ■

One notices than that setting $u_0(a) := \gamma a$, where $\gamma := f_\infty/(\mu_0 \int_{\mathbb{R}_+} ak(a)da)$,

$$p(a,t) = \int_0^t (u(a,\tau) - u_0(a)) d\tau$$

it solves

$$\begin{cases} (\partial_t + \partial_a)p(a,t) = \dfrac{f(t) - f(0)}{\mu_0} + \displaystyle\int_{\mathbb{R}_+} p(\tilde{a},t)\zeta(\tilde{a})k(\tilde{a})d\tilde{a} + u_I(a) - u_0(a), & a > 0, \ t > 0, \\ p(0,t) = 0 & a = 0, \ t > 0, \\ p(a,0) = 0 & a > 0, \ t = 0, \end{cases}$$

together with the compatibility condition :

$$\int_{\mathbb{R}_+} k(a)p(a,t)da = \frac{1}{\mu_0} \int_0^t \left(f(\tau) - f_\infty\right) d\tau.$$

Setting p_0 the steady state of the previous equation, it solves :

$$\begin{cases} \partial_a p_0(a) = \dfrac{f_\infty - f(0)}{\mu_0} + \displaystyle\int_{\mathbb{R}_+} p_0(\tilde{a})\zeta(\tilde{a})k(\tilde{a})d\tilde{a} + u_I(a) - u_0(a) & a > 0, \\ p_0(0) = 0 & a = 0, \end{cases}$$

and should fulfill as well :

$$\int_{\mathbb{R}_+} k(a)p_0(a)da = \frac{1}{\mu_0}\int_0^\infty f(\tau) - f_\infty d\tau.$$

setting $\hat{p}(a,t) := p(a,t) - p_0(a)$ it solves :

$$\begin{cases} (\partial_t + \partial_a)\hat{p}(a,t) = \dfrac{f(t) - f_\infty}{\mu_0} + \displaystyle\int_{\mathbb{R}_+} \hat{p}(\tilde{a},t)\zeta(\tilde{a})k(\tilde{a})d\tilde{a} =: g(t) & a > 0,\ t > 0 \\ \hat{p}(0,t) = 0 & a = 0,\ t > 0 \\ \hat{p}(a,0) = -p_0(a) & a > 0,\ t = 0 \end{cases} \tag{39}$$

together with the compatibility condition :

$$\int_{\mathbb{R}_+} k(a)\hat{p}da = -\frac{1}{\mu_0}\int_t^\infty (f(\tau) - f_\infty)d\tau$$

Theorem 22. Assume that $\int_{\mathbb{R}_+}(1+a)^2k(a)da < \infty$ and that

$$f' \in L^\infty(\mathbb{R}_+)$$

and moreover that

$$\|f - f_\infty\|_{L^2(\mathbb{R}_+)} < \infty, \quad \int_{\mathbb{R}_+} t|f(t) - f_\infty|dt < \infty.$$

Then

$$x(t) - \gamma t - x(0) - \omega \to 0$$

as $t \to \infty$ where the constant ω is defined as

$$\omega := \frac{\frac{1}{\mu_0}\int_0^\infty (f(\tau) - f_\infty)\, d\tau - \int_{\mathbb{R}_+} k(a) \int_0^a (u_I(\tilde{a}) - u_0(\tilde{a}))d\tilde{a}da}{\int_{\mathbb{R}_+} k(a)ada}.$$

Proof. Applying the same arguments as in Theorem 19 and Corollary 5 to (39), one obtains first that

$$\int_{\mathbb{R}_+} \zeta(a)k(a)\, |\hat{p}(a,t)|\, da < \infty$$

for all times. This proves as well that

$$\|\hat{p}\|_{X_\infty} < C,$$

where we recall that $X_\infty := L^\infty(\mathbb{R}_+\times\mathbb{R}_+;(1+a)^{-1})$. Multiplying (39) by $k(a)\hat{p}$ and integrating by parts one obtains :

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}_+} k(a)\hat{p}^2 da + \frac{1}{2}\int_{\mathbb{R}_+}\zeta(a)k(a)\hat{p}^2 da = \\ &\quad\int_{\mathbb{R}_+} k(a)\hat{p}(a,t)da\left\{\frac{(f(t)-f_\infty)}{\mu_0} + \int_{\mathbb{R}_+}\hat{p}(\tilde{a},t)\zeta(\tilde{a})k(\tilde{a})d\tilde{a}\right\} \\ &\quad = -\left(\int_t^\infty (f(\tau)-f_\infty)d\tau\right)\left\{\frac{(f(t)-f_\infty)}{\mu_0} + \int_{\mathbb{R}_+}\hat{p}(\tilde{a},t)\zeta(\tilde{a})k(\tilde{a})d\tilde{a}\right\} < C\int_t^\infty |f(\tau)-f_\infty|\,d\tau\end{aligned}$$

then this shows that

$$\int_{\mathbb{R}_+} k(a)\hat{p}(a,t)^2 da < \infty$$

for all times and that

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\zeta(a)k(a)\hat{p}(a,t)^2 da dt < \infty$$

this shows that $g(t)$ the rhs in (39) is in $L^2(0,T)$. Now one uses Duhamel's principle and writes :

$$|\hat{p}(a,t)| = \left|\int_{t-a}^t g(s)ds\right| \leq \sqrt{a}\|g\|_{L^2((t-a,t))}, \quad t > a$$

where we used Cauchy Schwartz. So for every fixed a, $|\hat{p}(a,t)|$ tends to zero as t grows large. Applying Lebesgue's Theorem this shows that

$$\lim_{t\to\infty}\int_{\mathbb{R}_+}\zeta(a)k(a)\,|\hat{p}(a,t)|\,\mathbb{1}_{(0,t)}(a)da = 0,$$

whereas

$$\int_t^\infty \zeta(a)k(a)\,|\hat{p}(a,t)|\,da \leq \zeta_{\max}\int_t^\infty k(a)(1+a)da\|\hat{p}\|_{X_\infty} \to 0$$

again by Lebesgue's Theorem when t goes to infinity.

The limit function p_0 is explicit. Indeed setting $q := p_0(a) - \int_0^a (u_I(\tilde{a}) - u_0(\tilde{a}))d\tilde{a}$ one has

$$\partial_a q = \frac{f_\infty - f(0)}{\mu_0} + \int_{\mathbb{R}_+}\zeta(a)k(a)q(a)da + \int_{\mathbb{R}_+}\zeta(a)k(a)\int_0^a (u_I(\tilde{a}) - u_0(\tilde{a}))d\tilde{a}da$$

as in the rhs there are only constants (wrt to a) this implies that there exists ω s.t. $q = \omega a$ and thus

$$p_0(a) = \int_0^a (u_I(\tilde{a}) - u_0(\tilde{a}))d\tilde{a} + \omega a$$

and the compatibility condition gives that

$$\int_{\mathbb{R}_+} k(a)p_0(a)da = \frac{1}{\mu_0}\int_0^\infty (f(\tau) - f_\infty)\,d\tau$$

which uniquely defines ω. Setting

$$\hat{x}(t) = x(t) - \gamma t, \quad \forall t \in \mathbb{R}$$

it solves :

$$\int_{\mathbb{R}_+}(\hat{x}(t) - \hat{x}(t-a))k(a)da = \frac{f(t) - f_\infty}{\mu_0}$$

and the associated elongation is $u(a,t)-u_0(a)$ so that one can write :

$$\begin{aligned}\hat{x}(t)-\hat{x}(0) &= \frac{f(t)-f(0)}{\mu_0} + \int_0^t \int_{\mathbb{R}_+} \zeta(a)k(a)\left(u(a,\tau)-u_0(a)\right) da d\tau \\ &= \frac{f(t)-f(0)}{\mu_0} + \int_{\mathbb{R}_+} \zeta(a)k(a)p(a,t)da \\ &= \frac{f(t)-f(0)}{\mu_0} + \int_{\mathbb{R}_+} \zeta(a)k(a)p_0(a)da + \int_{\mathbb{R}_+} \zeta(a)k(a)\left(p(a,t)-p_0(a)\right) da\end{aligned}$$

Leading to

$$x(t)-\gamma t - x(0) - \frac{f_\infty - f(0)}{\mu_0} - \int_{\mathbb{R}_+} \zeta(a)k(a)p_0(a)da = \frac{f(t)-f_\infty}{\mu_0} + \int_{\mathbb{R}_+} \zeta(a)k(a)\hat{p}(a,t)da$$

as the rhs tends to zero thanks to the assumptions and the previous results. Now we aim at understanding the structure of the previous Ansatz :

$$\int_{\mathbb{R}_+} \zeta k p_0 da = \int_{\mathbb{R}_+} k(a)\partial_a p_0 da = \int_{\mathbb{R}_+} k(a)\left\{(u_I(a)-u_0(a))+\omega\right\} da = \frac{f(0)-f_\infty}{\mu_0} + \omega$$

this explains why the two terms above cancel and ends the proof. ■

Remark 9. One should compare this result with the precise characterization (11) of Theorem 12. Denoting $\tilde{f}(t) := f(t) + \mu_0 \int_t^\infty x_p(t-a)k(a)da$ one has then thanks to (11) that

$$x(t) = \tilde{f}/\mu_0 + 1 \star \tilde{f}/\mu_1 + \gamma \star \tilde{f}/\mu_0$$

then when t grows large it is now possible to define explicitly the limit of the last term in the rhs.

3 From delayed Poisson problem to the friction heat equation

Here we consider the problem :

$$\begin{cases} \mathcal{L}_\varepsilon[x_\varepsilon](t) - \Delta_s x_\varepsilon = f(s,t), & s \in \Omega, \quad t > 0, \\ \partial_\nu x_\varepsilon(s,t) = 0 & s \in \partial\Omega, \quad t > 0, \\ x_\varepsilon(s,t) = x_p(s,t) & s \in \Omega, \quad t \leq 0. \end{cases} \tag{40}$$

where we denote

$$\mathcal{L}_\varepsilon[x](t) := \frac{1}{\varepsilon}\int_{\mathbb{R}_+} (x(s,t)-x(s,t-\varepsilon a))\varrho(a,s)da, \quad \Delta_s x := \sum_{j=1}^d \partial^2_{s_j^2} x$$

and we are interested in the limit when ε goes to zero. Indeed one aims to show that under reasonable hypotheses, when ε goes to zero x_ε converges towards x_0 solving

$$\begin{cases} \mu_1(s)\partial_t x_0 - \Delta_s x_0 = f(s,t), & s \in \Omega, \quad t > 0, \\ x_0(s,0) = x_p(s,0) & s \in \Omega, \quad t \leq 0, \end{cases} \tag{41}$$

where $\mu_1(s) := \int_{\mathbb{R}_+} \varrho(s,a)da$. We redefine the space where x_ε lives

$$X_T := L^\infty((0,T); H_0^1(\Omega)),$$

for every positive real T possibly infinite.

We assume that ϱ as in (33) solves an ODE in age that is nevertheless now depending on s in the following manner :

$$\begin{cases} \partial_a \varrho(s,a) + \zeta(s,a)\varrho(s,a) = 0, & \text{a.e. } a > 0, \quad s \in \Omega \\ \varrho(s,0) = \beta(s), & a = 0, \quad s \in \Omega. \end{cases} \tag{42}$$

Assumptions 4. The data of (42) satisfy

- there exists $\omega \subset \Omega$ s.t. $|\omega| \neq 0$ and $\exists\, a_0 : \Omega \to \mathbb{R}_+$ s.t.

$$\exists\, \underline{a_0} > 0\,;\ \text{a.e. } s \in \omega, \quad a_0(s) \geq \underline{a_0}$$

and

$$\text{a.e. } (s,a) \in \omega \times (0, a_0(s)),\ \zeta(a,t) \leq \zeta_{\max} \text{ and } \beta(s) \geq \beta_{\min} > 0$$

- there exist a decreasing function $m \in L^1(\mathbb{R}_+, (1+a)^2)$ and $a_1 \in \mathbb{R}_+$ s.t. for a.e. $s \in \Omega$

$$\zeta(s,a) \geq -\frac{m'(a)}{m(a)}, \quad \text{a.e. } a > a_1$$

Proposition 7. Under the previous assumptions, for a.e. $s \in \Omega$, there exists $c_1 > 0$ s.t.

$$\operatorname*{essinf}_{s\in\omega} \int_{\mathbb{R}_+} \varrho(s,a)da > c_1, \quad \operatorname*{essinf}_{s\in\omega} \int_{\mathbb{R}_+} a\varrho(s,a)da > c_2$$

Proof. One writes :

$$\begin{aligned} \int_{\mathbb{R}_+} \varrho(s,a)da &\geq \int_0^{a_0(s)} \varrho(s,a)da \geq \int_0^{a_0(s)} \beta(s)\exp\left(-a\zeta_{\max}\right)da \\ &= \frac{\beta(s)}{\zeta_{\max}}\left(1 - \exp\left(-\zeta_{\max} a_0(s)\right)\right) \geq \frac{\beta_{\min}}{\zeta_{\max}}\left(1 - \exp\left(-\zeta_{\max} a_0(s)\right)\right) \\ &\geq \frac{\beta_{\min}}{\zeta_{\max}}\left(1 - \exp\left(-\zeta_{\max}\underline{a_0}\right)\right) > 0 \end{aligned}$$

and the same proof holds for the lower bound of the first moment on ω. ■

We now discretize (42) using the implicit scheme already defined in (**??**) for almost every $s \in \Omega$.

$$\begin{cases} R_{j+1}(s) := \left(1 + \Delta a \zeta_{j+1}(s)\right) R_j(s), & j \geq 0, \quad \text{a.e. } s \in \Omega, \\ R_0 := \beta(s), & j = 0 \end{cases} \tag{43}$$

We extend Proposition **??**, to this new frame.

Lemma 5. Assume that $m(a) := 1/(1+a)^\sigma$, then

$$\Delta a \sum_{j\in\mathbb{N}} (1+j\Delta a)^\alpha \operatorname*{ess\,sup}_{s\in\omega} R_j(s) < \infty$$

for $\alpha < \sigma - 1$.

Proof. If m is defined as in the claim, then

$$\zeta(s,a) \geq \frac{\sigma}{(1+a)}, \quad \forall a > a_1, \quad \text{a.e. } s \in \Omega$$

and one has as in the proof of Proposition **??**, $\forall j > j_1 = \lfloor a_1/\Delta a \rfloor$,

$$p_j \leq \frac{C}{(1+j\Delta a)^\sigma}$$

thus $\forall j > j_1 = \lfloor a_1/\Delta a \rfloor$,

$$\operatorname*{ess\,sup}_{s\in\omega} R_j(s) \leq \|\beta\|_{L^\infty(\Omega)} \frac{C}{(1+j\Delta a)^\sigma}$$

and then :

$$\sum_{j\in\mathbb{N}} (1+j\Delta a)^\alpha \operatorname*{ess\,sup}_{s\in\omega} R_j(s) \leq C\|\beta\|_{L^\infty(\Omega)} \left\{ 1 + \sum_{j\geq j_1} \frac{1}{(1+j\Delta a)^{\sigma-\alpha}} \right\}$$

and the series in the rhs converges provided that $\sigma - \alpha > 1$ which ends the proof. ∎

Assumptions 5. The data related to (40) satisfies :

i) at time $t = 0$ we assume that $x_p(s,0)$ has the following properties :

- $x_p(\cdot,0)$ is in $H^1_0(\Omega)$,

ii) when $t \leq 0$ one assumes furthermore that :

$$x_p \in C^0(\mathbb{R}_-; L^2(\Omega)), \quad \partial_t x_p \in L^\infty(\mathbb{R}_-, L^2(\Omega)).$$

the latter hypotheses translates into a Lipschitz constant which is L^2 in space *i.e.* :

$$|x_p(s,t_2) - x_p(s,t_1)| \leq C_{x_p}(s)|t_2 - t_1|, \quad \forall (t_2,t_1) \in (\mathbb{R}_-)^2 \tag{44}$$

where $C_{x_p}(s) \in L^2(\Omega)$.

iii) the source term f is s.t.

$$f \in H^1((0,T); L^2(\Omega))$$

3.1 A few words on *Minimizing movements*

Assume $H := H^1(0,1)$ is a Hilbert space, and Φ a convex functional $H \to \mathbb{R}$ $\Phi(W) := \int_0^1 |W'(s)|\, ds$. One define a semi-discrete solution Z^n as

$$Z^n := \operatorname{argmin} \mathcal{E}_n(W), \quad \mathcal{E}_n(W) := \frac{\|W - Z^{n-1}\|^2_{L^2(0,1)}}{2\Delta t} + \Phi(W)$$

Since the energy is convex with respect to W one derives the Euler-Lagrange equation associated to this minimization problem :

$$\left(\frac{Z^n - Z^{n-1}}{2\Delta t}, V\right) + (Z^{n\prime}, V') = 0, \quad \forall V \in H$$

where the brackets denote the scalar product in $L^2(0,1)$. Then by induction and using Lax-Milgram, it is easy to show existence and uniqueness of $Z^n \in H$ provided that $Z^{n-1} \in H$. Moreover one has as well that

Proposition 8. If $Z_0 \in H$ then one has :

$$\sum_{i=0}^{N-1} \|\frac{Z^{n+1} - Z^n}{\Delta t}\|_L^2(0,1)^2 \Delta t \leq \Phi(Z^0) < \infty, \quad \Phi(Z^n) \leq \Phi(Z^0), \quad \forall n \in \{1, \ldots, N\}$$

and

$$\int_0^1 Z^n(s) ds = \int_0^1 Z^0(s) ds, \quad \forall n \in \{1, \ldots, N\}$$

Proof. One writes :

$$\mathcal{E}_n(Z^n) \leq \mathcal{E}_n(Z^{n-1})$$

which translates into :

$$\frac{\|Z^n - Z^{n-1}\|_2^2}{2\Delta t} + \Phi(Z^n) \leq \Phi(Z^{n-1})$$

which then summed up for $n \in \{1, \ldots, N\}$ provides the claim. ∎

Let's define the piecewise constant and linear interpolations

$$\begin{cases} z_\Delta := \sum_{n=0}^{N-1} Z^n(s) \mathbb{1}_{[n\Delta t, (n+1)\Delta t)}(t), \quad t \in (0,T). \\ \tilde{z}_\Delta(s,t) := \sum_{n=0}^{N-1} \left\{ Z^n(s) + \left(\frac{t}{\Delta t} - n\right)(Z^{n+1} - Z^n)\right\} \mathbb{1}_{[n\Delta t, (n+1)\Delta t)}(t), \quad t \in (0,T). \end{cases}$$

then the first claim in the previous Proposition translates into

Proposition 9. The piecewise linear interpolation of $(Z^n)_{n \in \mathbb{Z}}$ satisfies

$$\tilde{z}_\Delta \in C^{0, \frac{(1-\gamma)}{4}}([0,T]; C^{0,\gamma}(\overline{\Omega}))$$

for every $\gamma \in (0,1)$, the bound is uniform with respect to Δt and ε. Thus $\tilde{z}_\Delta$ converges strongly in $C^0(\overline{\Omega} \times [0,T])$ when Δt goes to zero. Moreover, z_Δ converges strongly in $L^\infty((0,T); C(\overline{\Omega}))$.

The proof of a similar result will be provided later on.

These estimates provide a two-fold convergence results :

$$\sum_n \frac{Z^n - Z^{n-1}}{\Delta t} \mathbb{1}_{[n\Delta t, (n+1)\Delta t]} \rightharpoonup \partial_t z \text{ in } L^2_{s,t} \text{ weak}$$

and

$$\sum_n \nabla Z^n \mathbb{1}_{[n\Delta t,(n+1)\Delta t]} \rightharpoonup \nabla z \text{ in } L^2_{s,t} \text{ weak}$$

when $\Delta t \to 0$. Showing that z_Δ converges in fact towards the solution of the heat equation with natural boundary conditions. In what follows we should try to make the previous computations more rigorous as well as adapt them to the case of the delayed problem presented above.

3.2 Delayed gradient flows

Here we use the ideas of *Minimizing movements* from De Giorgi [2]. To this aim we make a semi-discretization of our problem in time and age (but not in space).

We discretise the age domain $\mathbb{R}_+$ with a regular grid s.t. :

$$\mathbb{R}_+ = \cup_{j\in\mathbb{N}} [j\Delta a, (j+1)\Delta a)$$

and the time interval setting $\Delta t = \varepsilon \Delta a$ s.t. $N = \lfloor T/\Delta t \rfloor$ and thus

$$[0,T) = \cup_{n=0}^N [n\Delta t, (n+1)\Delta t) + [N\Delta t, T)$$

but we will consider values of ε s.t. the last interval is always the null-set.

So we define the energy :

$$\mathcal{E}_n(w) := \frac{\Delta a}{2\varepsilon} \sum_{j\in\mathbb{N}} \int_\Omega \left| w(s) - x_\varepsilon^{n-j}(s) \right|^2 R_{j+1}(s) ds + \frac{1}{2} \int_\Omega |\nabla_s w(s)|^2 ds - \int_\Omega f^{n+1}(s) w(s) ds. \tag{45}$$

And we define

$$x_\varepsilon^{n+1} := \underset{w\in H^1(\Omega)}{\operatorname{argmin}} \, \mathcal{E}_n(w).$$

We complement the definition of the scheme by setting the past values for x_ε^n when $n<0$:

$$x_\varepsilon^n(s) := \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} x_p(s,\tau) d\tau =: x_p^n(s), \quad \forall n \in \mathbb{Z}_-, \quad \text{a.e. } s \in \Omega.$$

Again as above we define the piecewise constant functions thanks to the subscript Δ :

$$x_{\varepsilon,\Delta}(s,t) := \sum_{n\in\mathbb{Z}} x_\varepsilon^n(s) \mathbb{1}_{[n\Delta t,(n+1)\Delta t)}(t), \quad \forall t \in \mathbb{R}, \quad \text{a.e. } s \in \Omega.$$

and afine extensions :

$$\tilde{x}_{\varepsilon,\Delta} := \sum_{n\in\mathbb{Z}} \left\{ x_\varepsilon^n + \left(\frac{t}{\Delta t} - n \right) \delta x_\varepsilon^{n+\frac{1}{2}} \right\} \mathbb{1}_{[n\Delta t,(n+1)\Delta t)}(t),$$

where $\delta x_\varepsilon^{n+\frac{1}{2}} := x_\varepsilon^{n+1} - x_\varepsilon^n$, while in what follows we denote as well $\delta R_{i+\frac{1}{2}} := R_{i+1} - R_i$ an so on.

3.3 Existence, uniqueness and stability of the discrete solution $x_{\varepsilon,\Delta}$

Existence of minimizers relies on the convexity of the Dirichlet norm and is standard as the few properties listed below (see for instance Lemma 1 and 2, p. 973 [21]).

Theorem 23. Under hypotheses 4, 5, for every $n \geq 0$ there exists a minimizer $x_\varepsilon^n \in H^1(\Omega)$ of (45), *i.e.* there exists a minimizing subsequence $(x_\varepsilon^{n,k})_{k\in\mathbb{N}}$ s.t. as $k \to \infty$,
1) $x_\varepsilon^{n,k} \rightharpoonup x_\varepsilon^n$ weak in $H^1(\Omega)$,
2) $x_\varepsilon^{n,k} \to x_\varepsilon^n$ strong in $L^2(\Omega)$,
3) $x_\varepsilon^{n,k} \to x_\varepsilon^n$ a.e. $s \in \Omega$.
the unique minimizer solves the Euler-Lagrange equation :

$$(\mathcal{L}_\varepsilon^{n+1}, v) + \left(\nabla_s x_\varepsilon^{n+1}, \nabla_s v\right) = (f^{n+1}, v), \quad \forall v \in H^1(\Omega), \tag{46}$$

where

$$\mathcal{L}_\varepsilon^{n+1}(s) := \frac{\Delta a}{\varepsilon} \sum_{j\in\mathbb{N}} \left(x_\varepsilon^{n+1}(s) - x_\varepsilon^{n-j}(s)\right) R_{j+1}(s), \quad \text{a.e. } s \in \Omega.$$

Proof. The functional $\mathcal{E}_n$ is convex wrt w, moreover it is also coercive since for every $w \in H^1(\Omega)$ one has :

$$\mathcal{E}_n(w) \geq \frac{1}{2} \int_\Omega |\nabla_s w(s)|^2 \, ds,$$

moreover, the adminissible set is $H^1(\Omega)$ which is non-empty. Thus by [5, Theorems 1 & 2, Chapter 8, Section 2] there exists a minimizer x_ε^{n+1} s.t. if $(x_\varepsilon^{n+1,k})_{k\in\mathbb{N}}$ is a minimizing sequence *i.e.*

$$\lim_{k\to\infty} \mathcal{E}_n(x_\varepsilon^{n+1,k}) = \ell = \inf_{w\in H^1(\Omega)} \mathcal{E}_n(w)$$

then there exists a sub-sequence $(x_\varepsilon^{n+1,k_j})_{j\in\mathbb{N}}$ and a function $x_\varepsilon^{n+1} \in H^1(\Omega)$ s.t. claims 1) and 2) follow and

$$\mathcal{E}_n(x_\varepsilon^{n+1}) = \ell.$$

By [5, Theorem 4, Chapter 8, Section 2], minimizers solve the Euler-Lagrange equation :

$$(\mathcal{L}_\varepsilon^{n+1}, v) + \left(\nabla_s x_\varepsilon^{n+1}, \nabla_s v\right) = (f^n, v), \quad \forall v \in H^1(\Omega).$$

where

$$\mathcal{L}_\varepsilon^{n+1}(s) := \frac{\Delta a}{\varepsilon} \sum_{j\in\mathbb{N}} \left(x_\varepsilon^{n+1}(s) - x_\varepsilon^{n-j}(s)\right) R_{j+1}(s), \quad \text{a.e. } s \in \Omega$$

By Lemma 10, and the Lax-Milgram theorem, x_ε^{n+1} is unique. ■

A way to insure convergence, when ε or Δa go to zero, is to obtain some control on a discrete time derivative of $x_{\varepsilon,\Delta}$, typically an $\mathbf{L}^2_{s,t}$-bound is obtained in the case of a classical gradient flow directly from the minimization principle (cf Appendix in [23] and references therein). Here the result is less immediate : first, in the next lemma, we obtain a dissipation term in the energy estimates. These estimates provide a bound on the dissipation term. It then appears as a source term in a closed equation on $\delta x_\varepsilon^{n+\frac{1}{2}}$ that finally provides these key estimates (cf. Proposition 10).

Lemma 6. If $(R_i)_{i\in\mathbb{N}}$ and $(x_\varepsilon^n)_{n\in\mathbb{N}}$, are defined as above, one has :

$$\begin{aligned} &\frac{\Delta a}{2\varepsilon}\sum_{j\in\mathbb{N}}\int_\Omega \left(x_\varepsilon^{n+1}-x_\varepsilon^{n-j}\right)^2 R_{j+1}ds + \int_\Omega \left|x_\varepsilon^{n+1\prime}\right| ds + \sum_{m=1}^{n}\Delta t\mathcal{D}_m \\ &\leq \mathcal{E}_{-1}(x_\varepsilon^{-1}) + \int_\Omega f^0 x_\varepsilon^0 ds + \sum_{m=0}^{n}\int_\Omega f^m\left(\delta x_\varepsilon^{m+\frac{1}{2}}\right)ds, \quad \forall n\in\mathbb{N} \end{aligned} \tag{47}$$

where the dissipation term reads :

$$\mathcal{D}_n := \frac{\Delta a}{2}\sum_{j\in\mathbb{N}}\int_\Omega \left|u_{\varepsilon,j}^n\right|^2 \zeta_{j+1}R_{j+1}ds, \quad u_{\varepsilon,j}^n := \frac{1}{\varepsilon}\left(x_\varepsilon^n - x_\varepsilon^{n-j}\right),$$

and we denote by $u_{\varepsilon,j}^n$ the discrete elongation variable for $(j,n)\in\mathbb{N}^2$. The generic constant C in (47) is independent either of ε or Δa.

Proof. By definition of the minimization process, one has

$$\mathcal{E}_n(x_\varepsilon^{n+1}) \leq \mathcal{E}_n(x_\varepsilon^n),$$

since x_ε^{n+1} minimises the energy at time step $t=(n+1)\Delta t$. This reads

$$\begin{aligned} \mathcal{E}_n(x_\varepsilon^{n+1}) &\leq \frac{\Delta a}{2\varepsilon}\int_\Omega\sum_{j=0}^{\infty}|x_\varepsilon^n - x_\varepsilon^{n-j}|^2 R_{j+1}ds + \frac{1}{2}\int_\Omega|\nabla_s x_\varepsilon^n|^2 ds - \int_\Omega f^{n+1}(s)x_\varepsilon^n(s)ds \\ &\leq \frac{\Delta a}{2\varepsilon}\int_\Omega\sum_{j=1}^{\infty}|x_\varepsilon^n - x_\varepsilon^{n-j}|^2 R_{j+1}ds + \frac{1}{2}\int_\Omega|\nabla_s x_\varepsilon^n|^2 ds - \int_\Omega f^{n+1}(s)x_\varepsilon^n(s)ds \\ &\leq \frac{\Delta a}{2\varepsilon}\int_\Omega\sum_{j=1}^{\infty}|x_\varepsilon^n - x_\varepsilon^{n-j}|^2\left(R_j - \Delta a\zeta_{j+1}R_{j+1}\right)ds + \frac{1}{2}\int_\Omega|\nabla_s x_\varepsilon^n|^2 ds - \int_\Omega f^{n+1}(s)x_\varepsilon^n(s)ds \\ &\leq -\frac{\Delta a^2}{2\varepsilon}\int_\Omega\sum_{j=1}^{\infty}|x_\varepsilon^n - x_\varepsilon^{n-j}|^2\zeta_{j+1}R_{j+1} + \frac{\Delta a}{2\varepsilon}\int_\Omega\sum_{j=1}^{\infty}|x_\varepsilon^n - x_\varepsilon^{n-j}|^2 R_j \\ &\quad + \frac{1}{2}\int_\Omega|\nabla_s x_\varepsilon^n|^2 ds - \int_\Omega f^{n+1}(s)x_\varepsilon^n(s)ds \\ &\leq -\Delta t\frac{\Delta a}{2}\int_0^1\sum_{j=1}^{\infty}\left|\frac{x_\varepsilon^n - x_\varepsilon^{n-j}}{\varepsilon}\right|^2\zeta_{j+1}R_{j+1}ds + \frac{\Delta a}{2\varepsilon}\int_\Omega\sum_{j=0}^{\infty}|x_\varepsilon^n - x_\varepsilon^{n-1-j}|^2 R_{j+1} \\ &\quad + \frac{1}{2}\int_\Omega|\nabla_s x_\varepsilon^n|^2 ds - \int_\Omega f^{n+1}(s)x_\varepsilon^n(s)ds \\ &\leq -\Delta t D_n + \mathcal{E}_{n-1}(x_\varepsilon^n) + \int_\Omega\left(f^n(s) - f^{n+1}(s)\right)x_\varepsilon^n(s)ds \end{aligned}$$

for all $n\in\mathbb{N}$. For x_ε^0, one has simply that

$$\mathcal{E}_{-1}(x_\varepsilon^0) \leq \mathcal{E}_{-1}(x_\varepsilon^{-1}) \leq \left\|x_\varepsilon^{-1}\right\|^2_{\mathbf{H}^1(\Omega)} + \frac{\Delta a}{2\varepsilon}\int_\Omega\sum_{j=0}^{\infty}R_{j+1}\left|x_\varepsilon^{-1} - x_\varepsilon^{-1-j}\right|^2 ds - \int_\Omega f^0(s)x_\varepsilon^{-1}ds.$$

Using (44), one has that for almost every $s\in\Omega$ and $j>1$

$$|x_\varepsilon^{-1} - x_\varepsilon^{-1-j}| \leq \frac{C_{x_p}(s)}{\Delta t}\int_0^{\Delta t}|x_p(s,t) - x_p(s,t+(1-j)\Delta t)|dt \leq C_{x_p}(s)\Delta t(j-1).$$

Together these facts allow to give a bound on $\mathcal{E}_0(x_\varepsilon^0)$ uniform with respect to ε and Δa :

$$\mathcal{E}_{-1}(x_\varepsilon^{-1}) \leq \left\|(1+a)^2 \varrho_\Delta\right\|_{L^1_a(\mathbb{R}_+;L^\infty_s(\Omega))} \left\|C_{x_p}\right\|^2_{L^2(\Omega)} + \left\|x_\varepsilon^{-1}\right\|^2_{\mathbf{H}^1(\Omega)} + \left\|f^0\right\|_{L^2(\Omega)} \left\|x_\varepsilon^{-1}\right\|_{L^2(\Omega)}.$$

and by induction one obtains that :

$$\mathcal{E}_n(x_\varepsilon^{n+1}) + \Delta t \sum_{p=0}^{n} \mathcal{D}_p \leq \mathcal{E}_{-1}(x_\varepsilon^{-1}) + \int_\Omega f^0 x_\varepsilon^0 + \sum_{p=0}^{n-1} f^{p+1} \delta x_\varepsilon^{p+\frac{1}{2}} ds - \int_\Omega f^{n+1} x_\varepsilon^n ds$$

which gives the final claim. ■

Here we show one of the key estimates of the paper.

Proposition 10. Under hypotheses above, and for Δt small enough, one has :

$$\sum_{n=1}^{N} \Delta t \left\{ \left\| \frac{x_\varepsilon^{n+1} - x_\varepsilon^n}{\Delta t} \right\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon \left\| \frac{\nabla\left(x_\varepsilon^{n+1} - x_\varepsilon^n\right)}{\Delta t} \right\|^2_{\mathbf{L}^2(\Omega)} \right\} \leq C,$$

where the constant does not depend neither on ε nor on Δt.

Proof. Recalling the definition of $u_{\varepsilon,j}^n$ one checks easily that

$$\varepsilon \delta u_{\varepsilon,j}^{n+\frac{1}{2}} + \Delta t \frac{\delta u_{\varepsilon,j-\frac{1}{2}}^n}{\Delta a} = \delta x_\varepsilon^{n+\frac{1}{2}} \quad \forall j \geq 1,$$

while $u_{\varepsilon,0}^n = 0$. Equivalently, because of the specific CFL condition, $u_{\varepsilon,j+1}^{n+1} = u_{\varepsilon,j}^n + \delta x_\varepsilon^{n+\frac{1}{2}}/\varepsilon$ for all $j \geq 1$. Setting $\mathbf{T}_{\varepsilon,j}^n = R_j u_{\varepsilon,j}^n$ for $j \in \mathbb{N}$, one obtains using (43) :

$$\varepsilon \delta \mathbf{T}_{\varepsilon,j}^{n+\frac{1}{2}} + \Delta t \frac{\delta \mathbf{T}_{\varepsilon,j-\frac{1}{2}}^n}{\Delta a} + \Delta t R_j \zeta_j u_{\varepsilon,j-1}^n = R_j \delta x_\varepsilon^{n+\frac{1}{2}},$$

which, summing over $j \in \mathbb{N}^*$, gives

$$\varepsilon \sum_{j \geq 1} \delta \mathbf{T}_{\varepsilon,j}^{n+\frac{1}{2}} \Delta a - \varepsilon \Delta a \mathbf{T}_{\varepsilon,0}^n + \Delta t \sum_{j \geq 1} R_j \zeta_j u_{\varepsilon,j-1}^n \Delta a = \mathfrak{s}_\varepsilon^n \delta x_\varepsilon^{n+\frac{1}{2}}.$$

where $\mathfrak{s}_\varepsilon^n := \Delta a \sum_{j \in \mathbb{N}^*} R_j$. By definition,

$$\varepsilon \Delta a \mathbf{T}_{\varepsilon,0}^{n+1} \equiv 0.$$

Adding both equations gives :

$$\varepsilon \delta \mathcal{L}_\varepsilon^{n+\frac{1}{2}} + \Delta t \sum_{j \in \mathbb{N}^*} R_j \zeta_j u_{\varepsilon,j-1}^n \Delta a = \mathfrak{s}_\varepsilon^n \delta x_\varepsilon^{n+\frac{1}{2}},$$

since $\sum_{j \in \mathbb{N}} \mathbf{T}_{\varepsilon,j}^n \Delta a = \sum_{j \in \mathbb{N}} R_j u_{\varepsilon,j}^n \Delta a = \mathcal{L}_\varepsilon^n$. Now we make the discrete difference of (46) between steps $n+1$ and n, in order to express $\delta \mathcal{L}_\varepsilon^{n+\frac{1}{2}}$ as a function of $\delta x_\varepsilon^{n+\frac{1}{2}}$. This reads :

$$(\delta \mathcal{L}_\varepsilon^{n+\frac{1}{2}}, \boldsymbol{v}) + \left(\partial_x \left(\delta x_\varepsilon^{n+\frac{1}{2}}\right), \partial_x \boldsymbol{v}\right) = (\delta f^{n+\frac{1}{2}}, \boldsymbol{v})$$

We now close the problem solved by $\delta x_\varepsilon^{n+\frac{1}{2}}$:

$$\left(\mathfrak{s}_\varepsilon^n \delta x_\varepsilon^{n+\frac{1}{2}}, \boldsymbol{v}\right) + \varepsilon\left(\nabla\left(\delta x_\varepsilon^{n+\frac{1}{2}}\right), \nabla \boldsymbol{v}\right) = \Delta t\left(\sum_{j\in\mathbb{N}^*} R_j \zeta_j u_{\varepsilon,j-1}^n \Delta a, \boldsymbol{v}\right) + \varepsilon\left(\delta f^{n+\frac{1}{2}}, \boldsymbol{v}\right). \quad (48)$$

where we assumed that $\mu_{0,\min} \leq \mathfrak{s}_\varepsilon(s)$ for a.e. $s \in \Omega$ which is a very restrictive hypothesis. This avoids possible detachment of adhesions on certain compact set of Ω. Thus setting $\boldsymbol{v} = \delta x_\varepsilon^{n+\frac{1}{2}}$ in the weak formulation above, one writes finally :

$$\begin{aligned}
&\mu_{0,\min}\left\|\delta x_\varepsilon^{n+\frac{1}{2}}\right\|^2_{L^2(\Omega)} + \varepsilon\left\|\nabla \delta x_\varepsilon^{n+\frac{1}{2}}\right\|^2_{L^2(\Omega)} \\
&\quad \leq \Delta t \left\|\sum_{j\in\mathbb{N}^*} R_j \zeta_j u_{\varepsilon,j-1}^n \Delta a\right\|_{L^2(\Omega)} \left\|\delta x_\varepsilon^{n+\frac{1}{2}}\right\|_{L^2(\Omega)} + \varepsilon\left\|\delta f^{n+\frac{1}{2}}\right\|_{L^2(\Omega)}\left\|\delta x_\varepsilon^{n+\frac{1}{2}}\right\|.
\end{aligned}$$

Using Young's inequality on the right hand side above, one has :

$$\begin{aligned}
\frac{1}{\Delta t}\sum_{n=0}^N \left\|\delta x_\varepsilon^{n+\frac{1}{2}}\right\|^2_{L^2(\Omega)} &\lesssim \sum_{n=0}^N \Delta t \left\|\sum_{j\in\mathbb{N}} \zeta_{j+1} R_{j+1} u_{\varepsilon,j}^n \Delta a\right\|^2_{L^2(\Omega)} + \frac{\varepsilon^2}{\Delta t}\left\|\delta f^{n+\frac{1}{2}}\right\|^2_{L^2(\Omega)} \\
&\lesssim \Delta t \sum_{n=0}^N \Delta a \sum_{j\in\mathbb{N}} \int_\Omega \zeta_{j+1} R_{j+1} (u_{\varepsilon,j}^n)^2 ds + C = \Delta t \sum_{n=0}^N \mathcal{D}_n + C \\
&\leq C + \sum_{m=0}^N \|f^m\|_{L^2(\Omega)} \left\|\delta x_\varepsilon^{m+\frac{1}{2}}\right\|_{L^2(\Omega)} \leq C + \frac{1}{\lambda}\|f_\Delta\|^2_{L^2((0,T)\times\Omega)} + \frac{\lambda}{\Delta t}\sum_{m=0}^N \left\|\delta x_\varepsilon^{m+\frac{1}{2}}\right\|^2_{L^2(\Omega)}.
\end{aligned}$$

where we used the estimates of Lemma 6 in order to bound the dissipation term with a constant and the product of the source term with the finite differences of x_ε^n. This ends the proof by choosing λ small enough. ■

3.4 Convergence when Δ goes to zero

Proposition 11. Under hypotheses 4 and 5, $\tilde{x}_{\varepsilon,\Delta}$, the piecewise linear interpolation of $(x_\varepsilon^n)_{n\in\mathbb{Z}}$ satisfies

$$\tilde{x}_{\varepsilon,\Delta} \in C^{0,\frac{(1-\gamma)}{4}}([0,T]; C^{0,\gamma}(\overline{\Omega}))$$

for every $\gamma \in (0,1)$, the bound is uniform with respect to Δt and ε. Thus $\tilde{x}_{\varepsilon,\Delta}$ converges strongly in $C^0(\overline{\Omega}\times[0,T])$ when Δt goes to zero. Moreover, $x_{\varepsilon,\Delta}$ converges strongly in $L^\infty((0,T); C(\overline{\Omega}))$.

Proof. Thanks to Lemma 6, $\tilde{x}_{\varepsilon,\Delta}$ belongs to $L^\infty_t \mathbf{H}^1_x$ uniformly with respect to ε, which shows weak-$\star$ convergence in this space. Weak convergence in $H^1_t \mathbf{L}^2_x$ follows from Proposition 10. The interpolation inequality

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} \leq c\|u\|_{H^{(\gamma+1)/2}(\Omega)} \leq c\|u\|^{\frac{(1-\gamma)}{2}}_{L^2(\Omega)} \|u\|^{\frac{(1+\gamma)}{2}}_{H^1(\Omega)}$$

holds for every $u \in H^1(\Omega)$ and for every $\gamma \in (0,1)$. Combined with the $L^\infty_t \mathbf{H}^1_x$ bound provided by Lemma 6, this leads to :

$$\|\tilde{x}_{\varepsilon,\Delta}(t_2) - \tilde{x}_{\varepsilon,\Delta}(t_1)\|_{C^{0,\gamma}(\overline{\Omega})} \leq c(t_2 - t_1)^{\frac{(1-\gamma)}{4}}.$$

We complete the convergence proof for $\tilde{x}_{\varepsilon,\Delta}$ by an application of the Ascoli-Arzela theorem. ■

Corollary 7. Under the previous hypotheses, the same result can be derived for $x_\varepsilon := \lim_{\Delta t \to 0} x_{\varepsilon,\Delta}$, *i.e.*

$$x_\varepsilon \in C^{0,\frac{(1-\gamma)}{4}}([0,T]; C^{0,\gamma}(\overline{\Omega}))$$

for every $\gamma \in (0,1)$, the bound is uniform with respect to ε. This implies that x_ε converges to x_0 strongly in $C^0(\overline{\Omega} \times [0,T])$ when ε goes to zero.

Proof. Considering $\tilde{x}_{\varepsilon,\Delta}$, the piecewise continuous function in time, $\partial_t \tilde{x}_{\varepsilon,\Delta}$ is bounded in $\mathbf{L}^2_{s,t}$ uniformly with respect to ε, thus $\partial_t \tilde{x}_{\varepsilon,\Delta} \rightharpoonup \partial_t x_\varepsilon$ weakly in $\mathbf{L}^2_{s,t}$ and one has that

$$\|\partial_t x_\varepsilon\|_{\mathbf{L}^2_{s,t}} \leq \liminf_{\Delta \to 0} \|\partial_t \tilde{x}_{\varepsilon,\Delta}\|_{\mathbf{L}^2_{s,t}} = \liminf_{\Delta \to 0} \left(\Delta t \sum_{n \in \mathbb{N}} \left\| \delta x_\varepsilon^{n+\frac{1}{2}} / \Delta t \right\|^2_{\mathbf{L}^2(\Omega)} \right)^{\frac{1}{2}}.$$

A similar argument provides an $L^\infty_t \mathbf{H}^1_s$ bound for x_ε. One can then follow again the same steps as in the proof of Proposition 11. ■

Next, we consider the convergence of $\mathcal{L}_{\varepsilon,\Delta}(s,t) := \sum_{n=0}^N \mathbb{1}_{(n,n+1)\Delta t}(t) \mathcal{L}^n_\varepsilon(s)$.

Proposition 12. Under hypotheses 4 and 5, for every fixed $\varepsilon > 0$, the discrete delay term converges to the continuous limit when Δa goes to zero, *i.e.*

$$\int_0^T \int_\Omega \mathcal{L}_{\varepsilon,\Delta}(s,t) \boldsymbol{\varphi}_\Delta(s,t) ds dt \to \int_0^T \int_\Omega \mathcal{L}_\varepsilon(s,t) \varphi(s,t) ds dt$$

for all $\varphi \in C^0([0,T]; L^2(\Omega))$ and $\boldsymbol{\varphi}_\Delta(s,t) := \sum_{n=0}^N \mathbb{1}_{(n,n+1)\Delta t}(t) \boldsymbol{\varphi}^n(s)$ where $\boldsymbol{\varphi}^n(s) := \int_{n\Delta t}^{(n+1)\Delta t} \boldsymbol{\varphi}(s,t) dt / \Delta t$.

Proof. Lets consider the term

$$L_\Delta(s,t) := \sum_{n=0}^N \mathbb{1}_{J_n}(t) \Delta a \sum_{j \in \mathbb{N}} x_\varepsilon^{n-1-j}(s) R_{j+1}(s) = \sum_{n=0}^N \mathbb{1}_{J_n}(t) \int_{\mathbb{R}_+} x_{\varepsilon,\Delta}(s, n\Delta t - \varepsilon a) R_\Delta(s, a + \Delta a) da$$

we aim at showing that L_Δ is close to $\tilde{L}_\Delta(s,t) := \int_{\mathbb{R}_+} x_{\varepsilon,\Delta}(s, t - \varepsilon a) R_\Delta(s, a + \Delta a) da$. Here we extend ideas of the convergence proof of Proposition **??**. The difficulties come from the presence of the space variable. The term L_Δ is well defined. Indeed, we start considering the following sum :

$$\begin{aligned}
\Delta t \sum_{n=0}^N \Delta a \sum_{j \in \mathbb{N}} \int_\Omega \left| x_\varepsilon^{n-1-j}(s) \right|^2 R_{j+1}(s) ds &\leq \Delta t \sum_{n=0}^N \Delta a \sum_{j \in \mathbb{N}} \left\| x_\varepsilon^{n-1-j} \right\|^2_{L^2(\Omega)} \|R_{j+1}\|_{L^\infty(\Omega)} \\
&\leq T \left\{ \|x_{\varepsilon,\Delta}\|^2_{L^\infty_t L^2_s} + \left\| C_{x_p} \right\|^2_{L^2(\Omega)} + \left\| x_\varepsilon^{-1} \right\|^2_{L^2_s} \right\} \Delta a \sum_{j \in \mathbb{N}} (1 + j\Delta a)^2 \|R_j\|_{L^\infty(\Omega)} \\
&\leq T \left\{ \|x_{\varepsilon,\Delta}\|^2_{L^\infty_t L^2_s} + \left\| C_{x_p} \right\|^2_{L^2_s} + \left\| x_\varepsilon^{-1} \right\|^2_{L^2_s} \right\} \|(1+a) R_\Delta\|_{L^1_a L^\infty_s}
\end{aligned}$$

where we split the sum wrt j in two parts : in the first part, for which $j \in \{0, n-1\}$, we use the $L^2_{t,s}$ bound provided by the derivative in 10, the second part uses the Lipschitz property of the past data (cf Assumptions 5 ii)). These estimates show that $L_\Delta \in L^2_{t,s}$. Indeed :

$$\int_s \int_t |L_\Delta(s,t)|^2 \, dtds = \int_s \sum_n \int_{J_n} \left| \int_{\mathbb{R}_+} x_{\varepsilon,\Delta}(s, n\Delta t - \varepsilon a) R_\Delta(s, a + \Delta a) da \right|^2 dtds$$
$$\leq \int_s \sum_n \int_{J_n} \int_{\mathbb{R}_+} R_\Delta(s, a+\Delta a) da \int_{\mathbb{R}_+} |x_{\varepsilon,\Delta}(s, n\Delta t - \varepsilon a)|^2 R_\Delta(s, a+\Delta a) dadtds$$

where one should notice that we have used, in the second line, Jensen's inequality in order to recover the square inside the integral. Since one manipulates positive quantities integrals and sums commute by the Monotone Convergence Theorem (Beppo Levi), this allows to write :

$$\int_s \int_t |L_\Delta(s,t)|^2 \, dtds$$
$$\leq \sum_n \int_{J_n} \int_\Omega \left\{ \int_{\mathbb{R}_+} R_\Delta(s, a+\Delta a) da \int_{\mathbb{R}_+} |x_{\varepsilon,\Delta}(s, n\Delta t - \varepsilon a)|^2 R_\Delta(s, a+\Delta a) \right\} dsdadt$$
$$\leq \sum_n \int_{J_n} \|R_\Delta\|_{L^1_a L^\infty_s} \int_{\mathbb{R}_+} \|x_{\varepsilon,\Delta}(\cdot, n\Delta t - \varepsilon a)\|^2_{L^2_s} \|R_\Delta(\cdot, a+\Delta a)\|_{L^\infty_s} dadt$$
$$= \|R_\Delta\|_{L^1_a L^\infty_s} \Delta t \sum_{n=0}^N \Delta a \sum_{j \in \mathbb{N}} \left\| x_\varepsilon^{n-1-j} \right\|^2_{L^2_s} \|R_{j+1}\|_{L^\infty_s}$$

the last term has already been estimated above which proves the claim. In the same way one shows that $\tilde{L}_\Delta \in L^2_{s,t}$. Combining the previous arguments and the proof of Proposition **??**, one has as well that

$$\int_s \int_t \left| L_\Delta(s,t) - \tilde{L}_\Delta(s,t) \right| dtds \leq C \sum_n \Delta t \Delta a \sum_{j \in \mathbb{N}} \left\| x_\varepsilon^{n-j-1} - x_\varepsilon^{n-j-2} \right\|^2_{L^2_s} \|R_{j+1}\|_{L^\infty_s}$$

and by the same kind of arguments as in Proposition **??**, we conclude using Proposition 10 and the Lipschitz properties of the past data that the latter rhs is $O(\Delta a)$. The rest of the proof follows similar arguments adapted to this framework. ■

Proposition 13. Under the previous hypotheses, $g_\Delta(s,t) := \sum_{n=0}^N \mathbb{1}_{J_n}(t) \delta x_\varepsilon^{n+\frac{1}{2}}(s)/\Delta t$ converges weakly in $L^2((0,T); H^1(\Omega))$ to $\partial_t x_\varepsilon$ solving

$$\int_0^T \int_\Omega \mu_0(s) \partial_t x_\varepsilon(s,t) \varphi(s,t) + \varepsilon \nabla_s \varphi \cdot \nabla_s \partial_t x_\varepsilon dsdt = \int_0^T \int_\Omega \partial_t f \varphi dsdt$$
$$+ \int_0^T \int_\Omega \int_{\mathbb{R}_+} \zeta(s,a) u_\varepsilon(s,a,t) \varrho(s,a) da \varphi(s,t) dsdt$$

Proof. At the discrete level it is enough to take (48) with $\boldsymbol{v}$ replaced by φ^n and integrate wrt n. Then since one has :

$$\begin{aligned} \mu_{0,\Delta} &\to \mu_0 && \text{in } L^\infty(\Omega) \text{ strong,} \\ g_\Delta &\rightharpoonup \partial_t x_\varepsilon && \text{in } L^2(\Omega \times (0,T)) \text{ weak,} \\ \nabla_s g_\Delta &\rightharpoonup \nabla_s \partial_t x_\varepsilon && \text{in } L^2(\Omega \times (0,T)) \text{ weak} \end{aligned}$$

one has that

$$\Delta t \sum_n \left(\mathfrak{s}_\varepsilon^n \frac{\delta x_\varepsilon^{n+\frac{1}{2}}}{\Delta t}, \varphi^n \right) \to \int_0^T \int_\Omega \mu_0 \partial_t x_\varepsilon \varphi ds dt.$$

Indeed adapting Theorem **??** to the space dependent case one obtains :

$$\sum_{j\in\mathbb{N}} \operatorname*{ess\,sup}_{s\in\Omega} |R_{j+1} - R_j| \leq \sum_{j\in\mathbb{N}} \zeta_{\max} \Delta a \operatorname*{ess\,sup}_{s\in\Omega} R_j$$

which means that $\rho_{0,\Delta} \in \mathrm{BV}_a L_s^\infty$ which is compactly embedded in $L_a^1 L_s^\infty$. Thus

$$\int_{\mathbb{R}_+} \int_\Omega \rho_{0,\Delta} \varphi ds da \to \int_{\mathbb{R}_+} \int_\Omega \rho_0 \varphi ds da, \quad \forall \varphi \in L_a^\infty L_s^1$$

in particular if φ is constant wrt a this proves that

$$\int_\Omega \int_{\mathbb{R}_+} \rho_{0,\Delta} \varphi ds da \to \int_\Omega \mu_0 \varphi ds, \quad \forall \varphi \in L_s^1.$$

Since ε is fixed one uses as well the estimates on the gradient of g_Δ in order to obtain the limit of the elliptic part. By strong convergence of $x_{\varepsilon,\Delta}$ in $L^\infty((0,T); C(\overline{\Omega}))$ the convergence of the rhs follows. ■

Theorem 24. *Under previous hypotheses, $x_{\varepsilon,\Delta}$ converges in $C([0,T], H^1(\Omega)) \cap H^1((0,T); L^2(\Omega))$ towards x_ε solving :*

$$\int_0^T \int_\Omega \{\mathcal{L}_\varepsilon[x_\varepsilon](s,t)\varphi(s,t) + \nabla_s x_\varepsilon \cdot \nabla_s \varphi\} ds dt = \int_0^T \int_\Omega f(s,t)\varphi(s,t) ds dt \tag{49}$$

for all $\varphi \in C([0,T]; L^2(\Omega))$, where the continuous delay operator reads :

$$\mathcal{L}_\varepsilon[x_\varepsilon](s,t) := \frac{1}{\varepsilon} \int_{\mathbb{R}_+} (x_\varepsilon(s,t) - x_\varepsilon(s,t-\varepsilon a)) \rho_0(s,a) da =: \int_{\mathbb{R}_+} u_\varepsilon(s,a,t) \rho_0(s,a) da.$$

3.5 Some more stability results

Now the problem solved by u_ε reads :

$$\begin{cases} (\varepsilon\partial_t + \partial_a) u_\varepsilon = (\mu_0 - \varepsilon\Delta_s)^{-1} \int_{\mathbb{R}_+} (\zeta \rho_0 u_\varepsilon)(s,a,t) da, & (s,a,t) \in \Omega \times \mathbb{R}_+ \times (0,T) \\ u_\varepsilon(s,0,t) = 0 & (s,a,t) \in \Omega \times \{0\} \times (0,T) \\ u_\varepsilon(s,a,t) = 0 & (s,a,t) \in \partial\Omega \times \mathbb{R}_+ \times (0,T) \\ u_\varepsilon(s,a,0) = u_{\varepsilon,I}(s,a) & (s,a,t) \in \Omega \times \mathbb{R}_+ \times \{0\} \end{cases} \tag{50}$$

where $u_{\varepsilon,I}(s,a) := \frac{x_\varepsilon(s,0) - x_p(s,-\varepsilon a)}{\varepsilon}$.

Theorem 25. Under hypotheses 4 and 5, and if

$$\int_{\Omega\times\mathbb{R}_+} \rho_0(s,a)|u_I(s,a)|dads < \infty,$$

one has :

$$\int_{\Omega\times\mathbb{R}_+} \rho_0(s,a)|u_\varepsilon(s,a,t)|dads \leq \int_{\Omega\times\mathbb{R}_+} \rho_0(s,a)|u_I(s,a)|dads$$

moreover if u_I s.t.

$$\sup_{a\in\mathbb{R}_+} \frac{\int_\Omega |u_I(s,a)|ds}{(1+a)} < \infty,$$

then

$$\int_\Omega |u_\varepsilon(s,a,t)|ds \in Y_T := L^\infty\left((0,T)\times\mathbb{R}_+, \frac{1}{1+a}\right)$$

and the bound is uniform wrt ε.

Proof. A simple use of Lemma 6, and convergence arguments above, one shows that

$$\int_{\Omega\times\mathbb{R}_+} \zeta\rho_0(u_\varepsilon)^2 dads \leq \frac{\zeta_{\max}}{\varepsilon}\mathcal{E}(x_p(\cdot,0)),$$

which then using again Jensen's inequality provides that

$$\int_\Omega \left(\int_{\mathbb{R}_+} \zeta\rho_0|u_\varepsilon|da\right)^2 ds \leq \frac{\zeta^2_{\max}}{\varepsilon}\mathcal{E}(x_p(\cdot,0)).$$

which then insures that for fixed ε, $f(s,t) := \int_{\mathbb{R}_+} \zeta\rho_0 u_\varepsilon da$ belongs to $L^\infty((0,T);L^2(\Omega))$. If we solve the problem : for a given $f(s,t)$ find $v(s,t)$ solving

$$\begin{cases} \mu_0 v - \varepsilon\Delta v = f, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

For almost every $t \in (0,T)$ one solves the elliptic problem, thus there exists a unique $v \in L^\infty((0,T);H^2(\Omega)\cap H^1_0(\Omega))$ by Lax-Milgram and standard elliptic regularity. These considerations allow to fulfill hypotheses of the main theorem in [3], namely for a.e. $t \in (0,T)$, $v(\cdot,t) \in L^1(\Omega)$, $\Delta v(\cdot,t) \in L^1(\Omega)$ and $\partial_\nu v\,(\cdot,t) \in L^1(\partial\Omega)$ which insures that $v(\cdot,t) \in \mathbb{X}$ where

$$\mathbb{X} := \left\{ u \in W^{1,1}(\Omega) \;\text{ s.t. }\; \left|\int \nabla u\cdot\nabla\psi ds\right| < C\|\psi\|_{L^\infty(\Omega)}\; \forall\psi \in C^1(\overline{\Omega})\right\}.$$

and thus a *Greens inequality* holds (cf. Theorem 1.3, [3]) :

$$\int_\Omega \nabla v^+\cdot\nabla\psi ds \leq \int_{\partial\Omega} H\psi - \int_\Omega G\psi, \quad \forall\varphi \in C^1(\overline{\Omega}),$$

where v^+ denotes the positive part of v and $G \in L^1(\Omega)$ and $H \in L^1(\partial\Omega)$ are given by :

$$G := \begin{cases} \Delta v & \text{on } \{v > 0\} \\ 0 & \text{on } \{v \leq 0\} \end{cases}, \quad H := \begin{cases} \partial_\nu v & \text{on } \{v > 0\} \\ 0 & \text{on } \{v < 0\} \\ \min(\partial_\nu v, 0) & \text{on } \{v = 0\}. \end{cases}$$

Applying the latter result to $|v| := v^+ - v^-$ and as v vanishes on the boundary, one obtains that

$$\int_\Omega \Delta v \ \mathrm{sgn} v \, ds \leq 0.$$

Returning to (50), one has

$$\varepsilon \partial_t u_\varepsilon + \partial_a u_\varepsilon = v$$

which gives, in the sense of characteristics :

$$\varepsilon \partial_t \int_{\mathbb{R}_+} \rho_0 |u_\varepsilon| da + \int_{\mathbb{R}_+} \zeta \rho_0 |u_\varepsilon| da \leq \mu_0 |v|$$

Integrating in space one obtains

$$\varepsilon \frac{d}{dt} \int_{\Omega \times \mathbb{R}_+} \rho_0 |u_\varepsilon| dads + \int_{\Omega \times \mathbb{R}_+} \zeta \rho_0 |u_\varepsilon| dads \leq \int_\Omega \mu_0 |v| ds$$

But then

$$\int_\Omega \mu_0 |v| ds = \int_\Omega \left(\int_{\mathbb{R}_+} \zeta \rho_0 u_\varepsilon da \right) \mathrm{sgn} v ds + \varepsilon \int_\Omega \Delta v \ \mathrm{sgn} v \, ds \leq \int_{\Omega \times \mathbb{R}_+} \zeta \rho_0 \, |u_\varepsilon| \, dads.$$

This leads to

$$\varepsilon \frac{d}{dt} \int_{\Omega \times \mathbb{R}_+} \rho_0 |u_\varepsilon| dads \leq 0$$

which applying Gronwall's Lemma gives :

$$\int_{\mathbb{R}_+} (\rho_0 |u_\varepsilon|)(s,a,t) da \, ds \leq \int_{\mathbb{R}_+} (\rho_0 |u_I|)(s,a) da \, ds$$

this gives the first result. Then, one has that $q(a,t) := \int_\Omega |u_\varepsilon| ds$ solves

$$\begin{aligned} \varepsilon \partial_t q + \partial_a q &\leq \frac{1}{\mu_{0,\min}} \int_\Omega \mu_0 |v| ds \leq \frac{1}{\mu_{0,\min}} \int_{\Omega \times \mathbb{R}_+} \zeta \rho_0 \, |u_\varepsilon| \, dads \\ &\leq \frac{\zeta_{\max}}{\mu_{0,\min}} \int_{\mathbb{R}_+} \rho_0 |u_I| dads < C \end{aligned}$$

applying then the same results as in theorem 6.1 [16], one concludes that $q \in Y_T$. ∎

The question is now to show that under hypotheses 5, the assumptions of theorem 25 are fulfilled.

Lemma 7. Under assumptions 5,

$$J := \int_{\Omega \times \mathbb{R}_+} \rho_0 |u_I| dads < C$$

where the generic constant C is finite and independent on ε.

Proof. A triangular inequality gives :

$$J \leq \int_\Omega \frac{|x_\varepsilon(s,0) - x_p(s,0)|}{\varepsilon} \mu_{0,I}(s) ds + \int_{\Omega \times \mathbb{R}_+} \frac{|x_p(s,0) - x_p(s,-\varepsilon a)|}{\varepsilon} \rho_0(s,a) dads$$

Then by similar arguments as in the theorem above, one considers the problem solved by $\hat{z}_\varepsilon(s,0) := x_\varepsilon(s,0) - x_p(s,0)$ which reads :

$$\mu \hat{z}_\varepsilon(s,0) - \varepsilon \Delta \hat{z}_\varepsilon(s,0) = \int_{\mathbb{R}_+} (x_p(s,0) - x_p(s,-\varepsilon a))\, \rho_0(s,a) da + \varepsilon \Delta x_p(s,0)$$

where, thanks to elliptic regularity and the assumption that $\Delta x_p(s,0) \in L^1(\Omega)$, one fulfills again the hypotheses of [3] and one states in the same way as above that

$$\|\mu_{0,I} \hat{z}_\varepsilon(\cdot,0)\|_{L^1(\Omega)} \leq \left\| \int_{\mathbb{R}_+} (x_p(s,0) - x_p(s,-\varepsilon a))\, \rho_0(s,a) da \right\|_{L^1(\Omega)} + \varepsilon \|\Delta x_p\|_{L^1(\Omega)}$$

which together with the Lipschitz-like assumption 5 (ii) ends the proof. ■

Lemma 8. Under assumptions 5, one has also that the second requirement on u_I holds :

$$K := \sup_{a \in \mathbb{R}_+} \frac{\int_\Omega |u_I(s,a)| ds}{(1+a)} < C,$$

where the generic constant is independent on ε.

Proof. The same triangular inequality holds but we do not integrate in age :

$$\begin{aligned}\int_\Omega |u_I| ds &\leq \int_\Omega \frac{|x_\varepsilon(s,0) - x_p(s,0)|}{\varepsilon} ds + \int_\Omega \frac{|x_p(s,0) - x_p(s,-\varepsilon a)|}{\varepsilon} ds \\ &\leq \int_\Omega \mu_{0,I} \frac{|x_\varepsilon(s,0) - x_p(s,0)|}{\varepsilon \mu_{0,\min}} ds + a \int_\Omega C_{x_p}(s) ds \leq C + a\sqrt{|\Omega|} \left\| C_{x_p} \right\|_{L^2(\Omega)}\end{aligned}$$

then dividing by $(1+a)$ and taking the supremum on $\mathbb{R}_+$ ends the proof. ■

3.6 Convergence when ε vanishes

We then prove the convergence result when ε goes to zero.

Theorem 26. Under the previous hypotheses, x_ε converges toward x_0 solving :

$$\int_0^T \int_\Omega \{\mu_1(s) \partial_t x_0(s,t) \varphi(s,t) + \nabla_s x_0 \cdot \nabla_s \varphi\}\, dsdt = \int_0^T \int_\Omega f(s,t) \varphi(s,t) dsdt \tag{51}$$

Proof. We only focus on the convergence of the delay part since the rest is standard by weak convergence and linearity of the gradient operator. Recalling that the definition of the elongation

$$\begin{aligned}&\int_{Q_T} \int_0^\infty \rho_0(s,a) u_\varepsilon(s,a,t) \varphi(s,t)\; da\; ds\; dt = \int_{\mathbb{R}_+} \int_{Q_T} \rho_0(s,a) u_\varepsilon(s,a,t)(s,a,t) \varphi(s,t)\; ds\; dt\; da \\ &= \int_0^{T/\varepsilon} \left\{ \left(\int_{\varepsilon a}^T \int_\Omega \rho_0 u_\varepsilon(s,a,t) \varphi dsdt \right) a + \int_0^{\varepsilon a} \int_\Omega \rho_0 u_\varepsilon \varphi ds \right\} da + \int_{T/\varepsilon}^\infty \int_{Q_T} \rho_0(s,a) u_\varepsilon(s,a,t) \varphi(s,t)\; ds\; dt\; da \\ &=: J_1 + J_2 + J_3\end{aligned}$$

where we set

$$u_\varepsilon(s,a,t) := \frac{(x_\varepsilon(s,t) - x_\varepsilon(s,t-\varepsilon a))}{\varepsilon}.$$

Now J_1 can be rewritten as :

$$J_1 = \int_0^{T/\varepsilon} \left\{ \int_{\varepsilon a}^T \int_\Omega \rho_0 \frac{(x_\varepsilon(s,t) - x_\varepsilon(s,t-\varepsilon a))}{\varepsilon a} \varphi ds dt \right\} a da$$

for almost every fixed $a \in (0, T/\varepsilon)$ one has convergence of the term

$$a \int_{\varepsilon a}^T \int_\Omega \rho_0 \frac{(x_\varepsilon(s,t) - x_\varepsilon(s,t-\varepsilon a))}{\varepsilon a} \varphi ds dt \to a \int_0^T \int_\Omega \rho_0 \partial_t x_0 \varphi ds dt$$

because of weak convergence in $L^2(Q_T)$ of the sequence $\frac{(x_\varepsilon(t) - x_\varepsilon(t-\varepsilon a))}{\varepsilon a}$ for every $\varphi \in L^2(Q_T)$. Moreover thanks to the estimates on ρ_0 one has that

$$\begin{aligned}
&\int_{\varepsilon a}^T \int_\Omega \rho_0 \frac{(x_\varepsilon(s,t) - x_\varepsilon(s,t-\varepsilon a))}{\varepsilon} \varphi ds dt = \int_{\varepsilon a}^T \int_\Omega \rho_0(s,a) u_\varepsilon(s,a,t) \varphi(s,t) ds dt \\
&\leq C\,(1+a) \operatorname*{ess\,sup}_{s \in \Omega} \rho_0(s,a) \left\| \frac{\int_\Omega |u_\varepsilon| \, ds}{1+a} \right\|_{L^\infty_{s,a}} \|\varphi\|_{L^\infty_{s,t}}
\end{aligned}$$

the rhs being a L^1 function in age. Applying then Lebesgue's Theorem to the function $f_\varepsilon(a) := a \int_{\varepsilon a}^T \int_\Omega \rho_0 \frac{(x_\varepsilon(s,t) - x_\varepsilon(s,t-\varepsilon a))}{\varepsilon a} \varphi ds dt \; \mathbb{1}_{(0,T/\varepsilon)}(a)$ gives the main limit term. For what concerns the rest :

$$J_2 + J_3 = \int_0^T \int_{t/\varepsilon}^\infty \int_\Omega \rho_0(s,a,t) u_\varepsilon(s,a,t) ds \; da \; \varphi(s,t) \; dt =: \int_0^T g_\varepsilon(t) dt$$

but

$$\begin{aligned}
g_\varepsilon(t) &\leq \int_{t/\varepsilon}^\infty C \rho_0(a)(1+a) \sup_{a \in \mathbb{R}_+} \frac{\int_\Omega |u_\varepsilon| ds}{(1+a)} \|\varphi\|_{L^\infty(Q_T)} da \\
&\leq C \left(1 + \frac{t}{\varepsilon}\right) \left\| (1+a)^2 \rho_0 \right\|_{L^1_a L^\infty_s} \sup_{a \in \mathbb{R}_+} \frac{\int_\Omega |u_\varepsilon| ds}{(1+a)} \|\varphi\|_{L^\infty(Q_T)}
\end{aligned}$$

which integrated in time gives that

$$|J_2 + J_3| \sim O(\varepsilon \, |\ln \varepsilon|).$$

On the other hand by standard arguments of weak convergence, one easily proves thanks to the energy estimates that

$$\int_{Q_T} \nabla x_\varepsilon \cdot \nabla \varphi \; ds \; dt \to \int_{Q_T} \nabla x_0 \cdot \nabla \varphi \; ds \; dt.$$

This proves that the limit equation is exactly the weak form of 51 stated in the claim. For the consistency with the initial condition it follows from Lemma 7. ■

Appendix

Lemma 9. If H is strictly convex function on $\mathbb{R}$ and μ a probability measure, if

$$\int_{\mathbb{R}_+} H(f(a))d\mu(a) = H\left(\int_{\mathbb{R}_+} f(a)d\mu(a)\right) \tag{52}$$

then f is constant on the support of μ.

Proof. Assume that there is a set $K \subset \operatorname{supp}\mu$ s.t. $\mu(K) > 0$ and $f(a) \neq \overline{f} := \int_{\mathbb{R}_+} f(\tilde{a})d\mu(\tilde{a})$ for almost every $a \in K$. As H is strictly convex on $\mathbb{R}$, it is also strictly convex at the point $\overline{f}$, *i.e.* there exists $\zeta \in \mathbb{R}$ s.t.

$$H(x) > H(\overline{f}) + \zeta(x - \overline{f}), \quad \forall x \neq \overline{f}$$

thus on K one has :

$$H(f(a)) > H(\overline{f}) + \zeta(f(a) - \overline{f}), \quad \text{a.e. } a \in K$$

then integrating over $\mathbb{R}_+$ one recovers :

$$\int_{\mathbb{R}_+} H(f(a))d\mu(a) > H(\overline{f})$$

which contradicts (52) so f must be equal to its average on the support of μ. ■

Using similar techniques as in the proof of the Poincaré-Wirtinger inequality we show that

Lemma 10. Let Ω be an open bounded connected set and $\omega \subset \Omega$. Define the norm : $\|x\|_V := \|x'\|_{L^2(\Omega)} + \|x\|_{L^2(\omega)}$ and $V := \{u \in L^2_{\text{loc}}(\Omega)\ ;\ \|u\|_V < \infty\}$ then there exists a constant c_V s.t.

$$\|x\|_{H^1(\Omega)} \leq c_V \|x\|_V, \quad \forall x \in V$$

Proof. Following [5, Theorem 5.8.1, p. 275], we argue by contradiction. Were the stated estimate false, there would exist for each integer $k \in \mathbb{N}$ a function $u_k \in V$ s.t.

$$\|u_k\|_{H^1(\Omega)} > k\|u_k\|_V.$$

we renormalize by defining

$$v_k := \frac{u_k}{\|u_k\|_{H^1(\Omega)}}$$

then $\|v_k\|_{H^1(\Omega)} = 1$ and the previous inequality becomes :

$$\|v_k\|_V < \frac{1}{k} \tag{53}$$

Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, there exists a subsequence $(v_{k_j})_{j\in\mathbb{N}} \subset (v_k)_{k\in\mathbb{N}}$ and a function $v \in L^2(\Omega)$ s.t.

$$v_{k_j} \to v \in L^2(\Omega).$$

From the normalization, it follows that $\|v\|_{H^1(\Omega)} = 1$. On the other hand, (53), implies that for each $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega v(s)\varphi'(s)ds = \lim_{j\to\infty} \int_\Omega v_{k_j}(s)\varphi'(s)ds = -\lim_{j\to\infty} \int_\Omega v'_{k_j}(s)\varphi(s)ds = 0.$$

Consequently $v \in H^1(\Omega)$, with $v' = 0$ a.e. $s \in \Omega$. Thus v is constant, since Ω is connected. However this conclusion contradicts the fact that $\|v\|_{H^1(\Omega)} = 1$, since $\int_\omega v^2(s)ds = 0$ and we must have $v \equiv 0$ on ω in which case $\|v\|_{H^1(\Omega)} = 0$. This contradiction establishes the claim. ∎

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